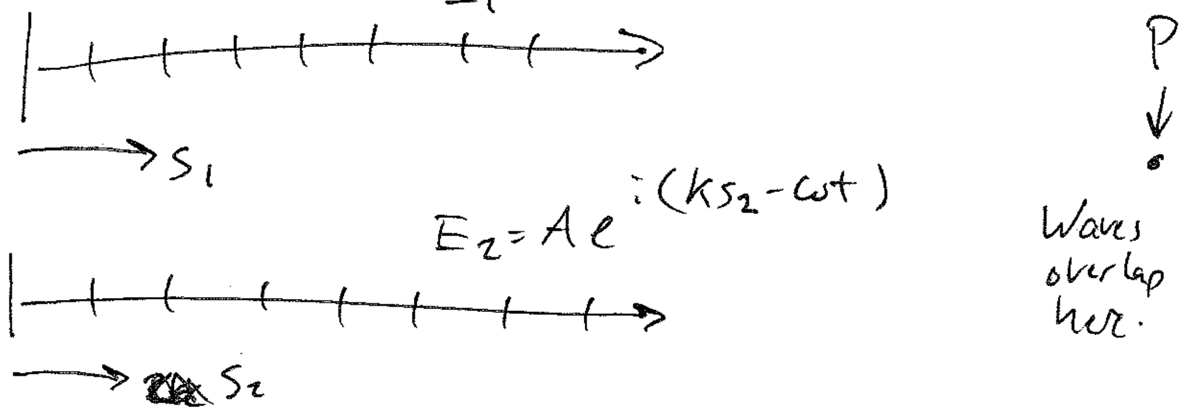
Interference & Diffraction

Two Beam interference

Suppose two traveling waves overlap at some point in space:

$$E_1 = Ae^{i(ks_1-\omega t)}$$



s_1: measures distance along travelling wave #1

s_2: " #2

Note: we assume that (1) Both waves have the same amplitude (A)

(2) Both waves have the same wavenumber (k).

Then the total electric field at point P is the linear sum:

$$E_p = E_1 + E_2 = Ae^{i(ks_1-\omega t)} + Ae^{i(ks_2-\omega t)}$$

$$E_p = A\left(e^{i(ks_1-\omega t)} + e^{i(ks_2-\omega t)}\right)$$

<u>Define:</u> $\bar{s} \equiv \frac{s_1+s_2}{2}$ (average of s_1 & s_2)

$\delta \equiv s_2 - s_1$ (difference between s_1 & s_2)

Then: $s_1 = \bar{s} - \frac{\delta}{2}$

$s_2 = \bar{s} + \frac{\delta}{2}$

So that

$$E_p = A e^{-i\omega t}\left(e^{ik(\bar{s}-\delta/2)} + e^{ik(\bar{s}+\frac{\delta}{2})}\right)$$

$$= A e^{i(k\bar{s}-\omega t)} \underbrace{\left(e^{-ik\delta/2} + e^{ik\delta/2}\right)}_{2\cos(k\delta/2)}$$

$$= 2A \underbrace{e^{i(k\bar{s}-\omega t)}}_{\text{"wave factor"}} \underbrace{\cos(k\delta/2)}_{\text{"Amplitude factor"}}$$

For optical waves, $\omega \approx 10^{14}$ Hz, so the wave factor is oscillating extremely quickly, too quickly to observe with any conventional instrumentation. Your eye will only notice

~~The~~ The average value of the wave factor, which is $\frac{1}{2}$.

What your eye sees is the Intensity, which is proportional to the electric field squared:

$$I \sim |E_p|^2 \sim \underbrace{4A^2\cos^2(k\delta/2)}$$

here I've ignored the factor of $(\frac{1}{2})^2$ that comes from the average value of the wave factor. I've absorbed this factor into the proportional sign ($\sim$).

Question: What is $(k\delta)$?

Answer: It is the phase difference between E_1 & E_2 due to the fact that they travelled different distances to reach point P. In other words,

$$\delta = S_2 - S_1 = \text{path length difference}$$
$$= \text{units of meters}$$

~~And we multiply~~

And to convert a path length difference into a phase we multiply it by K.

$$\Delta\phi = K\delta = \frac{2\pi}{\lambda}\delta = 2\pi\left(\frac{\delta}{\lambda}\right)$$

phase difference

For example, if $\delta = \lambda$, then $\Delta\phi = 2\pi$ radians

if $\delta = \frac{\lambda}{2}$, then $\Delta\phi = \pi$ radians

if $\delta = 4.5\lambda$, then $\Delta\phi = (4.5)(2\pi)$
$= 9\pi$ radians.

So the intensity at point P can be written

$$\boxed{I \sim 4A^2 \cos^2\left(\frac{\Delta\phi}{2}\right)}$$

This will be true for any 2-beam interference experiment where both beams have equal amplitude and wave number.
The only question is what's the phase difference between the waves? If the phase difference is due to a path length difference (δ), then

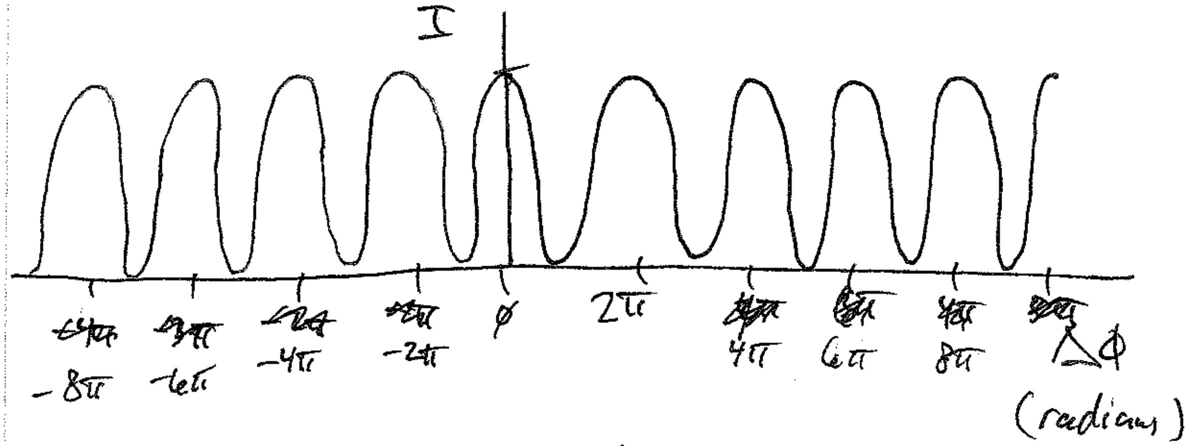
$$\Delta\phi = k\delta$$

But no matter what caused the phase difference, the intensity will be $I \sim 4A^2\cos^2(\Delta\phi/2)$

Note that:

① The maximum intensity is (4 x) larger than it would be if there were only one beam.

(2) The intensity depends strongly on the phase difference between the waves

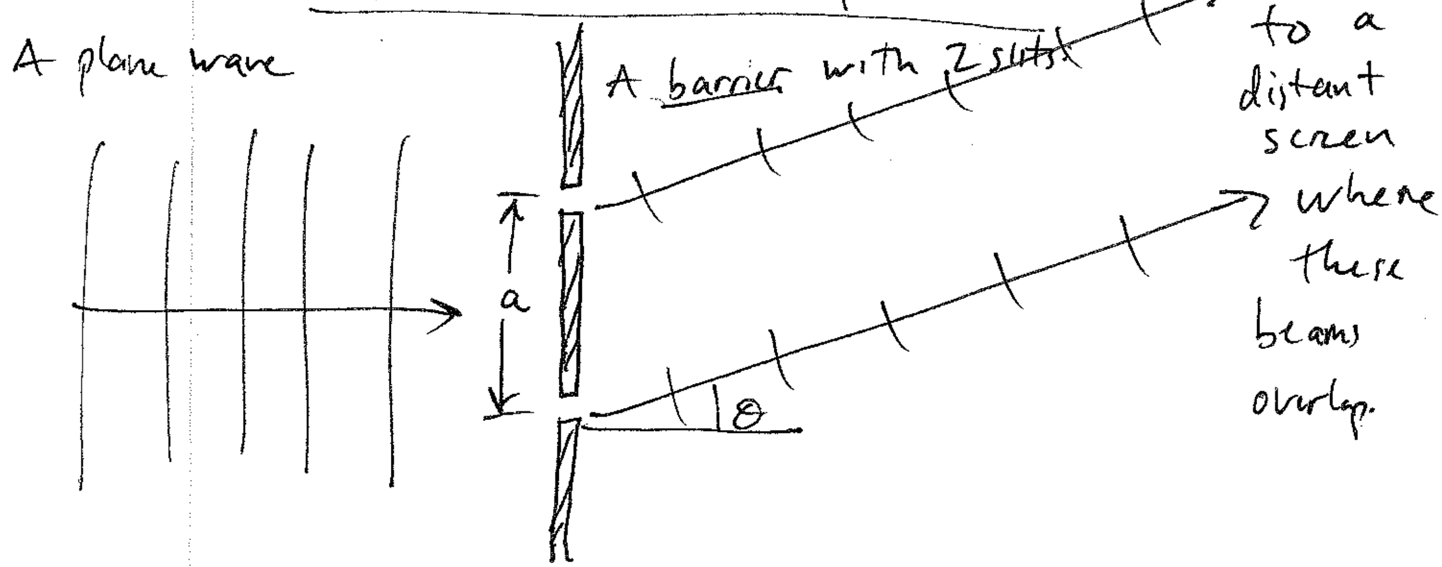


(3) When $\Delta\phi = m(2\pi)$, where m = integer, we have constructive interference, with an intensity maximum.

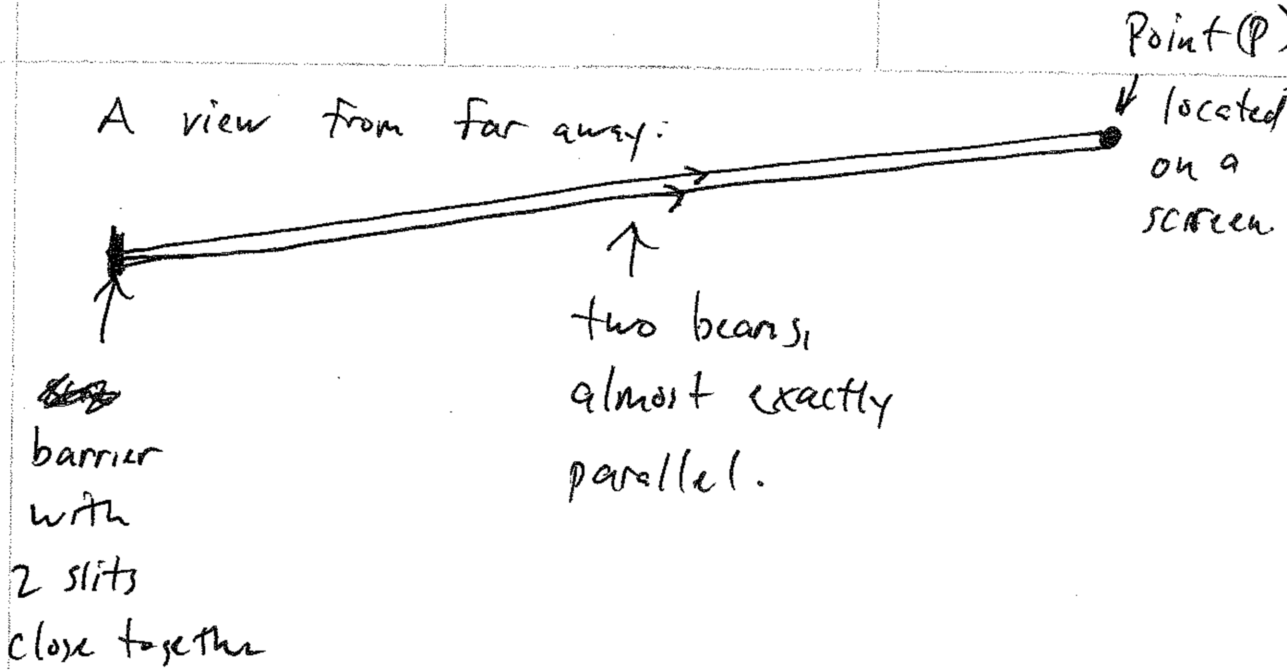
(4) When $\Delta\phi = \left(m + \frac{1}{2}\right)(2\pi)$, where m = integer we have destructive interference, with an intensity ~~minimum~~ zero.

An example of 2-beam interference:

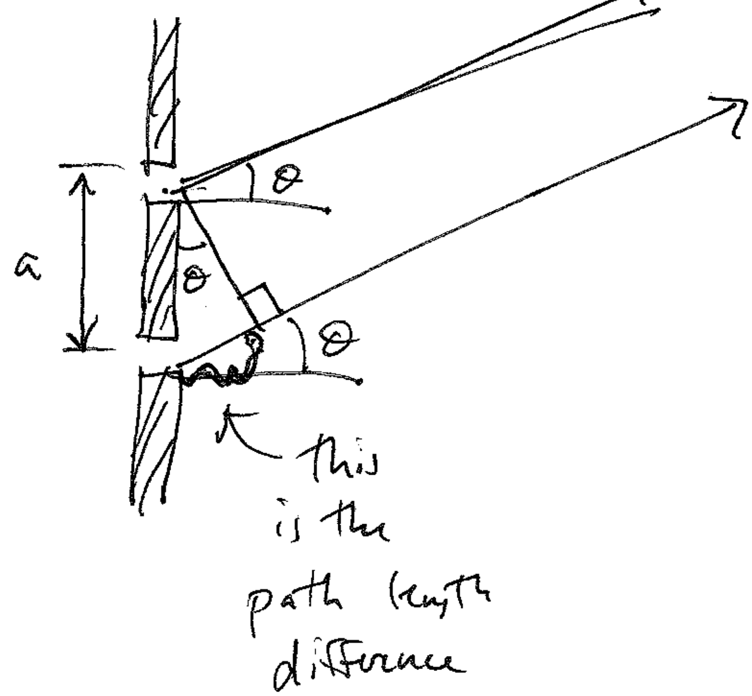
Young's Double Slit experiment:



A view from far away:



What's the path length difference between the 2 beams? Zoom in on the barrier again.

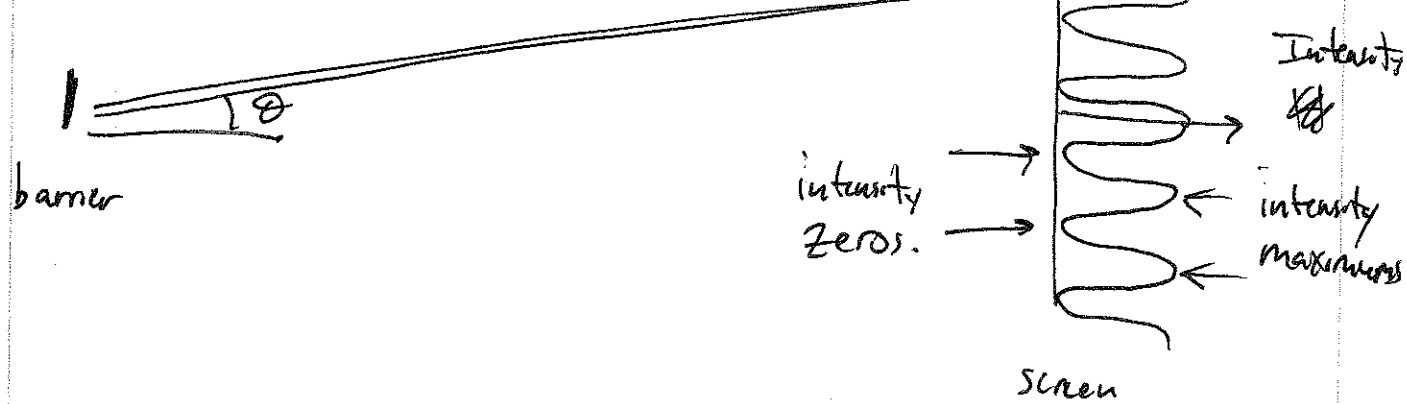


From the geometry, $\delta = a \sin\theta$, so

$$\Delta\phi = ka\sin\theta$$

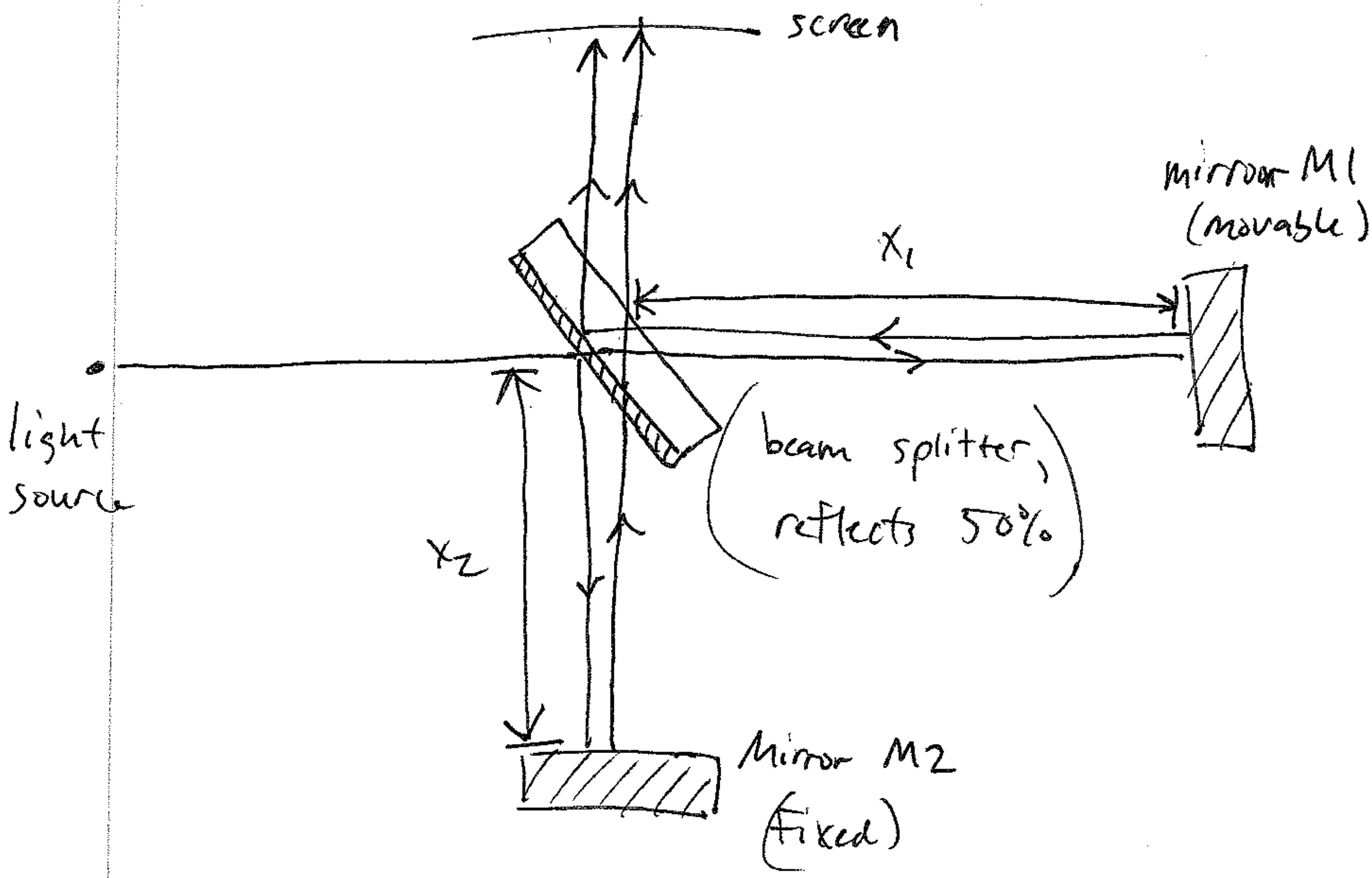
and $I \sim 4A^2 \cos^2\left(\frac{ka\sin\theta}{2}\right) = 4A^2\cos^2\left(\frac{\pi a \sin\theta}{\lambda}\right)$

On the screen we would see



Another example of 2-beam interference:

Michelson Interferometer:



Path difference: $\delta = 2(x_1 - x_2)$

↑ because each beam travels out-and-back in each arm.

phase difference due to path difference

$$= k\delta = 2k(x_2 - x_1)$$

phase difference due to internal reflection of beam travelling back from M1 $= \pi$.

Total phase difference $= \Delta\phi = 2k(x_2-x_1) + \pi$

Intensity = $I \sim 4A^2 \cos^2\left(\frac{\Delta\phi}{2}\right)$

$$\sim 4A^2 \cos^2\left(k(x_2-x_1) + \frac{\pi}{2}\right)$$

For constructive interference we should have

$$\Delta\phi = m(2\pi) \quad \text{where } m = \text{integer}$$

or $\quad \frac{\Delta\phi}{2} = m\pi$

$$\frac{2k(x_2-x_1) + \pi}{2} = m\pi$$

$$\left(\frac{2\pi}{\lambda}\right)(x_2-x_1) + \frac{\pi}{2} = m\pi$$

$$\boxed{\frac{2(x_2-x_1)}{\lambda} = m + \frac{1}{2}}, \quad m = \text{integer}.$$

condition for constructive interference.

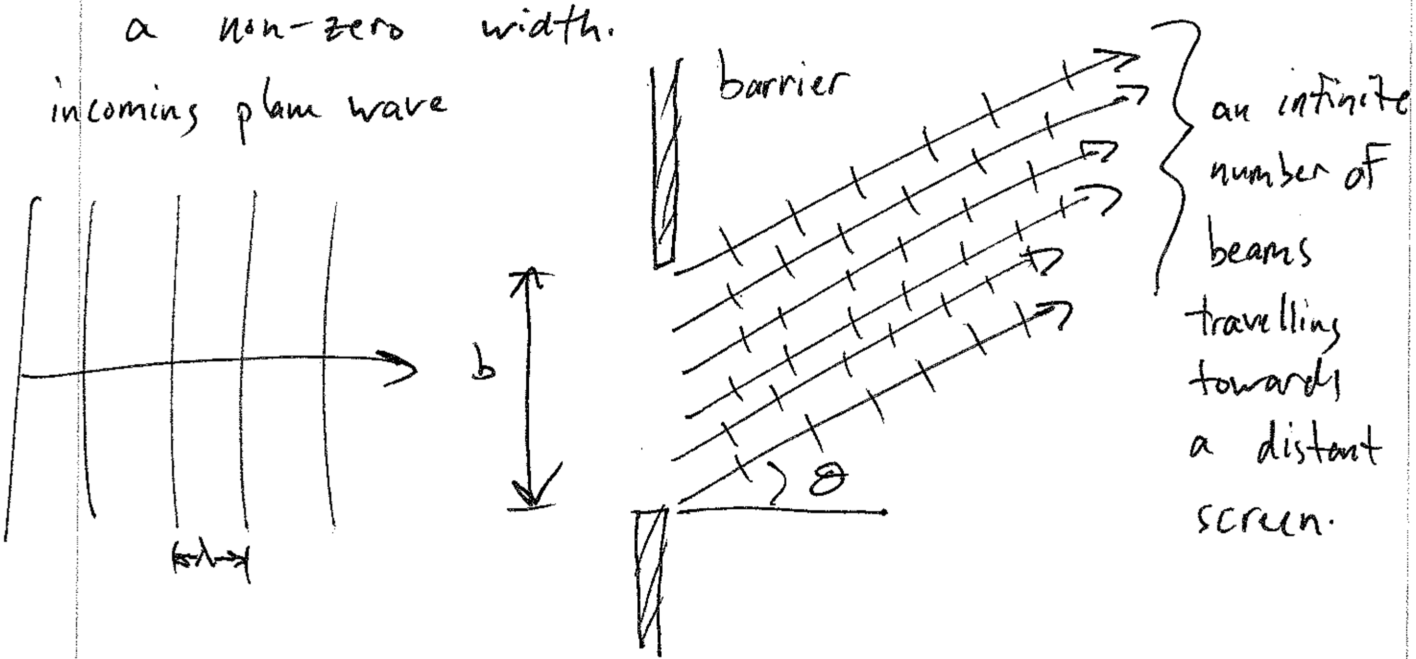
Destructive Interference:

$$\boxed{\frac{2(x_2 - x_1)}{\lambda} = m}$$

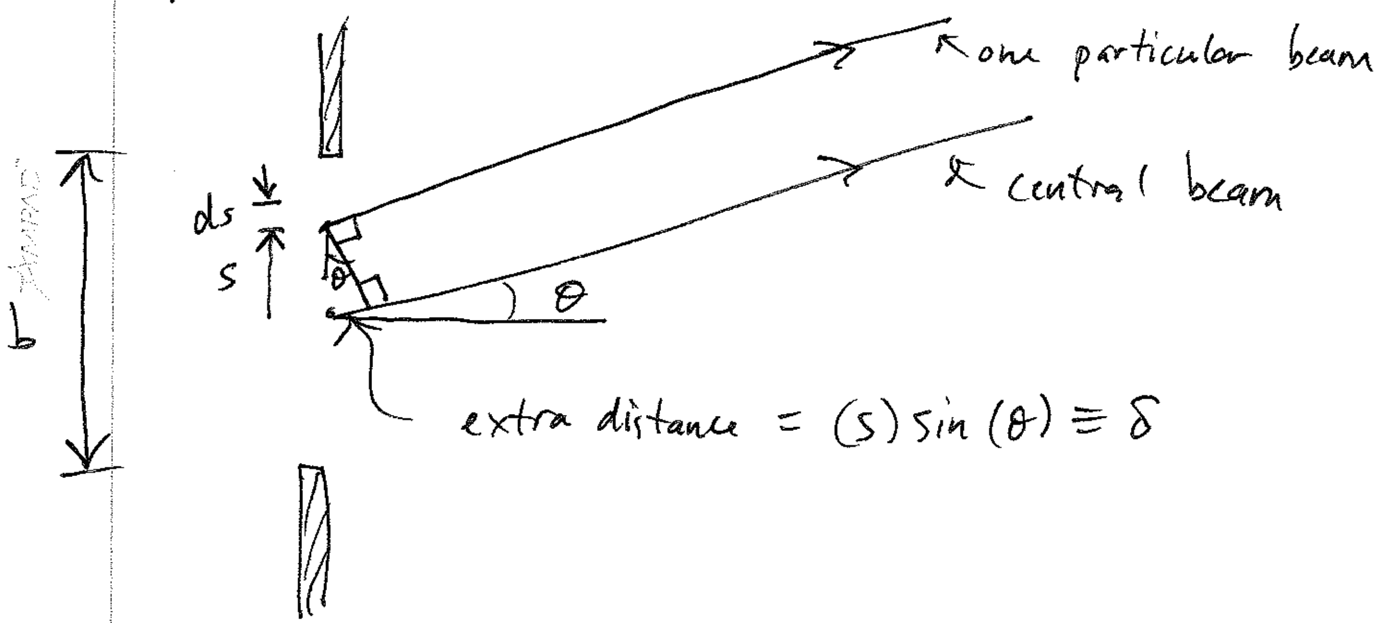
If M1 is movable, then we can observe the interference maxima and minima and count them as the mirror moves. By counting these maxima and measuring how far the mirror has moved, we can get a measurement of the wavelength.

Single Slit Diffraction

Diffraction is the interference of an infinite number of beams. The beams originate within a slit in a barrier, the slit having a non-zero width.



These beams will interfere with each other because they have relative path length differences. How much? Consider one particular beam compared to the central beam:



The total Electric field at a particular location on the screen is the sum over all such fields created by all the beams.

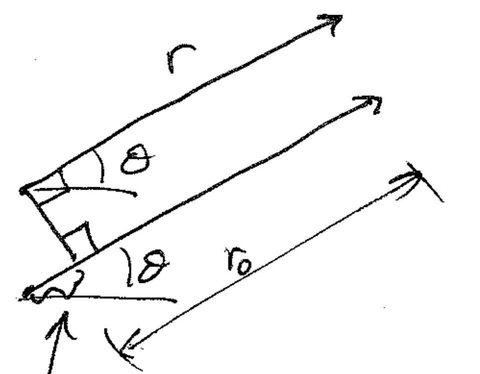
$$dE_p = \underbrace{\left(\frac{E_0\,ds}{r}\right)}_{\substack{\text{Amplitude}\\ \text{Factor for}\\ \text{spherical waves}}} \underbrace{e^{i(kr-\omega t)}}_{\text{wave Factor}}$$

r = distance travelled by beam located at position (s) in the slit.

The electric field amplitude drops like $\frac{1}{r}$ because the intensity drops like $\frac{1}{r^2}$ (and $I \sim E^2$).

Now, let $r_0 \equiv$ the distance travelled by the central beam. Then

$$r = r_0 + \delta$$



δ, could be (+) or (−)

Then

$$dE_p = \frac{E_0\,ds}{(r_0 + \delta)} e^{i(k(r_0+\delta)-\omega t)}$$

negligible compared to r_0

$$\approx \frac{E_0\,ds}{r_0} e^{i(kr_0-\omega t)} e^{ik\delta}$$

Also, $\delta = (s)\sin\theta$, so that

$$dE_p = \frac{E_0\,ds}{r_0} e^{i(kr_0-\omega t)} e^{iks\sin\theta}$$

Now we can add up all the beams ~~Be~~ by integrating over the entire slit (integrate (s) from $-\frac{b}{2}$ to $+\frac{b}{2}$

$$E_p = \text{total electric field} = \int dE_p$$

$$= \frac{E_0}{r_0} e^{i(kr_0-\omega t)} \int_{-\frac{b}{2}}^{+\frac{b}{2}} e^{iks\sin\theta}\, ds$$

$$= \frac{E_0}{r_0} e^{i(kr_0-\omega t)} \left. \frac{e^{iks\sin\theta}}{(ik\sin\theta)} \right|_{-b/2}^{+b/2}$$

$$= \frac{E_0}{r_0} e^{i(kr_0-\omega t)} \left(\frac{e^{ikb\sin\theta/2} - e^{-ikb\sin\theta/2}}{ik\sin\theta} \right)$$

$\sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i}$

Simplify notation: Define $\beta \equiv \frac{1}{2} kb\sin\theta$. Then

$$E_p = \frac{E_0 b}{r_0} e^{i(kr_0-\omega t)} \left(\frac{2i\sin(\beta)}{2i\beta} \right)$$

$$E_p = \frac{E_0 b}{r_0} \underbrace{e^{i(kr_0-\omega t)}}_{\text{wave factor}} \left(\frac{\sin\beta}{\beta} \right)$$

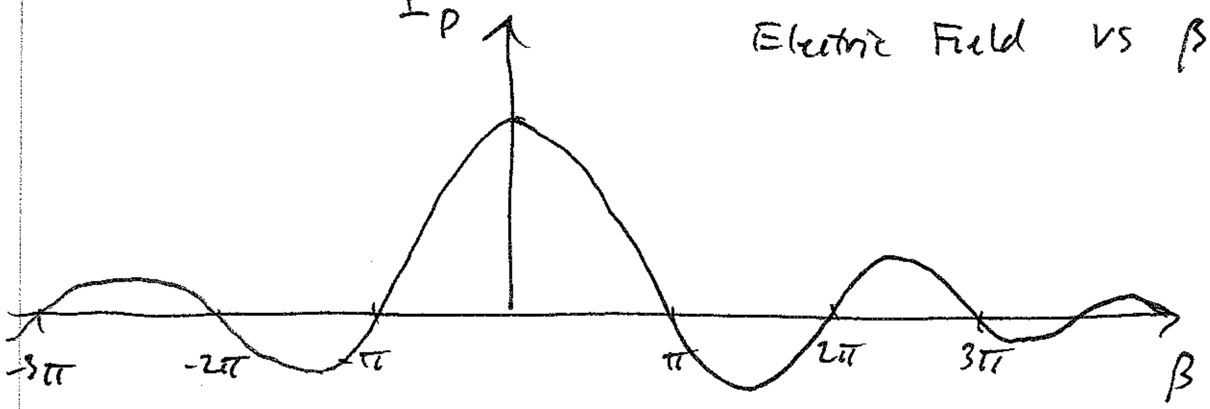
wave factor, can be ignored for optical waves when we average over time.

$$\text{Intensity} = I \sim |E_p|^2 \sim \left(\frac{E_0 b}{r_0}\right)^2 \left(\frac{\sin^2\beta}{\beta^2}\right)$$

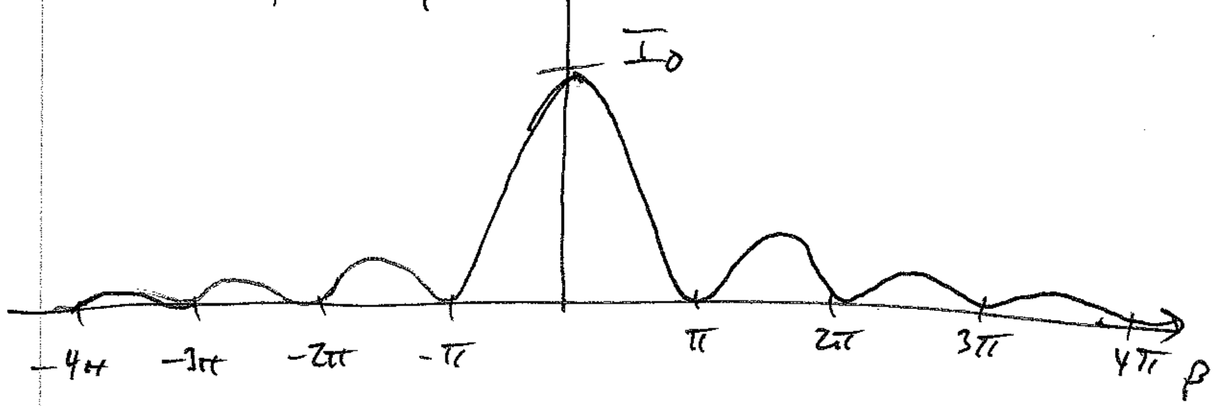
or $\boxed{I = I_0 \frac{\sin^2\beta}{\beta^2}}$ where I_0 = maximum intensity and $\beta = \frac{1}{2}kb\sin\theta$.

What does it look like on a screen?

Electric Field vs β



Intensity $\sim E_p^2$:



Zeros occur when $\beta = m\pi$, $m = \pm 1, \pm 2, \pm 3,$ but not $m = \emptyset$!

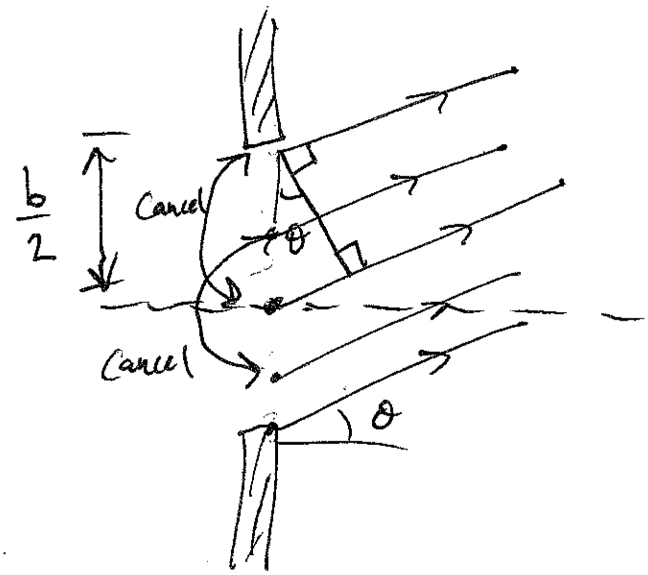
$$\downarrow$$

$$\frac{1}{2}kb\sin\theta = m\pi$$

$\frac{1}{2}k = \frac{\pi}{\lambda}$

$\boxed{b\sin\theta = m\lambda}$ Condition for diffraction zeros.

What's happening at the locations where the intensity is zero? Apparently all of the beams are cancelling each other. For example, imagine dividing the slit into 2 parts:



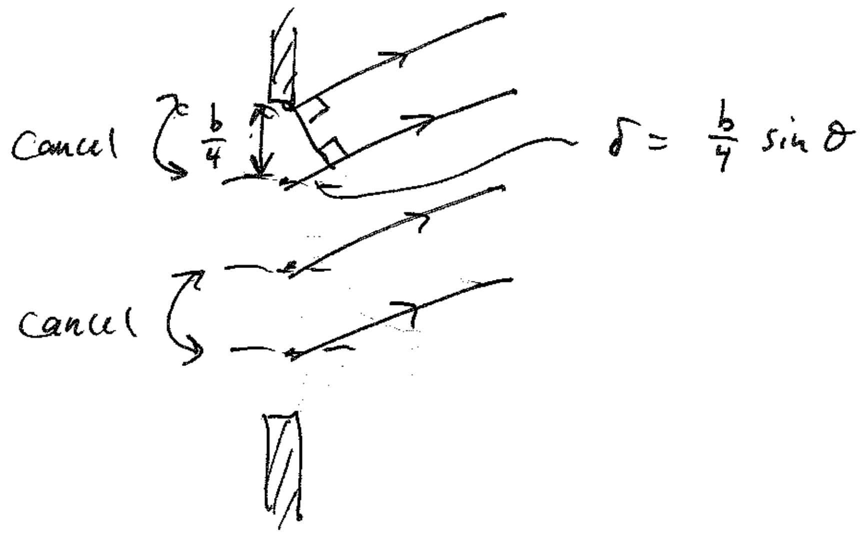
Can we find an angle θ where each beam in the lower half is exactly cancelled by a beam in the upper half? Yes, this will happen when

$$\delta = \frac{b}{2}\sin\theta = \frac{\lambda}{2}$$

In other words, when $\delta = \frac{\lambda}{2}$, the top beam will cancel the middle beam, and the same cancellation will happen for all pairs of beams in the slit. So the condition is

$$\boxed{b\sin\theta = \lambda}$$ This is the first zero.

What about the second zero? Well, we can also divide the screen into 4 parts:



This cancellation happens when $\delta = \frac{\lambda}{2}$

$$\frac{b}{4}\sin\theta = \frac{\lambda}{2}$$

$$\boxed{b\sin\theta = 2\lambda} \quad 2^{nd} \text{ zero.}$$

In general complete cancellation happens when

$$\boxed{b\sin\theta = m\lambda.}$$