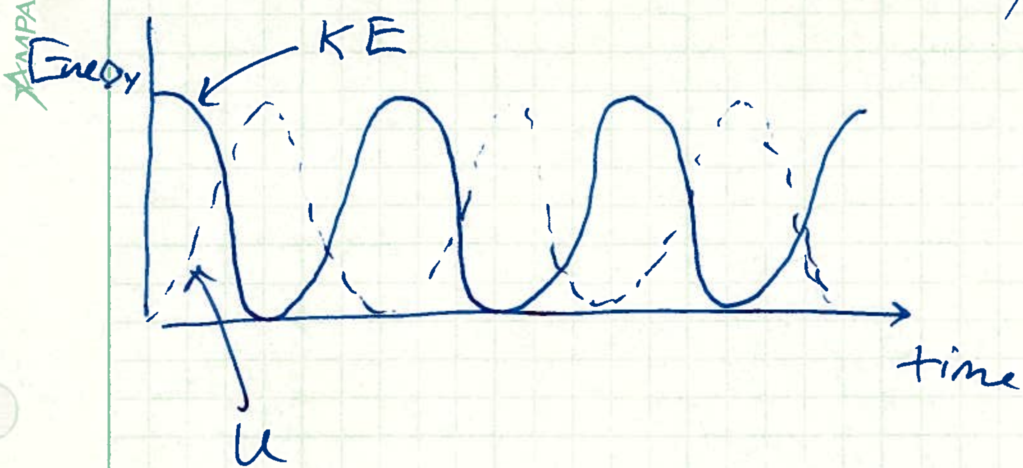
Phys 273 | Lecture 10 | 1

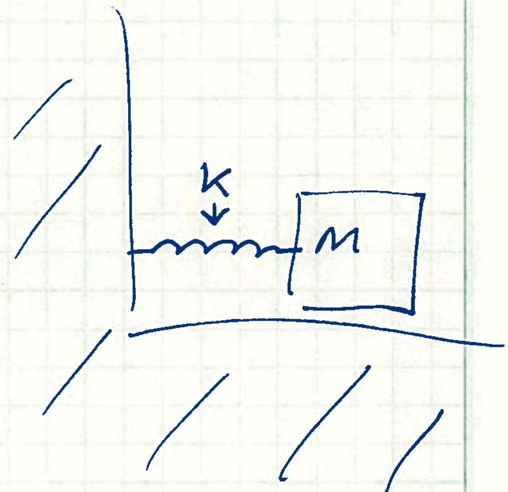
Electrical Oscillators

Mechanical Oscillator: energy is converted between kinetic and elastic potential

$KE = \frac{1}{2}mv^2 = \frac{1}{2}m\dot{x}^2$

$U = \frac{1}{2}kx^2$

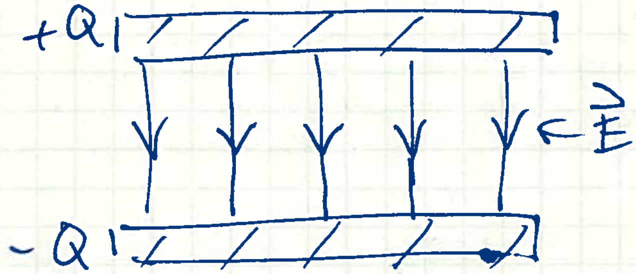




Electrical Oscillator: Energy is converted between electric ($\vec{E}$ field) and magnetic ($\vec{B}$ field).

(1) Capacitor: Device for storing energy in an electric field

Ex: Parallel Plate Capacitor



Each small volume of space (dV) with an electric field $\vec{E}$ stores a small amount of <u>electric</u> energy (dU_E):

$$dU_E = \underbrace{\tfrac{1}{2}\varepsilon_0 |\vec{E}|^2}_{u_E} dV \quad , \quad u_E \equiv \tfrac{1}{2}\varepsilon_0 |\vec{E}|^2$$

$= $ "electric energy density of free space" $= \dfrac{\text{Joules}}{\text{meter}^3}$

Energy, big U

energy density, little u

The total energy stored is

"big U" $\rightarrow U_E = \displaystyle\int_{\text{all space}} \tfrac{1}{2}\varepsilon_0 |\vec{E}|^2 dV$

If the electric field is created by a capacitor with charges $+Q$ and $-Q$ and voltage difference V, then the total energy can be written

$$U_E = \tfrac{1}{2}QV = \tfrac{1}{2}CV^2 = \frac{Q^2}{2C}$$

where $C = \text{capacitance} = \dfrac{Q}{V}$

Capacitance is a <u>constant</u> that only depends on the <u>shape</u> and <u>material</u> of the capacitor.

"$Q = CV$" says "charge on the capacitor is proportional to the voltage across it. C is the proportionality constant."

(2) Inductor: Device for storing energy in a magnetic field.

A simple solenoid:

The energy density of a magnetic field is

$$dU_B = \frac{1}{2\mu_0}|\vec{B}|^2 dV \quad , \quad u_B = \frac{1}{2\mu_0}|\vec{B}|^2$$

u_B ← "little u" = "magnetic energy density of free space" = $\frac{\text{Joules}}{\text{meter}^3}$

The total magnetic energy is

"big U" → $$U_B = \int_{\text{all space}} \frac{1}{2\mu_0}|\vec{B}|^2 dV$$

For an inductor, the total energy can be written as

$$U_B = \frac{1}{2}LI^2 \quad , \text{ where } L = \text{"self-inductance"}$$

L is determined by the shape and material of the inductor. It is the proportionality constant between magnetic flux and current:

Φ_B = magnetic flux through the inductor $= L\,I$

L: proportionality constant.

I: current

Similarities between C & L

Device	Circuit Symbol	MKS unit	Stores energy in:	Proportionality Constant:	Determined by
C		Farad	$\vec{E}$	$Q = CV$	Shape and material
L		Henry	$\vec{B}$	$\Phi_B = LI$	

If you know the shape & material of your $\left\{\begin{matrix}\text{capacitor}\\ \text{inductor}\end{matrix}\right\}$, then you can calculate $\left\{\begin{matrix}C\\ L\end{matrix}\right\}$.

Neither C nor L depends on: $\left.\begin{matrix}Q\\ V\\ I\end{matrix}\right\}$ These things depend on time, but C & L are constant.

Voltage Rules:

Capacitor: $|V_c| = \left|\frac{1}{C}Q\right|$ or $\boxed{V_C = \frac{1}{C}Q}$ (ignoring any sign)

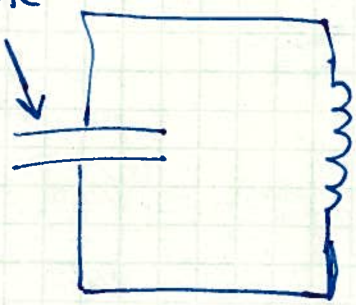
Inductor: $|V_L| = \left|-\frac{d\Phi_B}{dt}\right| = \left|-\frac{d}{dt}(LI)\right| = \left|L\frac{dI}{dt}\right|$

$\therefore \boxed{V_L = L\frac{dI}{dt}}$ (ignoring any sign)

LC oscillator

Simplest electrical oscillator.
Energy exchanges between electric & magnetic.

q = charge on plate

C  L

Voltage Loop rule:

$$V_C + V_L = \emptyset$$

$$\left(\frac{1}{C}q\right) \quad \left(L\frac{dI}{dt}\right) = L\frac{d^2q}{dt^2}$$

since $I = \frac{dq}{dt}$

$$\therefore \quad \frac{1}{C}q + L\frac{d^2q}{dt^2} = \emptyset$$

$$\boxed{\ddot{q} + \frac{1}{LC}q = \emptyset}$$

Simple harmonic oscillator equation

Solution: $q(t) = q_0 e^{i(\omega_0 t + \delta)}$, where $\omega_0 = \frac{1}{\sqrt{LC}}$ = "natural freq."

q_0 & δ are determined by the initial conditions

$$\dot{q}(t) = I(t) = (i\omega_0)(q_0 e^{i(\omega_0 t+\delta)})$$
$$= i\omega_0 q(t)$$

current has a phase shift of 90° compared to charge c

Energy: U_E = electric energy

$$= \frac{1}{2C} q^2 = \frac{1}{2C}\left[q_0 \cos(\omega_0 t + \delta)\right]^2$$

$$= \frac{q_0^2}{2C} \cos^2(\omega_0 t + \delta)$$

U_B = magnetic energy

$$= \frac{1}{2} L I^2$$

$$= \frac{1}{2} L \left[\underbrace{\mathrm{Re}\left(i\omega_0 q_0 e^{i(\omega_0 t+\delta)}\right)}_{-\omega_0 q_0 \sin(\omega_0 t+\delta)}\right]^2$$

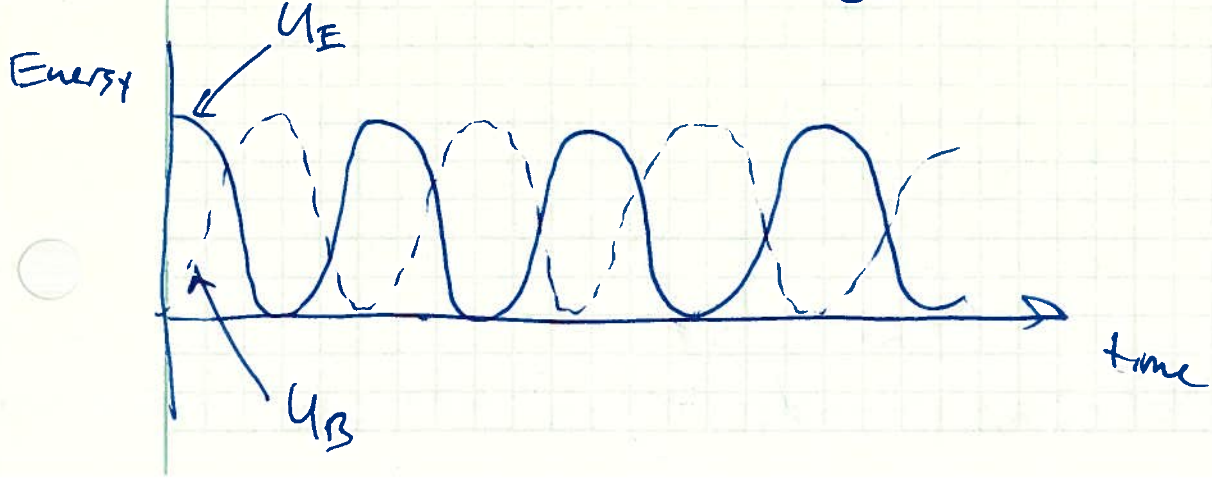
$$= \frac{1}{2} q_0^2 \underbrace{\omega_0^2 L}\, \sin^2(\omega_0 t + \delta)$$

$$\omega_0^2 = \frac{1}{LC} \quad \text{so} \quad \omega_0^2 L = \frac{1}{C}$$

$$= \frac{q_0^2}{2C} \sin^2(\omega_0 t + \delta)$$

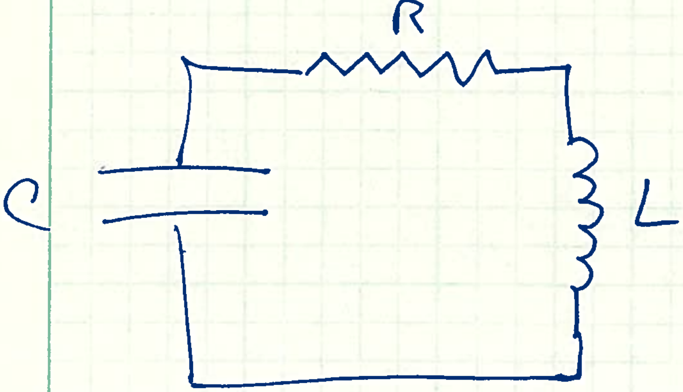
Total energy $= U_E + U_B = \frac{q_0^2}{2C}\left[\underbrace{\cos^2(\omega_0 t+\delta) + \sin^2(\omega_0 t+\delta)}_{1}\right]$

$$= \frac{q_0^2}{2C} = \text{constant}$$



LC oscillator with damping — RLC circuit

Add a resistor to the circuit: Electrical energy will be converted to heat in the resistor



Voltage Rule:

$$V_C + V_R + V_L = \emptyset$$

$V_C \downarrow \frac{q}{C}$

$V_R \downarrow IR = \dot{q}R$

$V_L \to L\frac{dI}{dt} = L\frac{d^2q}{dt^2} = L\ddot{q}$

$$\therefore \frac{q}{C} + R\dot{q} + L\ddot{q} = \emptyset$$

$$\boxed{\ddot{q} + \frac{R}{L}\dot{q} + \frac{1}{LC}q = \emptyset}$$

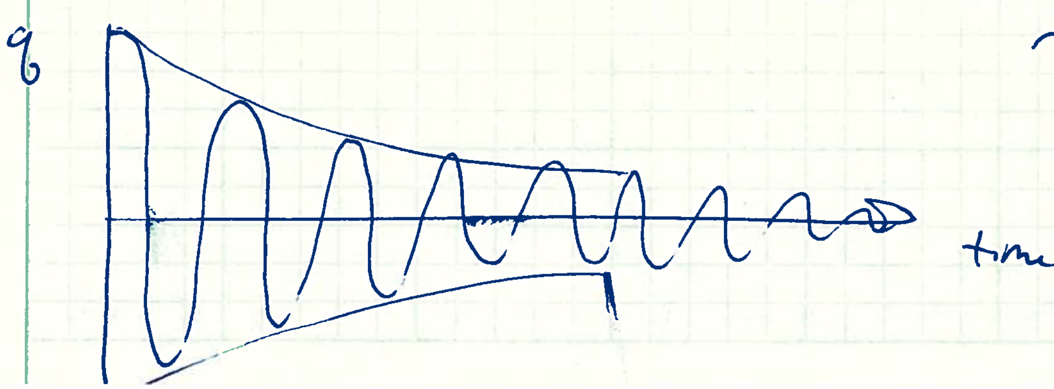
Simple Harmonic Oscillator with damping

Solution (light damping):

$$\boxed{q(t) = q_0 e^{-\gamma t/2} e^{i(\omega t + \delta)}}$$

where $\gamma = \frac{R}{L}$

and $\omega = \sqrt{\omega_0^2 - \frac{\gamma^2}{4}}$



Driven RLC circuit — Series

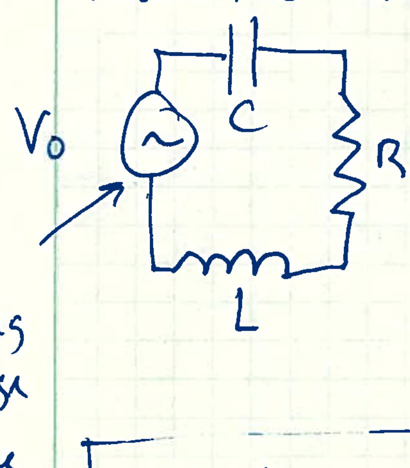
Suppose we have an oscillating circuit with a voltage source that varies in time as a cosine:

$$V_{source}(t) = V_0 \cos(\omega t) = V_0 e^{i\omega t}$$

forcing frequency

We choose δ (phase shift) $= \emptyset$ be choosing $t=\emptyset$ correctly.

Then the RLC series circuit looks like:



Voltage Loop Rule:

$$V_S = \cancel{V_R}\ V_C + V_R + V_L$$

$$\left(V_0 e^{i\omega t}\right) \quad \left(\tfrac{1}{C} q\right) \quad \left(R\dot{q}\right) \quad \left(L\ddot{q}\right)$$

$$\boxed{\ddot{q} + \left(\frac{R}{L}\right)\dot{q} + \left(\frac{1}{LC}\right)q = \left(\frac{V_0}{L}\right)e^{i\omega t}}$$

Driven Harmonic Oscillator.

Steady State Solution:

$$q(t) = q_0 e^{i(\omega t+\delta)} \text{ or } A e^{i(\omega t+\delta)}$$

where ~~$A = q_0$~~

$$q_0(\omega) = \frac{(V_0/L)}{\sqrt{(\omega_0^2-\omega^2)^2 + (\omega\gamma)^2}}$$

ω = driving frequency,

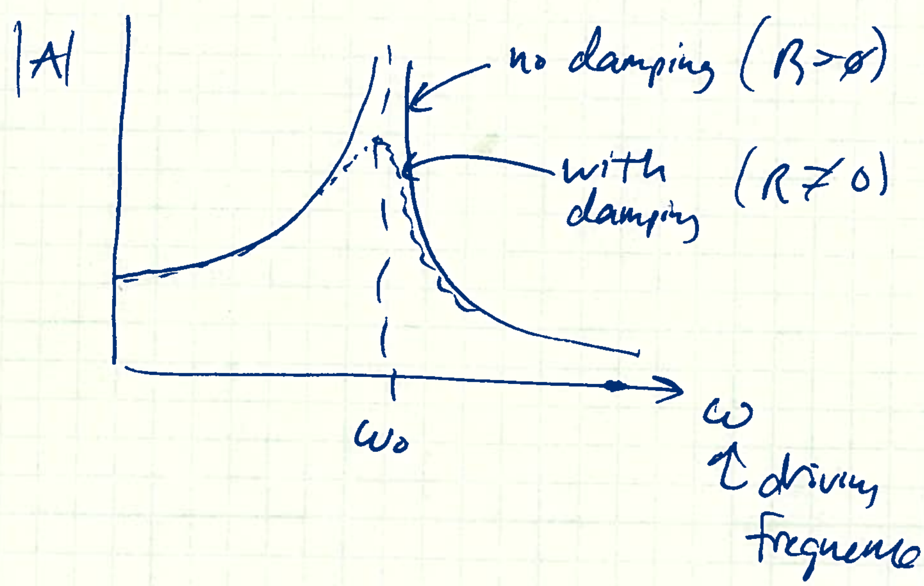
$\omega_0 = \frac{1}{\sqrt{LC}}$,

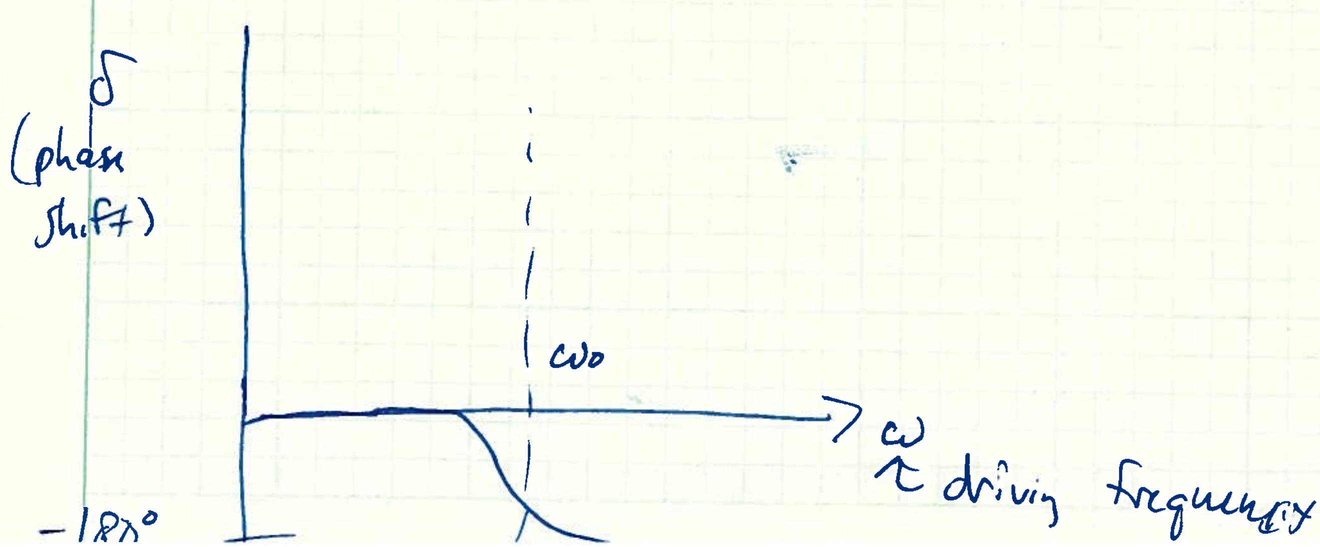
$\gamma = (R/L)$

and $$\delta(\omega) = -\tan^{-1}\left[\frac{\omega\gamma}{(\omega_0^2 - \omega^2)}\right]$$

Just like the forced mechanical oscillator we have a <u>resonance</u> when $\omega \approx \omega_0$.

When $\omega \approx \omega_0$, the following things become very large:
1) the peak charge on the capacitor
2) the peak current in the circuit
3) the total energy stored in the C & L.





Phasor Analysis of AC circuits

Phasors give us a geometric method for thinking about harmonic oscillators. Particularly useful for AC circuits.

Basic Principles

1. Resistors: Voltage Rule: $V = IR$

If $V_R = V_0 e^{i\omega t}$, then $I = \underbrace{\frac{V_0}{R}}_{\text{no phase shift}} e^{i\omega t}$

$$\boxed{V_R = IR}$$ Ohm's Law for resistors

2. Capacitors: $Q = CV$

$$I = \frac{dQ}{dt} = C\frac{dV}{dt}$$

If $V = V_0 e^{i\omega t}$, then $I = i\omega C \underbrace{V_0 e^{i\omega t}}_{V}$

or

$$V = I\left(\frac{1}{i\omega C}\right)$$

Define $\boxed{X_C \equiv \frac{1}{i\omega C}}$ "Capacitive Reactance"

Then $\boxed{V = IX_C}$ "Ohm's Law for Capacitors"

(3) Inductors: $V = L\frac{dI}{dt}$

If $V = V_0 e^{i\omega t}$

Then $I = \frac{1}{i\omega L}\underbrace{\left(V_0 e^{i\omega t}\right)}_{V}$

OR

$V = I\underbrace{(i\omega L)}$

Define $i\omega L \equiv X_L$ = "inductive reactance"

Then $\boxed{V = IX_L}$ Ohm's Law for Inductors

Summarizing

Resistors: $V = IR$, $\Rightarrow$ V & I are 100% in phase (no phase difference)

Capacitors: $V = IX_C$, $X_C = \frac{1}{i\omega C} = \frac{-i}{\omega C}$ phase difference

$\Rightarrow$ Voltage lags the current by 90° ($\pi/4$)

Inductors: $V = IX_L$, $X_L = i\omega L$ phase difference

$\Rightarrow$ Voltage leads the current by 90°.

In AC circuits, Capacitors and Inductors behave like resistors in that the peak voltage is proportional to the peak current. (Like Ohm's Law for resistors.) On the other hand,

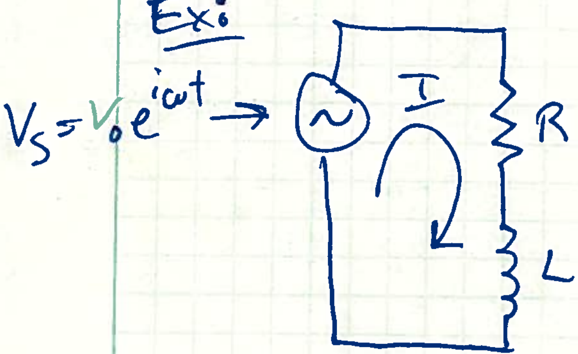
1. There is a phase shift of $+90°$ or $-90°$ between voltage & current.

2. The proportionality "constant" ~~for~~ depends on the frequency: $X_C = \frac{-i}{\omega C}$, $X_L = i\omega L$.

frequency dependent

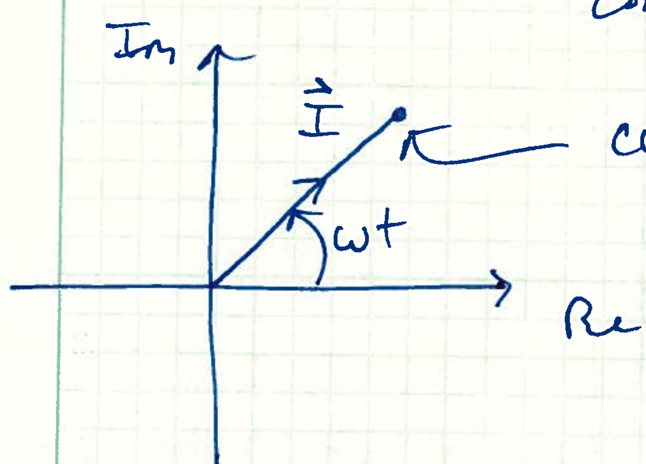
Application to Phasor Analysis

Ex:



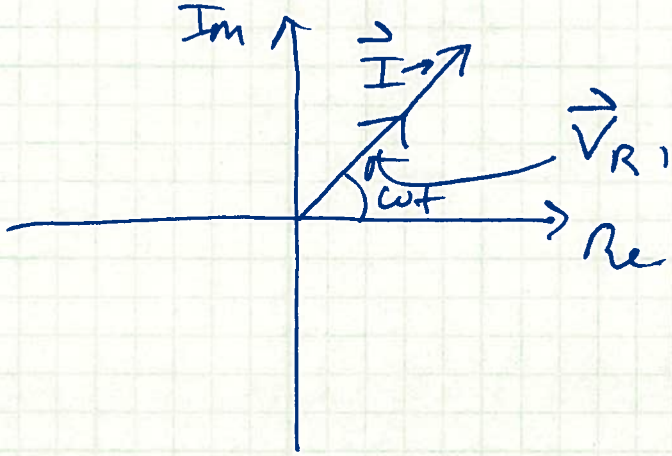
Driven RL circuit. This circuit has one and only one current.

Let's draw it in the complex plane:



current phasor, rotating at frequency ω. Makes an angle ωt with the real axis.

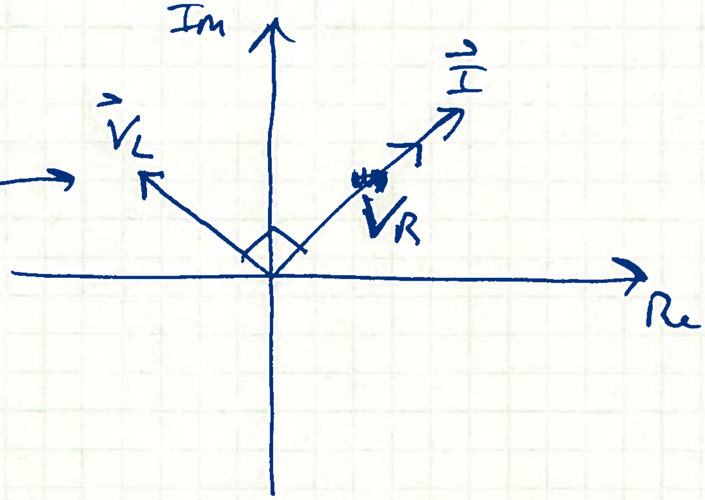
Add the voltage phasor for the resistor



voltage phasor for resistor In-phase with the current

Now add the voltage phasor for the inductor:

Voltage of Inductor leads the current by 90°.

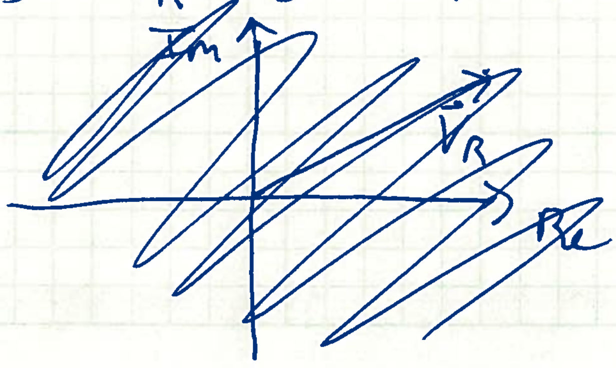


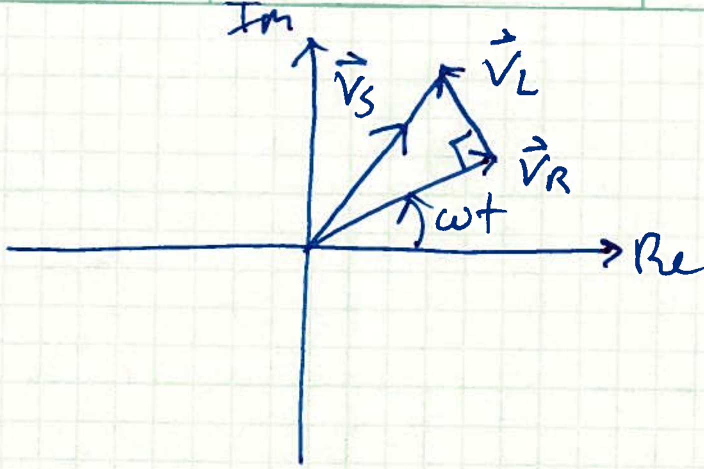
Now the voltage loop rule says:

$$V_S = V_R + V_L$$ which we can interpret geometrically.

$$\vec{V}_S = \vec{V}_R + \vec{V}_L$$ ⟸ phasors

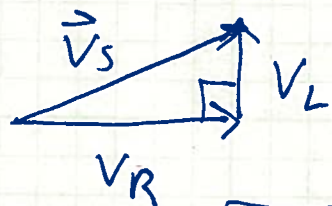
or





This is a phasor diagram for the voltages in the circuit

This is useful because we can use ~~the~~ geometry to figure out the relationships:



$$\therefore \quad |\vec{V}_S| = V_0 = \sqrt{|V_R|^2 + |V_L|^2}$$

$$|V_R| = I_0 R$$

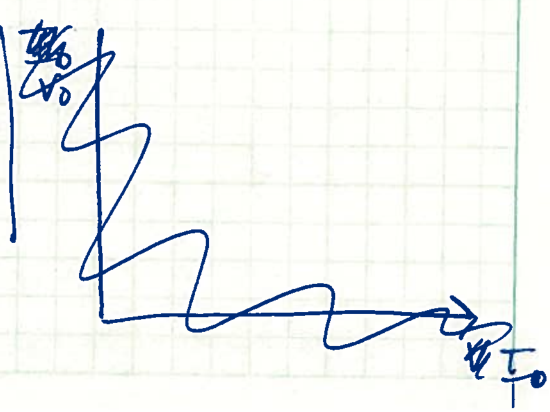
$$|V_L| = I_0 \, |(i\omega L)| = I_0(\omega L)$$

$$V_0 = \sqrt{(I_0 R)^2 + (\omega L I_0)^2}$$

$$V_0 = I_0 \underbrace{\left[\sqrt{R^2 + (\omega L)^2}\right]}_{} = I_0 Z$$

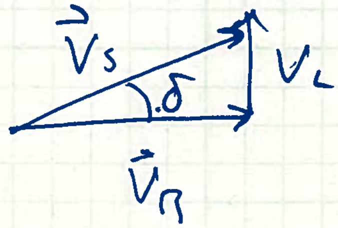
Define Z = "impedance"

Peak current $\boxed{I_0 = \dfrac{V_0}{\sqrt{R^2 + (\omega L)^2}}}$



Peak current drops as driving frequency increases

How about phase differences?



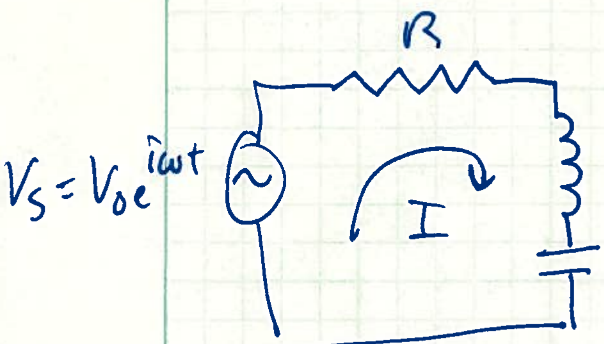
phase difference between V_R & V_S: (or I & V_S)

$$\delta = \tan^{-1}\left(\frac{|V_L|}{|V_R|}\right) = \tan^{-1}\left[\frac{\omega L I_0}{R I_0}\right]$$

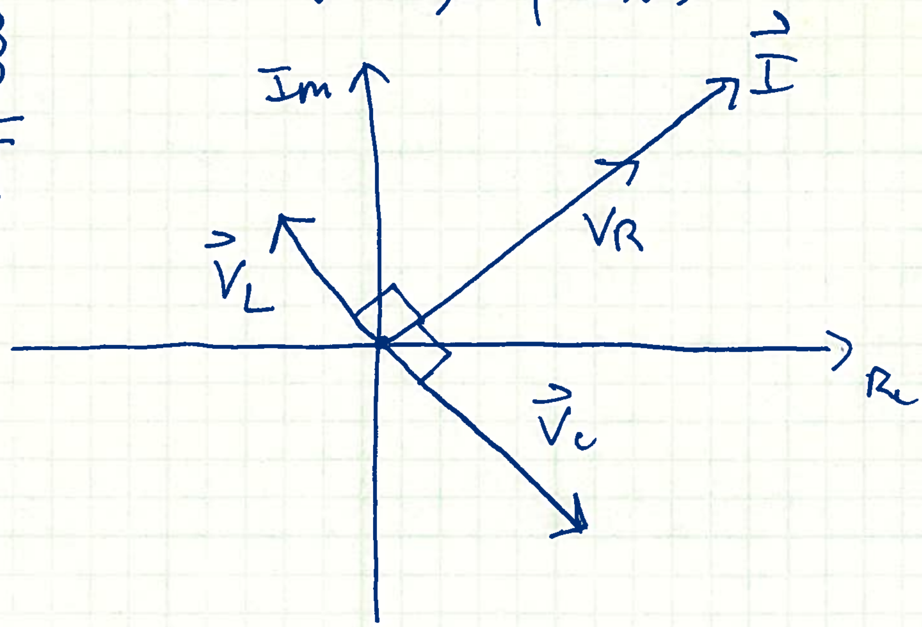
$$\boxed{\delta = \tan^{-1}\left(\frac{\omega L}{R}\right)}$$

Example

Driven RLC circuit:

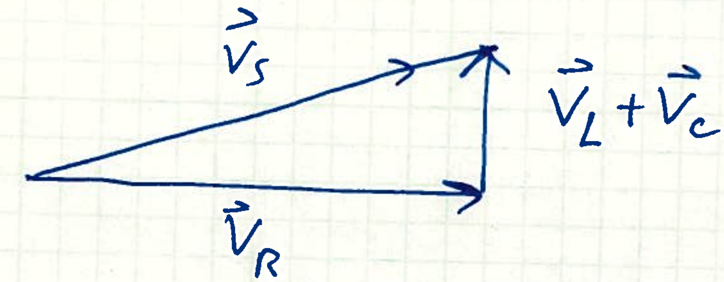


One current phasor,
four voltage phasors:



Voltage Loop Rule: $\vec{V}_S = \vec{V}_R + \vec{V}_L + \vec{V}_C$

Note that $\vec{V}_L$ & $\vec{V}_C$ are in opposite directions.



$$\therefore \ |\vec{V}_S| = V_0 = \sqrt{|V_R|^2 + (|\vec{V}_L + \vec{V}_C|)^2}$$

$$= \sqrt{(I_0 R)^2 + \left(I_0 \omega L - \frac{I_0}{\omega C}\right)^2}$$

$$= I_0 \underbrace{\sqrt{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2}}_{Z = \text{impedance of this circuit}}$$

$$\therefore \ I_0 = \text{peak current} = \frac{V_0}{\sqrt{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2}}$$

$$= \frac{(V_0/L)}{\sqrt{\left(\frac{R^2}{L^2}\right) + \left(\omega - \frac{1}{\omega L C}\right)^2}}$$

$$\frac{R}{L} = \gamma \ , \quad \frac{1}{LC} = \omega_0^2$$

$$\boxed{I_0 = \frac{\omega (V_0/L)}{\sqrt{(\omega\gamma)^2 + (\omega^2 - \omega_0^2)^2}}}$$

Resonance when $\omega \approx \omega_0$.

Previously we found by solving the differential equation that

$$q_0 = \frac{(V_0/L)}{\sqrt{(\omega\gamma)^2 + (\omega^2 - \omega_0^2)^2}} \quad \text{or} \quad q(t) = q_0 e^{i\omega t}$$

which means that

$$I(t) = \dot{q}(t) = i\underbrace{\omega q_0}_{I_0 = \omega q_0} e^{i\omega t}$$

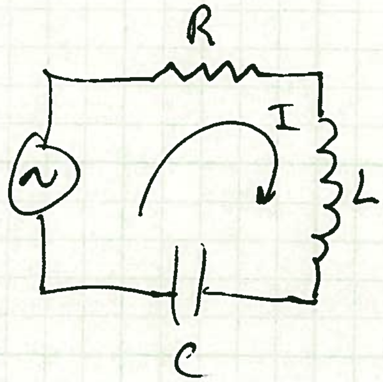
or

$$I_0 = \frac{\omega (V_0/L)}{\sqrt{(\omega\gamma)^2 + (\omega^2 - \omega_0^2)^2}}$$

So we got the same result without calculating with the differential equation.

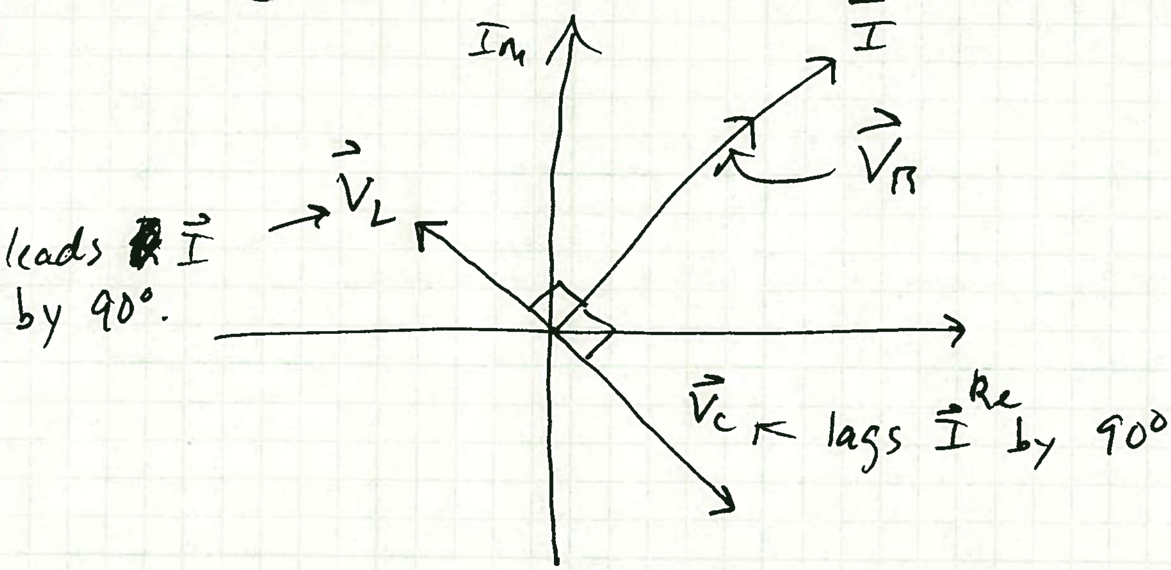
Driven RLC (series) circuit

$V_S = V_0 e^{i\omega t}$

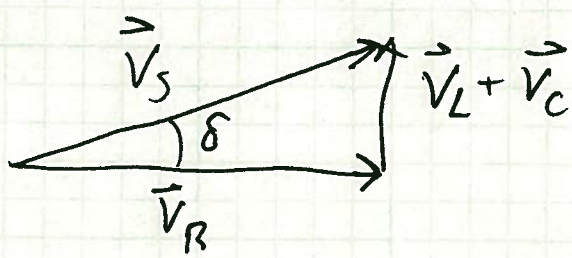


Only one current phasor: $\vec{I}$
(this circuit has only one current).

Voltage phasors can be drawn relative to $\vec{I}$: $\vec{V}_R, \vec{V}_L, \vec{V}_C$



Voltage Loop Rule: $\vec{V}_S = \vec{V}_R + \vec{V}_L + \vec{V}_C$



Phase shift:

$$\delta = \tan^{-1}\left[\frac{|\vec{V}_L + \vec{V}_C|}{|\vec{V}_R|}\right] \cancel{= \tan^{-1}\left[\frac{|i\omega L I_0 - \frac{i I_0}{\omega C}|}{|R I_0|}\right]}$$

2

$$= \tan^{-1}\left[\frac{\underbrace{\left|i\omega L I_0 - \frac{iI_0}{\omega C}\right|}}{\underbrace{|RI_0|}}\right]$$

$$\left|i\omega L I_0 - \frac{iI_0}{\omega C}\right| = I_0\left|i\omega L - \frac{i}{\omega C}\right| = I_0\left(\omega L - \frac{1}{\omega C}\right)$$

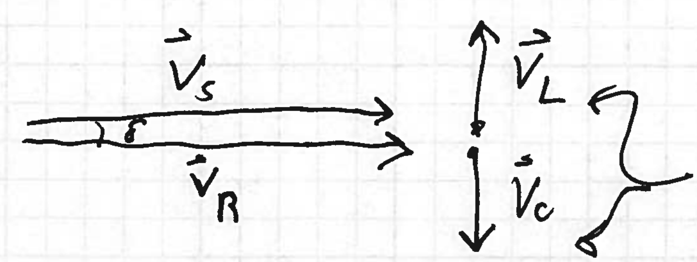
$$|RI_0| = RI_0$$

$\neq$ ~~tan~~

$$= \tan^{-1}\left[\frac{I_0\left(\omega L - \frac{1}{\omega C}\right)}{I_0\,(R)}\right]$$, or, using $\omega_0^2 = \frac{1}{LC}$ and $\gamma = \frac{R}{L}$,

$$\boxed{\delta = \tan^{-1}\left[\frac{\omega^2 - \omega_0^2}{\omega\gamma}\right]}$$

What does this mean? Well, if we choose to drive the circuit with frequency $\omega = \omega_0$, then the phase difference between $\vec{V}_S$ and V_R is zero. Then the phasor diagram looks like:



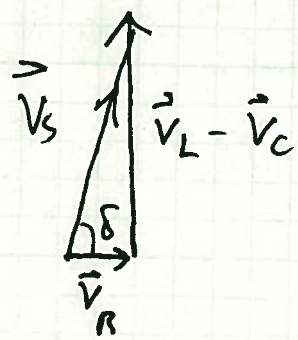
$\vec{V}_L$ & $\vec{V}_C$ are equal and opposite, when $\omega = \omega_0$.

Conversely, suppose we choose ω = very, very large.

Then $\vec{V}_c = \left(\frac{-i}{\omega C}\right)\vec{I} \approx \emptyset$

and $\vec{V}_L = (i\omega L)\vec{I} \approx$ large, and the

phasor diagram is

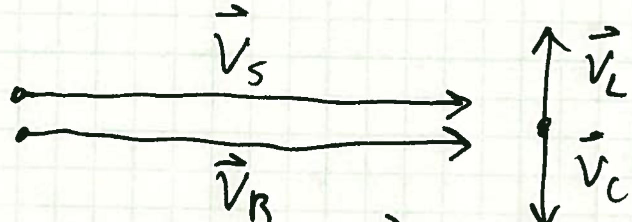


, and $\delta \to 90°$.

~~When does this~~

Question: At what driving frequency is the current maximal?

Answer: $\vec{I}$ is maximal when $\vec{V}_R$ is maximal, since $\vec{V}_R = \vec{I}R$. But $|\vec{V}_R|$ can never be larger that $|\vec{V}_S|$; ~~since they have to add to zero in the~~



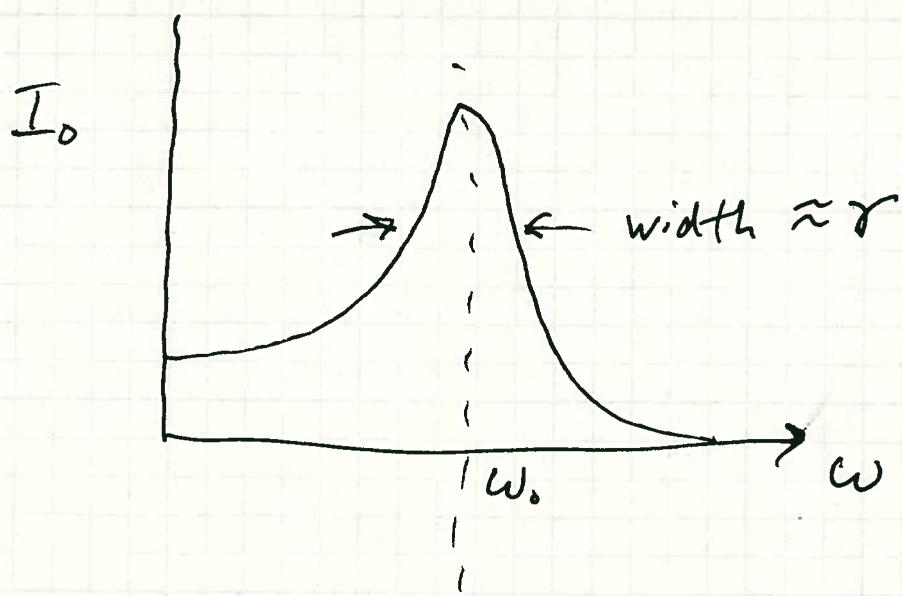
This happens when $\vec{V}_L + \vec{V}_c = \emptyset$

or $(i\omega L)I_0 + \left(\frac{-i}{\omega C}\right)I_0 = \emptyset$

or $\omega^2 = \frac{1}{LC}$

$$\boxed{\omega = \frac{1}{\sqrt{LC}} = \omega_0}$$ ← condition for maximal current.

The amplitude of the current displays a resonanance near $\omega = \omega_0$:



Series and Parallel impedance

We have the following rules for impedances in AC circuits:

① Resistors: $V = I Z_R$, where $Z_R = R$

② Capacitors: $V = I Z_C$, where $Z = \frac{-i}{\omega C}$

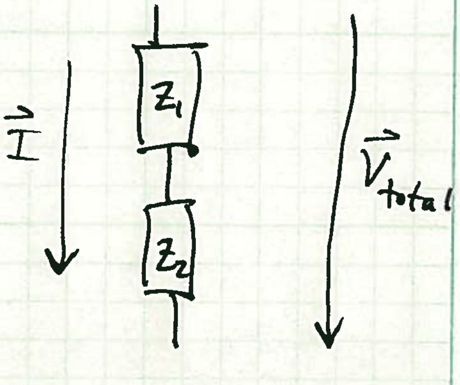
③ Inductors: $V = I Z_L$, where $Z_L = i\omega L$

phase shift between V & I

IF we combine two elements <u>in series</u>, the
the <u>total voltage drop</u> is

$$\vec{V}_{total} = \vec{I}Z_1 + \vec{I}Z_2 \leftarrow \text{because}$$
$$= \vec{I}(Z_1 + Z_2)$$

Z_1 & Z_2 share the same $\vec{I}$



$\therefore \; \vec{V}_{total} = \vec{I}\left(Z_{\cancel{total}\,series}\right)$ where $\boxed{Z_{series} = Z_1 + Z_2.}$

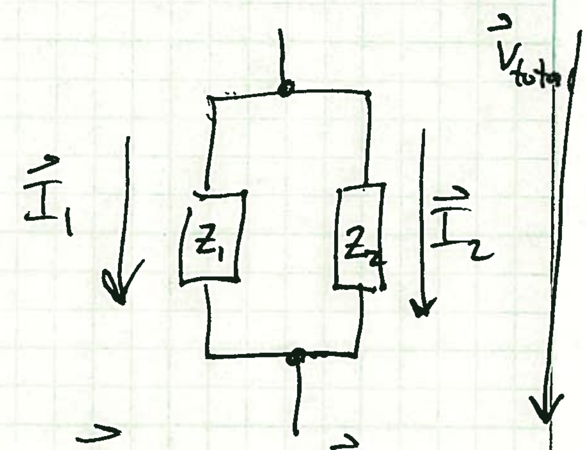
$\therefore$ The impedance of two elements in series is the simple sum of Z_1 & Z_2.

If we combine two elements <u>in parallel</u>, then the <u>total ~~current~~ voltage drop</u> is

$$\vec{V}_{total} = \vec{I}_1 Z_1$$

<u>and</u>

$$\vec{V}_{total} = \vec{I}_2 Z_2 .$$



The total current is

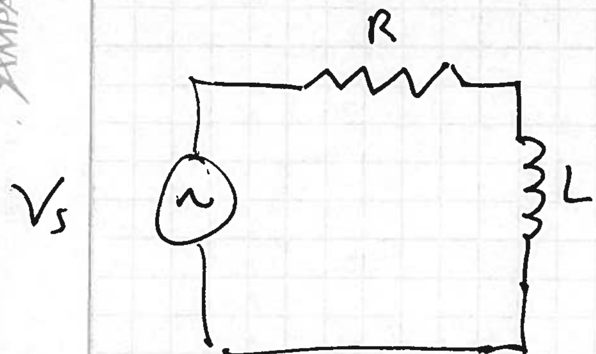
$$\vec{I}_{total} = \vec{I}_1 \cancel{Z_1} + \vec{I}_2 \cancel{Z_2} = \frac{\vec{V}_{total}}{Z_1} + \frac{\vec{V}_{total}}{Z_2}$$

$$\therefore \; \vec{V}_{total} = \vec{I}_{total}\left(\frac{1}{Z_1^{-1} + Z_2^{-1}}\right)$$

$$\vec{V}_{total} = \vec{I}_{total}\, Z_{parallel}$$, where $\boxed{Z_{parallel} = \frac{1}{Z_1^{-1} + Z_2^{-1}}}$

$$\therefore \boxed{\begin{aligned} Z_{series} &= Z_1 + Z_2 \\ Z_{parallel} &= \frac{1}{Z_1^{-1} + Z_2^{-1}} \end{aligned}}$$

Example RL ~~par~~. series circuit (driven)



$$Z_{total} = Z_R + Z_L$$

$$Z_{total} = R + i\omega L$$

$$\therefore \vec{V}_S = \vec{I}_{total} Z_{total}$$

$$\vec{V}_S = \vec{I}_{total} (R + i\omega L)$$

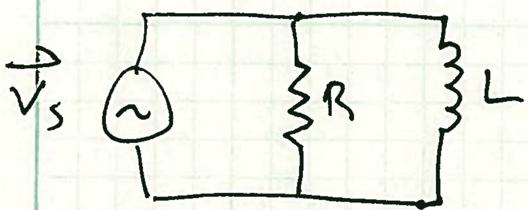
$\therefore$ phase difference between $\vec{V}_S$ & $\vec{I}_{total}$:

$$\delta = \text{phase of } (R + i\omega L) = \tan^{-1}\left(\frac{\omega L}{R}\right)$$

$\therefore$ magnitude of current:

$$|\vec{I}_{total}| = I_0 = \frac{\overbrace{|\vec{V}_S|}^{V_0}}{\underbrace{|R + i\omega L|}_{\sqrt{R^2 + (\omega L)^2}}} = \frac{V_0}{\sqrt{R^2 + (\omega L)^2}}$$

Example RL parallel circuit (driven)



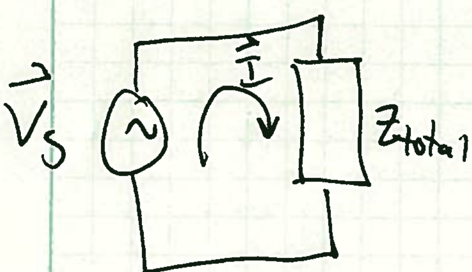
$$Z_{total} = \frac{1}{Z_R^{-1} + Z_L^{-1}}$$

$$= \frac{1}{R^{-1} + (i\omega L)^{-1}}$$

$$= \frac{(R\omega L) \times (\omega L + iR)}{(\omega L - iR) \times (\omega L + iR)}$$

$$= \left(\frac{R\omega L^2}{(\omega L)^2 + R^2}\right)\left(\omega + i\frac{R}{L}\right)$$

or



$$Z_{total} = \left(\frac{R\omega L^2}{(\omega L)^2 + R^2}\right)(\omega + i\gamma)$$

$$\therefore \vec{V}_{s} = \vec{I}_{total}\left[\left(\frac{R\omega L^2}{(\omega L)^2 + R^2}\right)(\omega + i\gamma)\right]$$ where $\gamma \equiv \frac{R}{L}$

Phase difference between $\vec{V}_s$ & $\vec{I}_{total}$:

$$\delta = \tan^{-1}\left(\frac{\gamma}{\omega}\right)$$ ← For very high ω, $\delta \to \emptyset$, like a resistor

← For very low ω, $\delta \to 90°$, like an inductor.