Maxwell's Equations: (In Integral Form)

$$\oint_{surface} \vec{E}\cdot\hat{n}\,da = \frac{Q_{enclosed}}{\varepsilon_0} \qquad \text{Gauss' Law}$$

$$\oint_{surface} \vec{B}\cdot\hat{n}\,da = 0 \qquad \text{Gauss' Law for Magnetism}$$

$$\oint_{curve} \vec{E}\cdot d\vec{l} = -\frac{d\Phi_B}{dt} \qquad \text{Faraday's Law}$$

$$\oint_{curve} \vec{B}\cdot d\vec{l} = \mu_0 I + \mu_0\varepsilon_0 \frac{d\Phi_E}{dt} \qquad \text{Modified Ampere's Law}$$

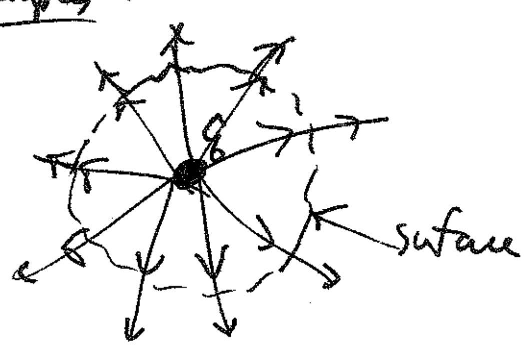
Roughly speaking, the left side of the equations tell us what the $\vec{E}$ & $\vec{B}$ field do, and the right side ~~says~~ tells us what causes them to do it

Gauss Law:

$$\underbrace{\oint_{surface} \vec{E}\cdot\hat{n}\,da}_{\text{Flux of } \vec{E} \text{ through a closed surface ....}} = \frac{Q_{enclosed}}{\varepsilon_0}$$

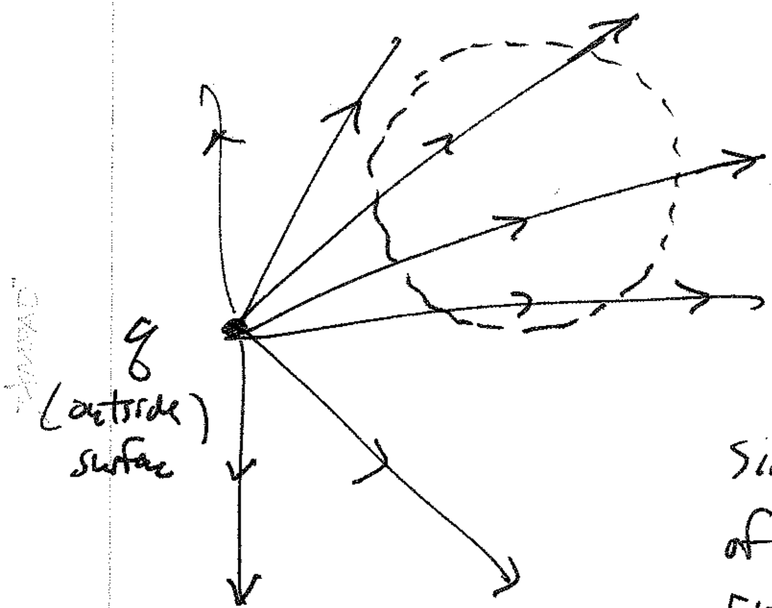
↑ Is caused by the presence of charge inside that surface.

Examples:



← Non-zero flux caused by charge q.

Conversely, if there ~~is~~ are no charges inside, then the flux must be zero:

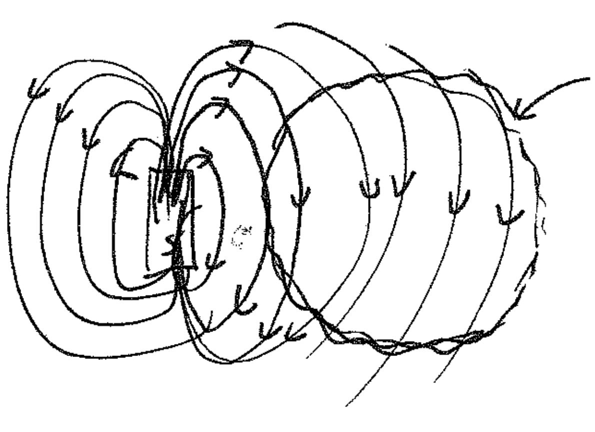


Net flux is zero: every electric field line penetrates through.

Since the flux is a measure of the number of electric field lines which ~~penetrate~~ originate inside, here the flux is zero.

Gauss' Law for Magnetism

$$\underbrace{\oint_{\text{surface}} \vec{B}\cdot\hat{n}\, da}_{\text{Flux through a closed surface}} = \underset{\text{is always zero}}{\underset{\uparrow}{\emptyset}}$$



Flux of $\vec{B}$ must be zero $\Rightarrow$ there are no magnetic monopoles which can act as a source of $\vec{B}$ field lines.

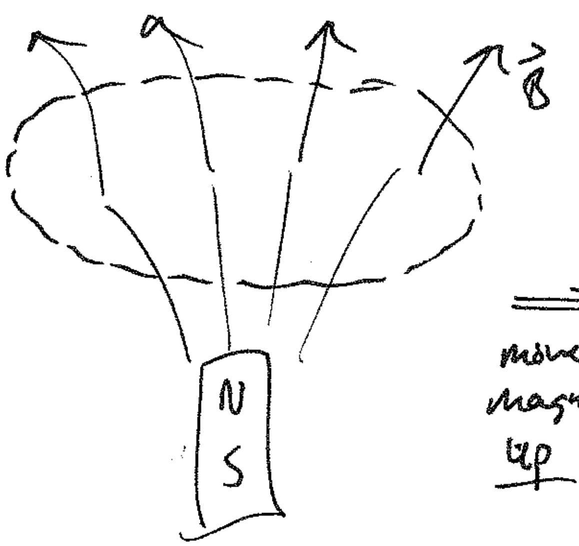
So Gauss' Law can be summarized:
" Electric Field lines can begin and end on charges. If charge is present, then there will be a non-zero flux through a surface

Magnetic Field lines can never begin or end anywhere, because there are no magnetic monopoles. They can only go in circles"

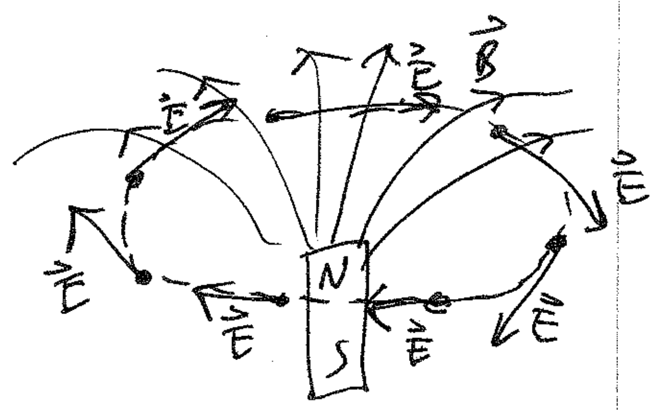
Faraday's Law

$$\underbrace{\oint \vec{E}\cdot d\vec{\ell}}_{\text{"Electric Field lines will go in circles..."}} = -\frac{d\Phi_B}{dt} \quad , \quad \Phi_B = \int_{\text{open surface}} \vec{B}\cdot\hat{n}\, da$$

"when the magnetic flux is changing in time."

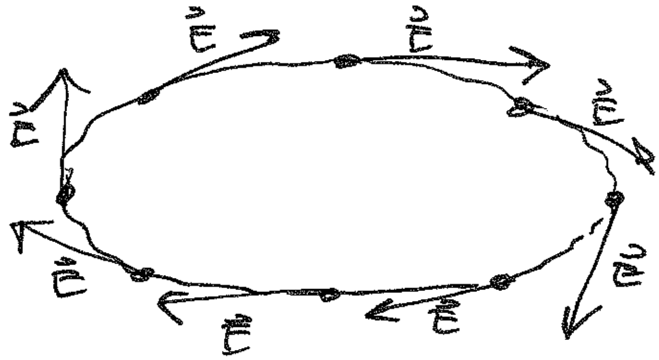


$\Longrightarrow$ move magnet up



Φ_B is larger now.
This creates a circulating electric field

Just the $\vec{E}$ field:

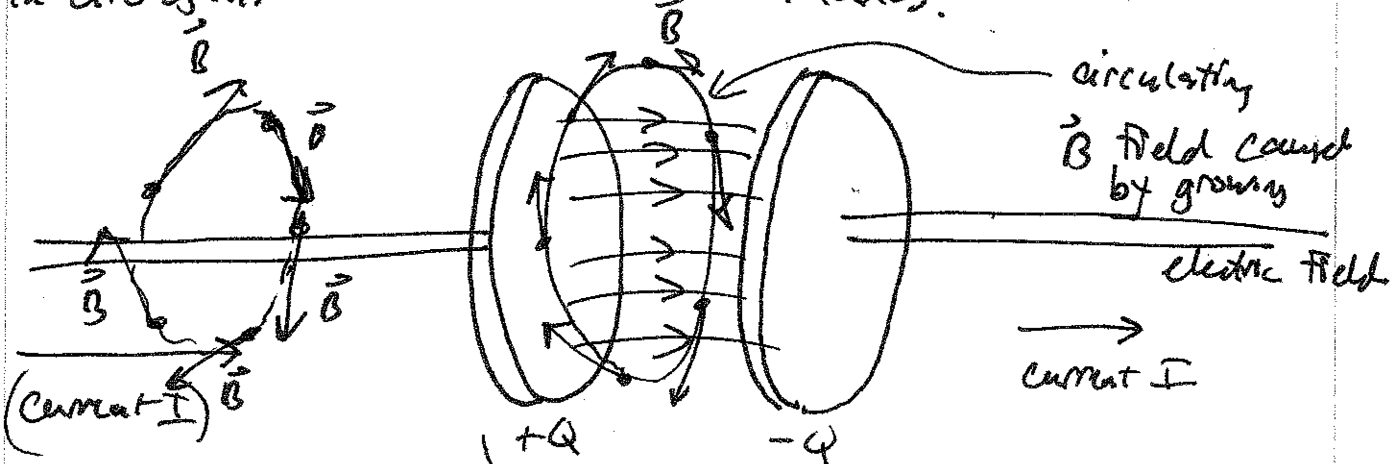


⇐ $\vec{E}$ field is going in a circle due to the changing magnetic flux.

Modified Ampere's Law

$$\oint_{\text{curve}} \vec{B} \cdot d\vec{l} = \mu_0 I + \mu_0 \varepsilon_0 \frac{d\Phi_E}{dt}$$

"Magnetic fields go in circles...." "caused by currents..." "and/or caused by changing electric fluxes."



Charging capacitor plates, electric field is growing.

<u>In Vacuum</u>, Maxwell's Equations look like:

(set all charges and currents equal to zero)

$$\left.\begin{aligned} \oint_{\text{surface}} \vec{E}\cdot\hat{n}\,da &= \emptyset \\ \oint_{\text{surface}} \vec{B}\cdot\hat{n}\,da &= \emptyset \end{aligned}\right\}$$

"Neither $\vec{E}$ nor $\vec{B}$ can start or stop anywhere"

$$\left.\begin{aligned} \oint \vec{E}\cdot d\vec{\ell} &= -\frac{d\phi_B}{dt} \\ \oint \vec{B}\cdot d\vec{\ell} &= \mu_0 \varepsilon_0 \frac{d\phi_E}{dt} \end{aligned}\right\}$$

"Both $\vec{E}$ & $\vec{B}$ will go in circles, caused by a changing flux of the other field"

Faraday's Law & Ampere's Law work together to create propagating waves in the $\vec{E}$ & $\vec{B}$ fields. It works like this:

(changing $\vec{B}$ Flux) $\underset{\text{Faraday}}{\Longrightarrow}$ (creates a changing $\vec{E}$ flux) $\underset{\text{Ampere}}{\Longrightarrow}$ (creates a changing $\vec{B}$ Flux)

$\Big\Downarrow$ ← Faraday

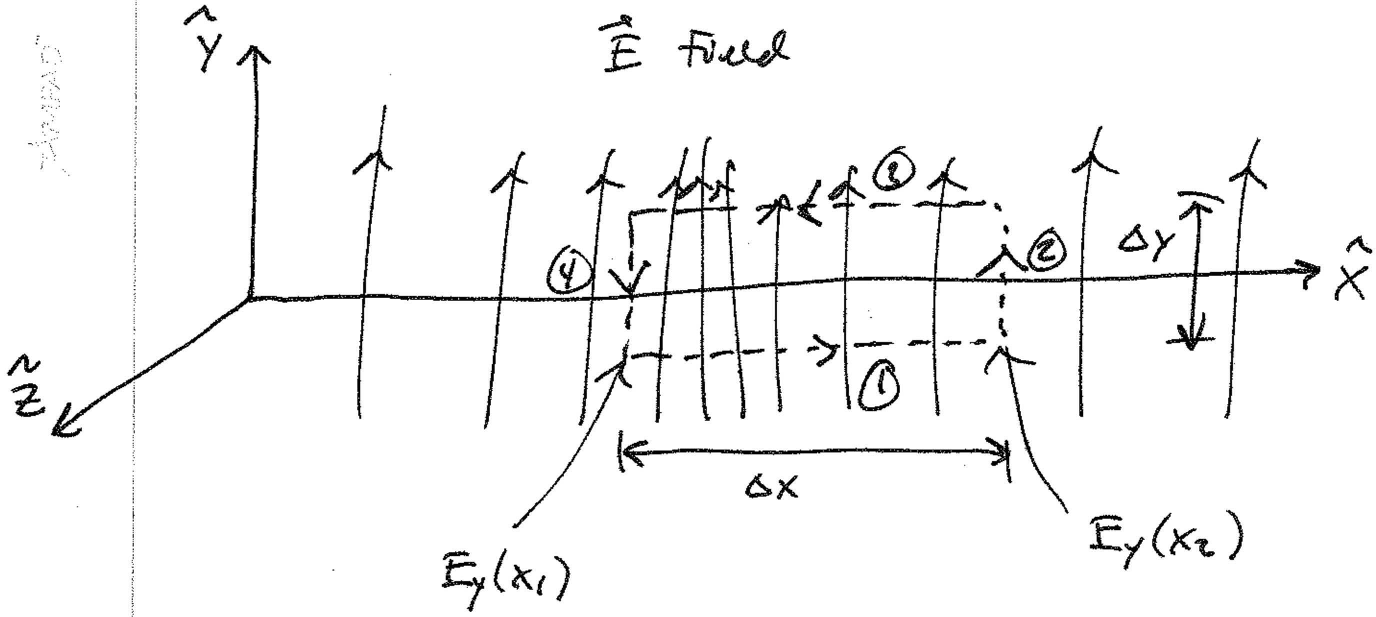
(creates a changing $\vec{E}$ Flux)

$\Big\Downarrow$ ← Ampere

see 6

We can show that Faraday's Law & Ampere's Law <u>imply</u> that $\vec{E}$ and $\vec{B}$ both satisfy the Classical Wave Equation.

Argument: Apply Faraday's Law around a square region with an electric field in the $\hat{y}$ direction.



∴ The left hand side of Faraday's Law says

$$\oint_{\text{square}} \vec{E}\cdot d\vec{l} = \underbrace{E_y(x_2)\,\Delta y}_{\text{segment } ②} - \underbrace{E_y(x_1)\,\Delta y}_{\text{segment } ④} = \left[E_y(x_2) - E_y(x_1)\right]\Delta y$$

(segments ① & ③ contribute zero because there $d\vec{l}$ is perpindicular to $\vec{E}$).

Now $E_y(x_2) - E_y(x_1)$ is the change in E_y over the small distance Δx.

Letting $\Delta X \to \phi$,
We can write this change as

$$E_y(x_2) - E_y(x_1) = \Delta E_y \approx \frac{\partial E_y}{\partial x} \Delta X$$

Then the left hand side of Faraday's Law says

$$\oint_{\text{square}} \vec{E} \cdot d\vec{\ell} = \underbrace{\left(\frac{\partial E_y}{\partial x} \Delta X\right) \Delta y}_{\text{Left hand side of Faraday's Law}} \quad \text{(LHS)}$$

The Right hand side of Faraday's Law says that this circulation in $\vec{E}$ must be caused by a <u>changing</u> <u>flux</u> <u>of $\vec{B}$</u>:

$$\frac{d\Phi_B}{dt} = \frac{d}{dt} \int_{\substack{\text{open surface} \\ \text{bounded by} \\ \text{square}}} \vec{B} \cdot \hat{n} \, da \approx \frac{d}{dt}\left(B_z \Delta X \Delta Y \right) \quad \text{(RHS)}$$

Appoximate $\vec{B}$ as roughly constant over the small area

Putting the Left Hand Side together with the Right Hand Side:

$$\frac{\partial E_y}{\partial x} \Delta X \Delta Y = \frac{dB_z}{dt} \Delta X \Delta Y$$

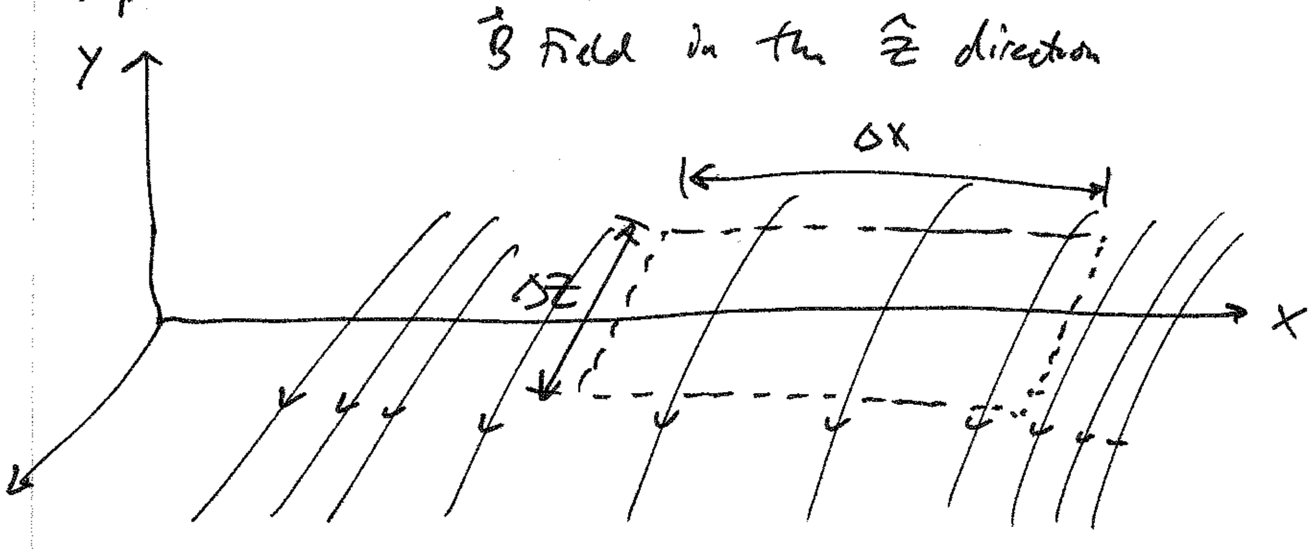
or

$$\frac{\partial E_y}{\partial X} = \frac{dB_z}{dt} = \frac{\partial B_z}{\partial t}$$

$$\boxed{\frac{\partial E_y}{\partial x} = -\frac{\partial B_z}{\partial t}} \quad (1)$$

This is Faraday's Law.

We can make a similar argument using modified Ampere's Law:



The mathematics is identical because the modified Ampere's Law is completely analogous to Faraday's Law (in the absence of charges & currents.)

The Result is

$$\boxed{\frac{\partial B_z}{\partial x} = -\mu_0 \varepsilon_0 \frac{\partial E_y}{\partial t}} \quad (2)$$

This is the modified Ampere's Law

Now put 1 & 2 together. Take $\frac{\partial}{\partial x}$ of Eq. 1:

$$\frac{\partial}{\partial x}\left(\frac{\partial E_y}{\partial x}\right) = -\frac{\partial}{\partial x}\left(\frac{\partial B_z}{\partial t}\right) = -\frac{\partial}{\partial t}\underbrace{\left(\frac{\partial B_z}{\partial x}\right)}_{\text{substitute from Eq. 2}}$$

$$= \frac{-\partial}{\partial t}\left(-\mu_0\varepsilon_0\frac{\partial E_y}{\partial t}\right)$$

$$= \mu_0\varepsilon_0\frac{\partial^2 E_y}{\partial t^2}$$

$$\therefore \quad \boxed{\frac{\partial^2 E_y}{\partial x^2} = \mu_0\varepsilon_0\frac{\partial^2 E_y}{\partial t^2}}$$ Classical Wave Equation

We can immediately see that:

1. EM waves propagate with a phase velocity of $v_p = \frac{1}{\sqrt{\mu_0\varepsilon_0}} = \underbrace{3\times10^8 \text{ m/s}}_{\text{speed of light}}$ in vacuum
2. EM waves can be identified with light.
3. EM waves will display <u>no dispersion in vacuum.</u> ⟹ Pulses will propagate forever.
   ⟹ Group velocity equal phase velocity
   ⟹ V_p is independent of wavelength

Maxwell's Equations in <u>Integral</u> <u>form</u> tell us about the <u>global</u> <u>properties</u> of $\vec{E}$ & $\vec{B}$. This can be very useful, but in many cases it is also useful to know how $\vec{E}$ & $\vec{B}$ are behaving at a single point in space. For this we need to re-cast Maxwell's Equations in <u>Differential Form</u>.

The Operator $\vec{\nabla}$ (or ∇) (Gradient)

Let $f(x,y,z)$ be a <u>scalar</u> function of position. Then

$$\vec{\nabla} f = \nabla f \equiv \cancel{\frac{\partial f}{\partial x}\hat{x}, \frac{\partial f}{\partial y}\hat{y}, \frac{\partial f}{\partial z}\hat{z}}$$

$$= \frac{\partial f}{\partial x}\hat{x} + \frac{\partial f}{\partial y}\hat{y} + \frac{\partial f}{\partial z}\hat{z}$$

$$= \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right) \text{ a vector.}$$

sometimes we put the arrow above ∇, and

sometimes we don't

We can think of $\vec{\nabla}$ as being a vector:

$$\vec{\nabla} \equiv \hat{x}\frac{\partial}{\partial x} + \hat{y}\frac{\partial}{\partial y} + \hat{z}\frac{\partial}{\partial z}$$

$$= \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)$$

~~cases~~

$\vec{\nabla}$ can act in 3 ways:

1. Operate on a scalar function, producing a vector:

$$\vec{\nabla}F = \text{vector} = \left(\frac{\partial F}{\partial x}, \frac{\partial F}{\partial y}, \frac{\partial F}{\partial z}\right) \quad \text{(The Gradient)}$$

2. Operate on a vector function, via a dot-product:

$$\vec{\nabla}\cdot\vec{v} = \text{scalar} = \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z}$$

This is called "the Divergence"

3. Operate on a vector function, via a cross-product

$$\vec{\nabla}\times\vec{V} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ v_x & v_y & v_z \end{vmatrix}$$

$$= \hat{x}\left(\frac{\partial v_z}{\partial y} - \frac{\partial v_y}{\partial z}\right) + \hat{y}\left(\frac{\partial v_x}{\partial z} - \frac{\partial v_z}{\partial x}\right) + \hat{z}\left(\frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y}\right)$$

This is called "the curl"

or "the circulation". (older terminology).

Example Divergences

1) Let $\vec{V} = \vec{r} = x\hat{x} + y\hat{y} + z\hat{z}$.

Then $\vec{\nabla}\cdot\vec{V} = \frac{\partial}{\partial x}(x) + \frac{\partial}{\partial y}(y) + \frac{\partial}{\partial z}(z)$

$= 1 + 1 + 1$

$= 3$

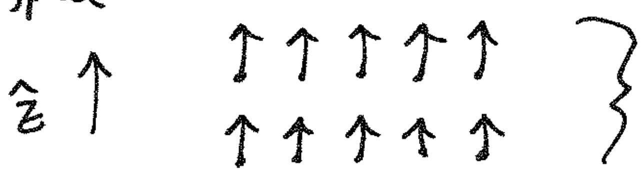
2) Let $\vec{v} = \hat{z}$

Then $\vec{\nabla}\cdot\vec{v} = \frac{\partial}{\partial x}(\emptyset) + \frac{\partial}{\partial y}(\emptyset) + \frac{\partial}{\partial z}(1) = \emptyset$

3) Let $\vec{v} = z\hat{z}$

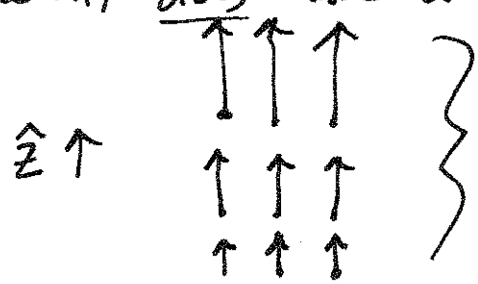
Then $\vec{\nabla}\cdot\vec{v} = \frac{\partial}{\partial x}(\emptyset) + \frac{\partial}{\partial y}(\emptyset) + \frac{\partial}{\partial z}(z) = 1.$

The Divergence is a measure of whether ~~the~~ any particular point in space is acting like a "source" of the vector field. So a uniform field (like $\vec{v} = \hat{z}$) has no divergence anywhere in space:

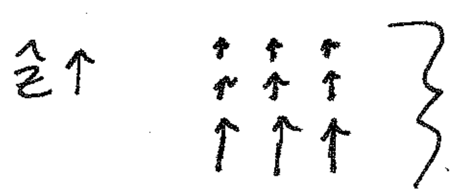


No divergence

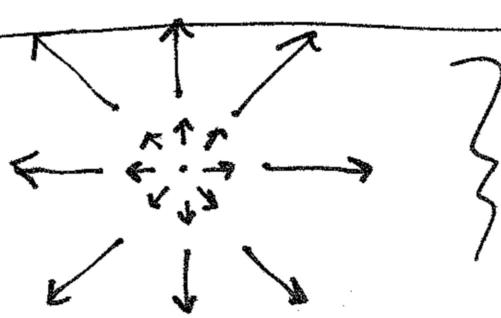
But a field which increases in intensity (magnitude) generally does have a non-zero divergence.



Positive Divergence



Negative Divergence

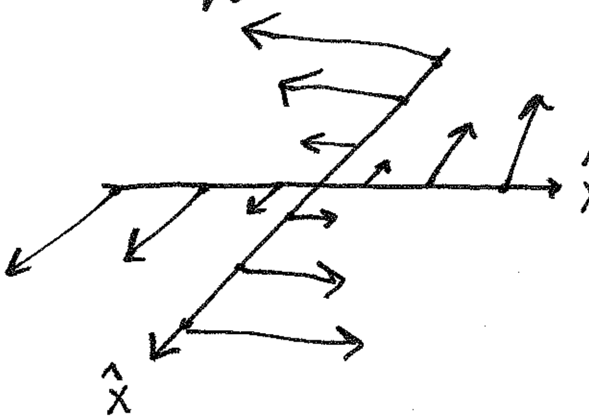


Positive Divergence

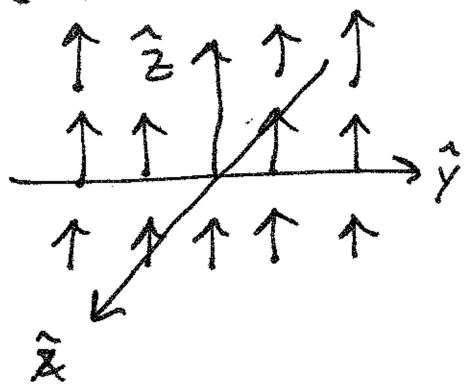
Example Curls

① Let $\vec{v} = -y\hat{x} + x\hat{y}$:

$\vec{v}$:



Curl of $\vec{v}$:



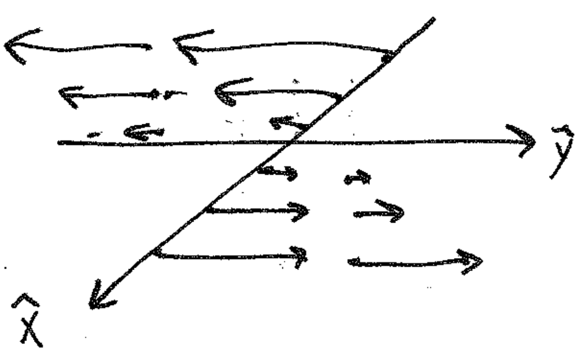
Then $$\vec{\nabla} \times \vec{v} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -y & x & \emptyset \end{vmatrix} = \left(\frac{\partial}{\partial x}(x) - \frac{\partial}{\partial y}(-y) \right) \hat{z}$$

$$= 1+1\,\hat{z}$$

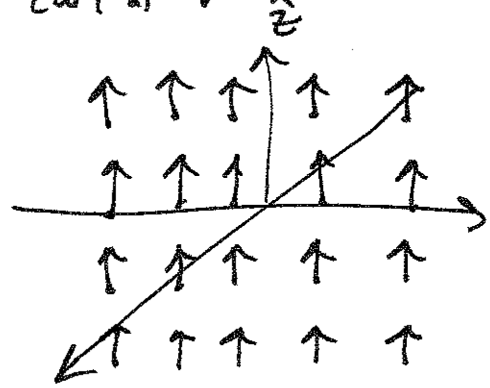
$$= 2\hat{z}$$

② Let $\vec{v} = x\hat{y}$

$\vec{v}$:



Curl of $\vec{v}$:



Then $\vec{\nabla} \times \vec{v} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \emptyset & x & \emptyset \end{vmatrix}$

$$= (1)\,\hat{z}$$

$$= \hat{z}$$

The curl is a measure of the tendency of the vector field to <u>rotate at each point in space</u>. Imagine putting a tiny paddle wheel in the vector field. If it wants to rotate, then the curl is non-zero

<u>2nd Derivatives</u>

① The Divergence of a Gradient:

$$\vec{\nabla} \cdot \vec{\nabla} f = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right) \cdot \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right)$$

$$= \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}$$

We call this the Laplacian Operator:

$$\vec{\nabla} \cdot \vec{\nabla} f = \nabla^2 f = \text{Laplacian of } f.$$

② The curl of a Gradient:

$\vec{\nabla} \times (\vec{\nabla} f) \leftarrow$ this is always zero
(prove from definition of $\vec{\nabla}$ and curl).

3 Gradient of a Divergence:

$$\vec{\nabla}(\vec{\nabla}\cdot F) \leftarrow$$ does not occur very often in physics

4 Divergence of a curl:

$$\vec{\nabla}\cdot(\vec{\nabla}\times\vec{v}) \leftarrow$$ this is always zero

5 Curl of a Curl:

$$\vec{\nabla}\times(\vec{\nabla}\times\vec{v}) = \vec{\nabla}(\vec{\nabla}\cdot\vec{v}) - \underbrace{\nabla^2\vec{v}}$$

This is the Laplacian of a vector:

$$\nabla^2\vec{v} = \nabla^2 v_x\,\hat{x} + \nabla^2 v_y\,\hat{y} + \nabla^2 v_z\,\hat{z}$$

$$\nabla^2\vec{v} = \left(\frac{\partial^2 v_x}{\partial x^2} + \frac{\partial^2 v_x}{\partial y^2} + \frac{\partial^2 v_x}{\partial z^2}\right)\hat{x}$$

$$+\left(\frac{\partial^2 v_y}{\partial x^2} + \frac{\partial^2 v_y}{\partial x^2} + \frac{\partial^2 v_y}{\partial z^2}\right)\hat{y}$$

$$+\left(\frac{\partial^2 v_z}{\partial x^2} + \frac{\partial^2 v_z}{\partial y^2} + \frac{\partial^2 v_z}{\partial z^2}\right)\hat{z}$$

The Fundamental Theorem of Calculus for Divergences

Also known as "Gauss' Theorem" and/or "the Divergence Theorem".

Recall the fundamental theorem of ordinary 1D calculus.

$$\int_a^b F(x)\,dx = f(b) - f(a) \qquad \text{where } \frac{df}{dx} = F(x)$$

or

$$\int_a^b \frac{dF}{dx}\,dx = F(b) - F(a)$$

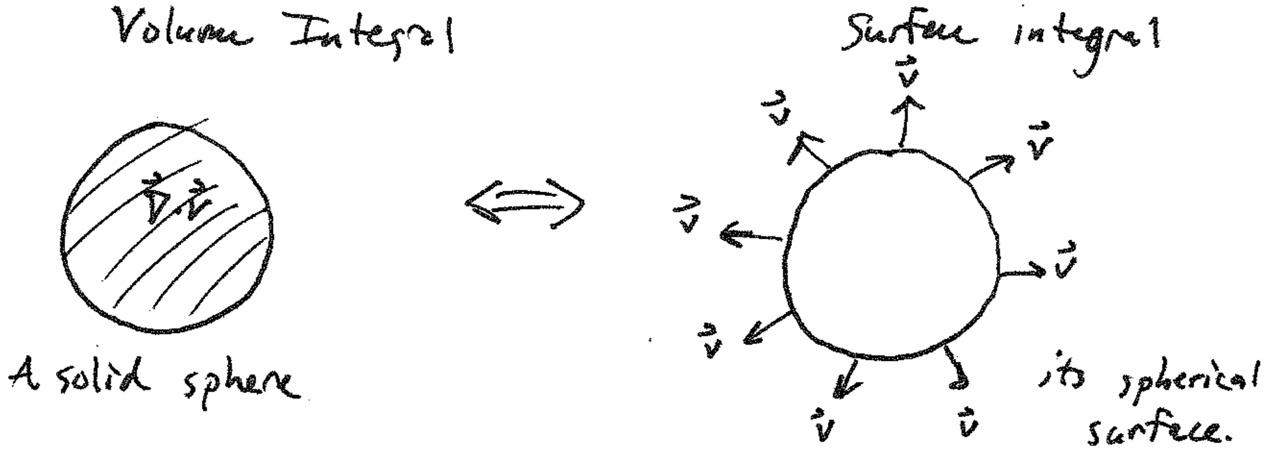
In vector calculus we have several types of derivatives, so we have several versions of the fundamental theorem.

For divergences, the Fundamental theorem says

$$\boxed{\int_{\text{volume}} (\vec{\nabla}\cdot\vec{v})\,dV = \oint_{\text{surface}} \vec{v}\cdot\hat{n}\,da}$$

"Gauss' ~~the~~ Theorem" or "Divergence Theorem"

Interpretation: Since $\vec{\nabla}\cdot\vec{v}$ measures how much the vector field $\vec{v}$ spreads out at each point in space, if we integrate over a ~~vo~~ volume it should be equal to the Flux of $\vec{v}$ out of the surface of the volume.



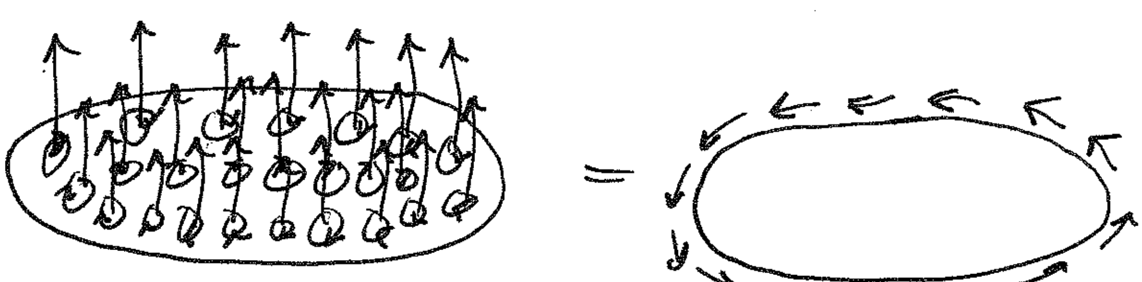
Fundamental Theorem of Calculus for Curls

Also known as "Stokes Theorem"

$$\underbrace{\int_{\text{surface}} (\vec{\nabla}\times\vec{v})\cdot\hat{n}\,da}\;=\;\oint_{\text{curve}} \vec{v}\cdot d\vec{l}$$

Flux of the curl through the surface = Integral of $\vec{v}$ around the edge

Interpretation: The curl measures how much the vector field $\vec{v}$ tends to rotate at each point in space. If we add up lots of small rotation vectors, we should get the line integral around the edge.



surface integral of the curl

line integral around the boundary.

Maxwell's Equations in Differential Form

The integral form of Maxwell's Equations tell us how the $\vec{E}$ field and $\vec{B}$ field behave on average over a surface or around a closed curve. This is sometimes useful, but it can also be clumsy. It is often more useful to know how the fields are behaving at each point in space. This form is called the differential form.

To convert from integral form to differential form, we use the Divergence Theorem and Stokes Theorem:

$$\boxed{\int_{\text{volume}} (\vec{\nabla}\cdot\vec{v})\, dV = \oint_{\text{surface}} \vec{v}\cdot\hat{n}\, da}$$ "Divergence Theorem"

and

$$\boxed{\int_{\text{surface}} (\vec{\nabla}\times\vec{v})\cdot\hat{n}\, da = \oint_{\text{curve}} \vec{v}\cdot d\vec{l}}$$ "Stokes Theorem"

Gauss' Law in Differential Form

$$\oint_{\text{surface}} \vec{E}\cdot\hat{n}\, da = \frac{Q_{enclosed}}{\varepsilon_0}$$

$$= \int_{\text{volume}} (\vec{\nabla}\cdot\vec{E})\, dV$$ according to the Divergence Theorem

$$\int_{\text{volume}} (\vec{\nabla} \cdot \vec{E})\, dV = \frac{Q_{\text{enclosed}}}{\varepsilon_0}$$

Now define $\rho(x,y,z) \equiv$ charge density at each point in space

$= \dfrac{\text{Coulombs}}{m^3}$ in SI units

Then $Q_{\text{enclosed}} = \displaystyle\int_{\text{volume}} \rho \, dV$

So that

$$\int_{\text{volume}} (\vec{\nabla} \cdot \vec{E})\, dV = \frac{1}{\varepsilon_0} \int_{\text{volume}} \rho \, dV = \int_{\text{volume}} \frac{\rho}{\varepsilon_0}\, dV$$

For this to be true for any volume, it must be true that the integrands on the left hand side and right hand side are equal:

$$\boxed{\vec{\nabla} \cdot \vec{E} = \frac{\rho}{\varepsilon_0}}$$ "Gauss' Law in Differential Form"

This says that at each point in space, the divergence of $\vec{E}$ is proportional to the charge density at the same point in space.

Similarly, the Divergence Theorem can be used to convert Gauss' Law for magnetism:

$$\oint \vec{B}\cdot\hat{n}\, da = \emptyset$$

$$\Downarrow$$

$$\boxed{\vec{\nabla}\cdot\vec{B} = \emptyset}$$

"Gauss' Law for Magnetism in Differential Form"

Faraday's Law and Ampere's Law in Differential Form

Use stoke's theorem to convert:

$$\underbrace{\oint_{curve} \vec{E}\cdot d\vec{l}}_{\hookrightarrow \int_{surface}(\vec{\nabla}\times\vec{E})\cdot\hat{n}\, da \text{ by Stokes Theorem}} = -\frac{\partial \Phi_B}{\partial t} = -\frac{\partial}{\partial t}\int_{surface} \vec{B}\cdot\hat{n}\, da$$

$$\int_{surface}(\vec{\nabla}\times\vec{E})\cdot\hat{n}\, da = \int_{surface}\left(-\frac{\partial \vec{B}}{\partial t}\right)\cdot\hat{n}\, da$$

Apparently the Integrands are equal:

$$\boxed{\vec{\nabla}\times\vec{E} = -\frac{\partial B}{\partial t}}$$

"Faraday's Law in Differential Form"

At every point in space, the curl of $\vec{E}$ is proportional to the time rate change of $\vec{B}$ at that same point.

Similarly for Ampere's Law we have

$$\oint_{\text{curve}} \vec{B}\cdot d\vec{l} = \mu_0 I + \mu_0 \varepsilon_0 \frac{d\Phi_E}{dt}$$

$$\Downarrow$$

$$\boxed{\vec{\nabla}\times\vec{B} = \mu_0 \vec{J} + \mu_0\varepsilon_0 \frac{\partial \vec{E}}{\partial t}}$$

"Ampere's Law in Differential Form."

where $\vec{J}$ = current per unit area perpendicular to the flow.

In vacuum, with no charges and no currents, the Four Maxwell Equations are even simpler.

$$\boxed{\begin{aligned} \vec{\nabla}\cdot\vec{E} &= 0 \\ \vec{\nabla}\cdot\vec{B} &= 0 \\ \vec{\nabla}\times\vec{E} &= -\frac{\partial \vec{B}}{\partial t} \\ \vec{\nabla}\times\vec{B} &= \mu_0\varepsilon_0 \frac{\partial \vec{E}}{\partial t} \end{aligned}}$$

In vacuum, no charges or currents

$$\boxed{\begin{aligned} \vec{\nabla}\cdot\vec{E} &= \rho/\varepsilon_0 \\ \vec{\nabla}\cdot\vec{B} &= 0 \\ \vec{\nabla}\times\vec{E} &= -\frac{\partial \vec{B}}{\partial t} \\ \vec{\nabla}\times\vec{B} &= \mu_0\vec{J} + \mu_0\varepsilon_0 \frac{\partial \vec{E}}{\partial t} \end{aligned}}$$

with charges and currents

Wave Equation from Differential Form of Maxwell's Eq.

Start with Faraday's Law:

$$\vec{\nabla}\times\vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

Take the curl of both sides:

$$\underbrace{\vec{\nabla}\times\left(\vec{\nabla}\times\vec{E}\right)} = \vec{\nabla}\times\left(\frac{-\partial\vec{B}}{\partial t}\right) = -\frac{\partial}{\partial t}\left(\vec{\nabla}\times\vec{B}\right)$$

↳ vector calculus identity:

$$\vec{\nabla}\times\vec{\nabla}\times\vec{v} = \vec{\nabla}(\vec{\nabla}\cdot\vec{v}) - \vec{\nabla}^2\vec{v}$$

so $$\vec{\nabla}\times\vec{\nabla}\times\vec{E} = \vec{\nabla}\underbrace{(\vec{\nabla}\cdot\vec{E})} - \vec{\nabla}^2\vec{E}$$

↳ zero in vacuum (no charges)

$$\vec{\nabla}\times\vec{\nabla}\times\vec{E} = -\vec{\nabla}^2\vec{E}$$

~~So the curl of Faraday's law says~~

So we have

$$-\vec{\nabla}^2\vec{E} = -\frac{\partial}{\partial t}\underbrace{\left(\vec{\nabla}\times\vec{B}\right)}$$

↑ $\mu_0\varepsilon_0\dfrac{\partial\vec{E}}{\partial t}$ by ~~Fara~~ Ampere's Law

$$-\vec{\nabla}^2\vec{E} = -\mu_0\varepsilon_0\frac{\partial^2\vec{E}}{\partial t^2}$$

or $$\boxed{\vec{\nabla}^2\vec{E} = \mu_0\varepsilon_0\frac{\partial^2 E}{\partial t^2}}$$ This is the Wave Equation for $\vec{E}$.

Component-by-component it reads as

x-component : $\frac{\partial^2 E_x}{\partial x^2} + \frac{\partial^2 E_x}{\partial y^2} + \frac{\partial^2 E_x}{\partial z^2} = \mu_0 \varepsilon_0 \frac{\partial^2 E_x}{\partial t^2}$

y-component : $\frac{\partial^2 E_y}{\partial x^2} + \frac{\partial^2 E_y}{\partial y^2} + \frac{\partial^2 E_y}{\partial z^2} = \mu_0 \varepsilon_0 \frac{\partial^2 E_y}{\partial t^2}$

z-component : $\frac{\partial^2 E_z}{\partial x^2} + \frac{\partial^2 E_z}{\partial y^2} + \frac{\partial^2 E_z}{\partial z^2} = \mu_0 \varepsilon_0 \frac{\partial^2 E_z}{\partial t^2}$

Similarly, by taking the curl of Ampere's Law, we get

$$\boxed{\vec{\nabla}^2 \vec{B} = \mu_0 \varepsilon_0 \frac{\partial^2 \vec{B}}{\partial t^2}}$$ Wave Equation for $\vec{B}$ Field.

Monochromatic Plane Wave Solution

A plane wave solution traveling in the z-direction can be written as

$$\vec{E}(z,t) = \vec{E}_0 e^{i(kz-\omega t)} \quad , \quad \vec{B}(z,t) = \vec{B}_0 e^{i(kz-\omega t)}$$

Clearly these harmonic functions will satisfy the wave equation. But Maxwell's Eqs place some additional constraints on:

1) The direction that the amplitude vectors $\vec{E}_0$ and $\vec{B}_0$ are allowed to point
2) The relationship between $\vec{E}$ & $\vec{B}$.

Keep in mind that, in principle, $\vec{E}_0$ and $\vec{B}_0$ could be complex, so that there is an additional phase hidden inside them

Additional constraints on $\vec{E}$ & $\vec{B}$

i) EM waves are <u>transverse</u>: $\vec{E}_0$ and $\vec{B}_0$ should be exactly perpendicular to the direction of travel.

This follows from $\vec{\nabla}\cdot\vec{E} = 0$ and $\vec{\nabla}\cdot\vec{B} = 0$.
For example, for our plane wave solution we have

$$\vec{\nabla}\cdot\left(\vec{E}_0 e^{i(kz-\omega t)}\right) = 0$$

$$\underbrace{\frac{\partial}{\partial x}\left(E_{0x} e^{i(kz-\omega t)}\right)}_{\text{no } x\text{-dependence}} + \underbrace{\frac{\partial}{\partial y}\left(E_{0y} e^{i(kz-\omega t)}\right)}_{\text{no } y\text{-dependence}} + \frac{\partial}{\partial z}\left(E_{0z} e^{i(kz-\omega t)}\right) = 0$$

So this reduces to $ik E_{0z} e^{i(kz-\omega t)} = 0$

$$\therefore \boxed{E_{0z} = 0}$$

Similarly $\boxed{B_{0z} = 0}$ (follows from $\vec{\nabla}\cdot B = 0$)

for a plane wave travelling in the z-direction.

For wave solutions, $\vec{E}$ and $\vec{B}$ should have no components in the direction of travel.

2) $\vec{E}$ and $\vec{B}$ are in phase with each other and mutually perpendicular. This follows from Faraday's Law:

$$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

The x-component of this equation is

$$\frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial z} = -\frac{\partial B_x}{\partial t}$$

For our plane wave solution this requirement is

$$\frac{\partial}{\partial y}\underbrace{\left(E_{0z} e^{i(kz-\omega t)}\right)}_{\text{no } y\text{-dependence}} - \frac{\partial}{\partial z}\left(E_{0y} e^{i(kz-\omega t)}\right) = -\frac{\partial}{\partial t}\left(B_{0x} e^{i(kz-\omega t)}\right)$$

$$-ikE_{0y} = i\omega B_{0x}$$

$$\boxed{-kE_{0y} = \omega B_{0x}}$$

Similarly, the y-component of Faraday's Law requires that

$$\boxed{kE_{0x} = \omega B_{0y}}$$

And we already know that $\boxed{B_{0z} = E_{0z} = \emptyset}$

We can write all three of these equation on one line using a cross product:

$$\vec{B}_0 = \frac{k}{\omega}\left(\hat{z} \times \vec{E}_0\right)$$

Or, since $\frac{k}{\omega} = \frac{1}{v_{phase}} = \frac{1}{c}$,

$$\vec{B}_0 = \frac{1}{c}(\hat{z} \times \vec{E}_0)$$

or $$\boxed{\hat{z} \times \vec{E}_0 = c\vec{B}_0}$$

This equation says that

- $\vec{B}_0$ is perpendicular to $\hat{z}$ (direction of travel)
- $\vec{E}_0$ is perpendicular to $\vec{B}_0$
- $\vec{E}_0$ is perpendicular to $\hat{z}$ (direction of travel)

Also $\vec{E}_0$ and $\vec{B}_0$ are related in their magnitudes by

$$\boxed{E_0 = c B_0}$$

And they are in phase $\Rightarrow$ when $\vec{E}$ is maximal, $\vec{B}$ is also maximal at that same location in space

Using the relationship between $\vec{E}$ & $\vec{B}$ we can write our plane wave solution as

$$\vec{E}(z,t) = |\vec{E}_0| e^{i(kz-\omega t)} \hat{x} \leftarrow \text{choose } x\text{-direction as the } \vec{E} \text{ field direction.}$$

$$\vec{B}(z,t) = \frac{1}{c}|\vec{E}_0| e^{i(kz-\omega t)} \hat{y}$$

These equations correctly describe that

1) $\vec{E}$ is $\perp$ to direction of travel
2) $\vec{B}$ is $\perp$ to direction of travel
3) $\vec{E}$ is $\perp$ to $\vec{B}$
4) $\vec{B}$ has magnitude $\left|\frac{\vec{E}}{c}\right|$

5) $\vec{E}$ & $\vec{B}$ are in phase with each other.

Question: What if the direction of travel is not the $\hat{z}$ direction?

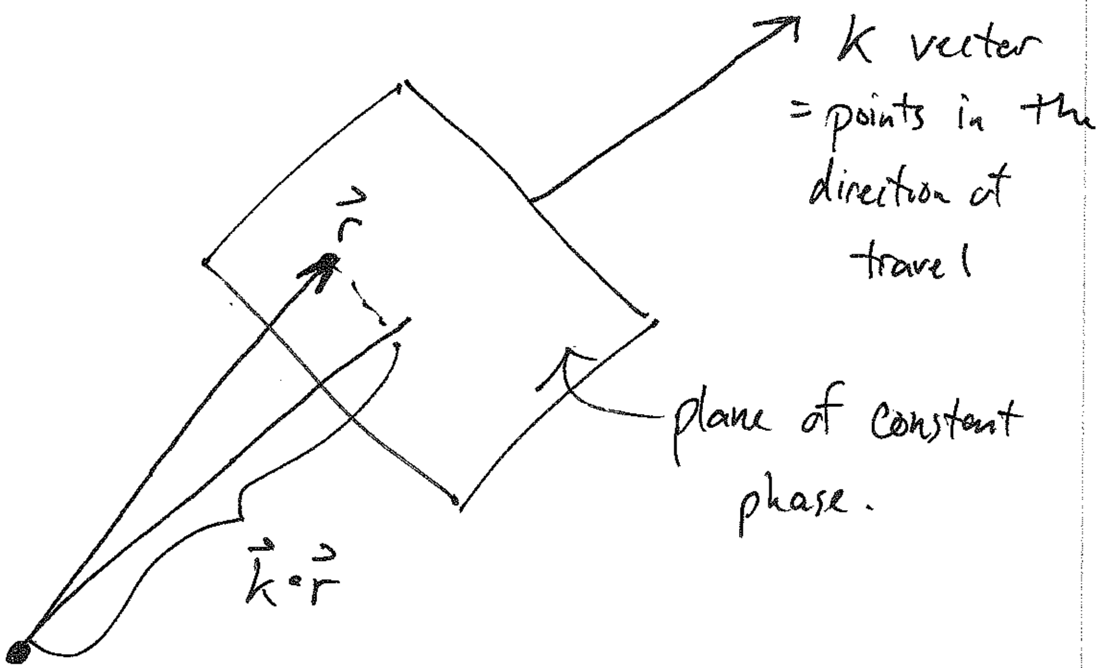
Answer: We need to generalize the "kz" part of the phase to 3 dimensions. First define a $\vec{k}$ vector:

$\vec{k}$: points in the direction of travel

and $|\vec{k}| = k =$ wavenumber for the plane wave.

Also: Let $\vec{r}$ be an arbitrary position vector.

Now imagine a plane of constant phase which is perpindicular to the direction of travel:



The projection of $\vec{r}$ onto $\vec{k}$ will be constant everywhere in this plane. So

$\vec{k}\cdot\vec{r}$ is the 3-dimensional generalization of kz.

So we can write our plane wave solution as

$$\vec{E}(\vec{r},t) = |\vec{E}_0|\,\hat{n}\,e^{i(\vec{k}\cdot\vec{r}-\omega t)}$$

$\hat{n}$ is a unit vector perpindicular to $\vec{k}$.

$$\vec{B}(\vec{r},t) = \frac{1}{c}|\vec{E}_0|\underbrace{(\hat{k}\times\hat{n})}\,e^{i(\vec{k}\cdot\vec{r}-\omega t)}$$

$\hat{k}\times\hat{n}$ is perpindicular to both $\vec{k}$ and $\hat{n}$.

Poynting Vector

Recall the energy density due to $\vec{E}$:

$$u_E = \frac{1}{2}\varepsilon_0 |\vec{E}|^2 = \text{energy density of space due to } \vec{E}.$$

and Recall

$$u_B = \frac{1}{2}\frac{1}{\mu_0}|\vec{B}|^2 = \text{energy density of space due to } \vec{B}.$$

For our monochromatic plane wave we have

$$|\vec{B}|^2 = \frac{1}{c^2}|\vec{E}|^2 = \mu_0\varepsilon_0|\vec{E}|^2$$

So that

$$u_B = \frac{1}{2}\frac{1}{\mu_0}|\vec{B}|^2 = \frac{1}{2}\frac{1}{\mu_0}\left(\mu_0\varepsilon_0|\vec{E}|^2\right)$$

$$u_B = \frac{1}{2}\varepsilon_0|\vec{E}|^2 = u_E \text{ !!}$$

So the energy density due to $\vec{E}$ is identical to that due to $\vec{B}$, (for a plane wave.).

The total energy density is

$$u = u_E + u_B = \frac{1}{2}\varepsilon_0|\vec{E}|^2 + \frac{1}{2}\varepsilon_0|\vec{E}|^2 = \varepsilon_0|\vec{E}|^2$$

$$u = \varepsilon_0 |\vec{E}|^2 = \varepsilon_0 |\vec{E}_0|^2 \cos^2(kz-\omega t)$$

↑ for a wave travelling in the z-direction.

Now Define the "Poynting Vector":

$$\boxed{\vec{S} \equiv \frac{1}{\mu_0} \vec{E} \times \vec{B}}$$

For our plane wave solution, the ~~poynting~~ Poynting Vector is

$$\vec{S} = \frac{1}{\mu_0}\left(|\vec{E}_0| \frac{|\vec{E}_0|}{c}\right) \cos^2(kz-\omega t)\, \hat{z}$$

Also: $\frac{1}{\mu_0 c} = \frac{\varepsilon_0}{\underbrace{(\mu_0 \varepsilon_0)}_{\frac{1}{c^2}} c} = \frac{\varepsilon_0 c^2}{c} = \varepsilon_0 c$

So $\vec{S} = c \underbrace{\left(\varepsilon_0 |\vec{E}_0|^2\right)}_{\text{energy density } u} \cos^2(kz-\omega t)\, \hat{z}$

$$\vec{S} = c u \hat{z}$$

So $\vec{S}$ has units of energy density times velocity, In other words,

$\vec{S}$ is the energy per unit area transported by the wave.

Also, since $\vec{S} = \frac{1}{\mu_0} \vec{E} \times \vec{B}$, $\vec{S}$ also points in the direction of travel.

If we take the time average of $\vec{S}$, we get

$$\underbrace{\langle \vec{S} \rangle}_{\text{time average}} = \text{Intensity} = I = c\varepsilon_0 |\vec{E}_0|^2 \underbrace{\langle \cos^2(kz-\omega t) \rangle}_{\text{time average of Cosine squared is } \frac{1}{2}.}$$

$$\boxed{\langle \vec{S} \rangle = I = \frac{1}{2} c\varepsilon_0 |\vec{E}_0|^2}$$

Dielectrics : EM waves in matter.

In a linear dielectric material, the atoms become electrically and magnetically polarized by the applied $\vec{E}$ and $\vec{B}$ fields.

We can accomodate this in Maxwell's Equations simply by replacing $\mu_0 \to \mu$

and $\varepsilon_0 \to \varepsilon$

in Ampere's Law. So we have

$$\nabla \cdot \vec{E} = 0 \leftarrow \text{no free charges}$$
$$\nabla \cdot \vec{B} = 0$$
$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

$$\nabla \times \vec{B} = \mu\varepsilon \frac{\partial E}{\partial t} \leftarrow \text{no free charge currents}$$

Then inside this dielectric material EM waves will be allowed. The modified wave equation will be

$$\nabla^2 \vec{E} = \mu\varepsilon \frac{\partial^2 \vec{E}}{\partial t^2}$$

So we can identify the phase velocity as

$$v_p = \frac{1}{\sqrt{\mu\varepsilon}}$$

We define the index of refraction to be

$$n \equiv \frac{c}{v_p} \quad \begin{array}{l} \leftarrow \text{speed of light in vacuum} \\ \leftarrow \text{speed of light in the material} \end{array}$$

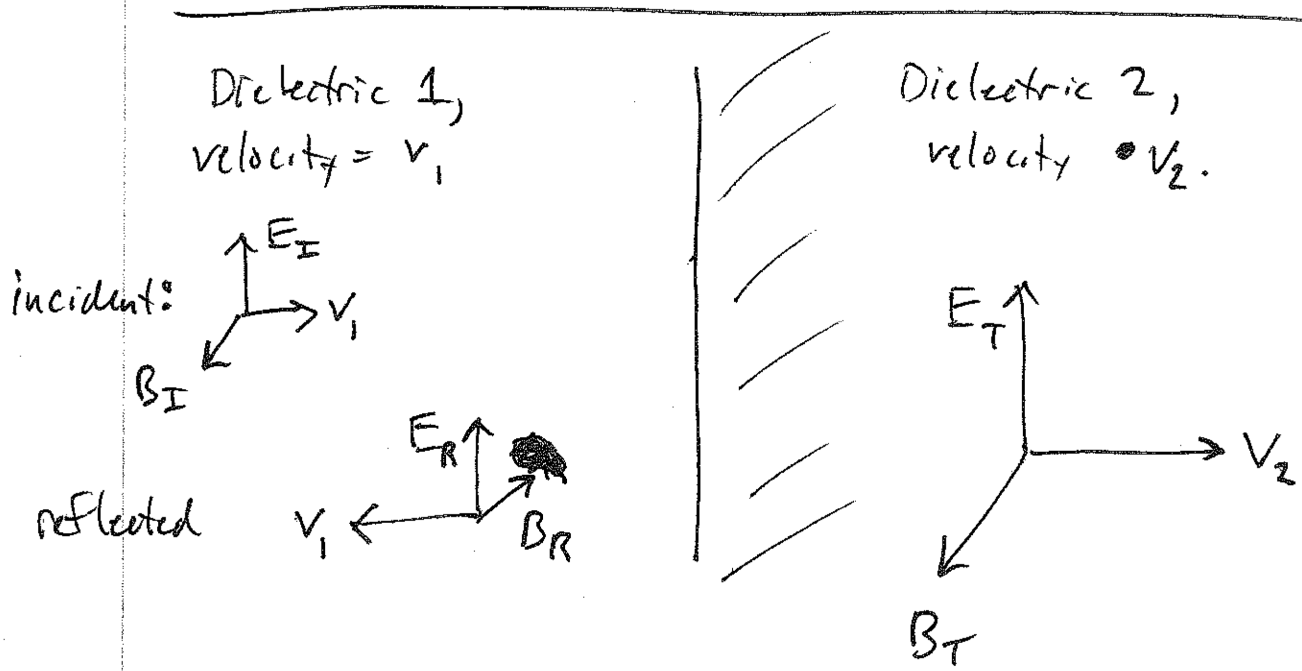
So $n = \sqrt{\dfrac{\varepsilon\mu}{\varepsilon_0\mu_0}}$

Usually $\mu \approx \mu_0$, so $n \approx \sqrt{\dfrac{\varepsilon}{\varepsilon_0}} > 1$

The Poynting Vector in the material will be

$$\vec{S} = \frac{1}{\mu} \vec{E} \times \vec{B}$$

Reflection & Transmission at a Dielectric Boundary



Incident: $\vec{E}_I = |\vec{E}_{0I}| e^{i(k_1 z - \omega t)} \hat{x}$

$$\vec{B}_I = \frac{1}{v_1} |\vec{E}_{0I}| e^{i(k_1 z - \omega t)} \hat{y}$$

Reflected:

$$\vec{E}_R = |\vec{E}_{0R}| e^{i(-k_1 z - \omega t)} \hat{x}$$

$$\vec{B}_R = -\frac{1}{v_1} |\vec{E}_{0R}| e^{i(-k_1 z - \omega t)} \hat{y}$$

Transmitted:

$$\vec{E}_T = |\vec{E}_{0T}| e^{i(k_2 z - \omega t)} \hat{x}$$

$$\vec{B}_T = \frac{1}{v_2} |\vec{E}_{0T}| e^{i(k_2 z - \omega t)} \hat{y}$$

Boundary Conditions

① $\vec{E}$ should be continuous across the boundary.

$$\boxed{E_{0I} + E_{0R} = E_{0T}} \quad ①$$

② $\frac{\vec{B}}{\mu}$ should be continuous across the boundary:

$$\frac{1}{\mu_1}\left(\frac{1}{v_1} E_{0I} - \frac{1}{v_1} E_{0R}\right) = \frac{1}{\mu_2} \frac{1}{v_2} E_{0T} \quad ②$$

or

$$\boxed{E_{0I} - E_{0R} = \frac{\mu_1 v_1}{\mu_2 v_2} E_{0T}} \quad ②$$

Solve ① and ② simultaneously. We did this before for waves on a string, and this is very similar.

Result:

$$E_{0R} = \left(\frac{\mu_2 v_2 - \mu_1 v_1}{\mu_2 v_2 + \mu_1 v_1} \right) E_{0I}$$

and $$E_{0T} = \left(\frac{2\mu_2 v_2}{\mu_2 v_2 + \mu_1 v_1} \right) E_{0I}$$

We can make this look exactly like transmission and reflection of waves on a string by defining the impedance for EM waves as

$$\boxed{Z \equiv \mu v = \mu\left(\frac{1}{\sqrt{\mu\varepsilon}}\right) = \sqrt{\frac{\mu}{\varepsilon}}}$$

Then $$E_{0R} = \left(\frac{Z_2 - Z_1}{Z_1 + Z_2} \right) E_{0I}$$

and $$E_{0T} = \left(\frac{2Z_2}{Z_1 + Z_2} \right) E_{0I}$$

Under this definition of Z, we can calculate the impedance of free space:

$$Z_{\text{free space}} = \sqrt{\frac{\mu_0}{\varepsilon_0}} = 376.7\ \Omega$$

we can find a simple expression to describe the intensity of the reflected beam in terms of the indices of refraction:

For most simple dielectrics, $\mu \approx \mu_0$, so that

$$E_{0R} = \left(\frac{\mu_2 v_2 - \mu_1 v_1}{\mu_2 v_2 + \mu_1 v_1}\right) E_{0I}$$

$$\approx \left(\frac{v_2 - v_1}{v_2 + v_1}\right) E_{0I}$$

$$= \left(\frac{\frac{c}{n_2} - \frac{c}{n_1}}{\frac{c}{n_2} + \frac{c}{n_1}}\right) E_{0I}$$

Multiply top and bottom by $n_1 n_2$

$$= \left(\frac{n_1 - n_2}{n_1 + n_2}\right) E_{0I}$$

The Intensity is the square of E, so

$$I_{reflected} = \left(\frac{n_1 - n_2}{n_1 + n_2}\right)^2 I_{incident}.$$