

Normal Mode: In a multi-particle system, a normal mode is a type of motion where all particles oscillate at the same frequency.

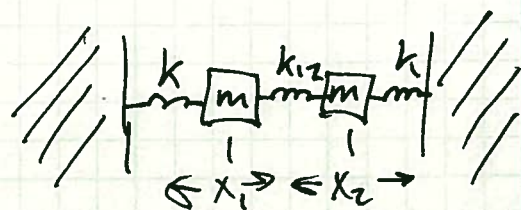
- The number of normal modes is equal to the number of particles.
- Each normal mode goes at its own frequency.
- The general solution is a sum over normal modes:

$$\vec{x}(t) = \sum_{n=1}^N a_n \vec{q}_n e^{i\omega_n t}$$

where $\vec{q}_n = (q_{1n}, q_{2n}, \dots, q_{Nn}) = n^{\text{th}}$ normal mode
 = "normal mode eigenvector"

2 coupled oscillators

(N=2)



Two normal modes:

$$\vec{q}_1 = (1, 1) = \text{"symmetric mode"}$$

$$\vec{q}_2 = (1, -1) = \text{"anti-symmetric mode"}$$

Then
$$\vec{x}(t) = a_1 (1, 1) e^{i\omega_1 t} + a_2 (1, -1) e^{i\omega_2 t}$$

↓

<p>Eqs. of Motion</p> $m\ddot{x}_1 + (k+k_{12})x_1 - k_{12}x_2 = 0$ $m\ddot{x}_2 + (k+k_{12})x_2 - k_{12}x_1 = 0$	$(x_1(t), x_2(t)) = a_1 (1, 1) e^{i\omega_1 t} + a_2 (1, -1) e^{i\omega_2 t}$
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For this system we called $\omega_1 = \omega_{small} = \omega_s$
 and $\omega_2 = \omega_{large} = \omega_L$

The frequencies are $\omega_1 = \omega_s = \sqrt{\frac{k}{m}}$

$\omega_2 = \omega_L = \sqrt{\frac{k+2k_{12}}{m}}$

Explicitly the solution is

$x_1(t) = a_1 e^{i\omega_1 t} + a_2 e^{i\omega_2 t}$

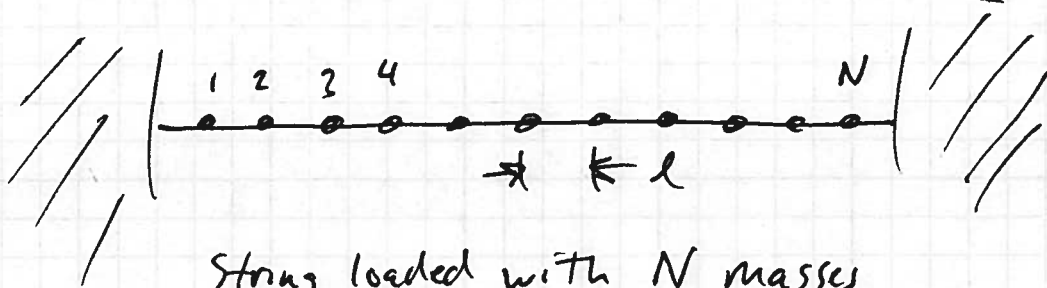
$x_2(t) = a_1 e^{i\omega_1 t} - a_2 e^{i\omega_2 t}$

or, taking the real part,

$x_1(t) = a_1 \cos(\omega_1 t) + a_2 \cos(\omega_2 t)$

$x_2(t) = a_1 \cos(\omega_1 t) - a_2 \cos(\omega_2 t)$

N-Coupled oscillator - The loaded string.



String loaded with N masses,
 each of mass (m).

String tension = T.

Transverse oscillations: Each mass has y-displacement (y_p).

String is fixed at each end, so $y_{p=0} = 0$ and $y_{p=N+1} = 0$. } boundary conditions

Equation of motion for mass (p):

$$\ddot{y}_p + 2\omega_0^2 y_p - \omega_0^2 (y_{p+1} + y_{p-1}) = 0$$

where $\omega_0^2 = T/ml$.

Normal mode solutions:

$$\vec{q}_n = \left(\sin\left(\frac{n\pi}{N+1}\right), \sin\left(\frac{2n\pi}{N+1}\right), \sin\left(\frac{3n\pi}{N+1}\right), \dots, \sin\left(\frac{Nn\pi}{N+1}\right) \right)$$

$$\omega_n = 2\omega_0 \sin\left(\frac{n\pi}{2(N+1)}\right)$$

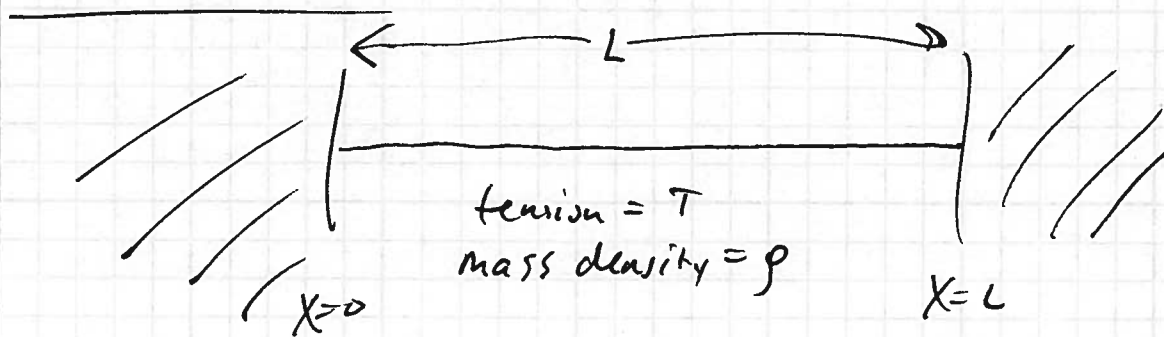
We also write the \vec{q}_n vectors on a particle-by-particle basis as

$$A_{pn} = \sin\left(\frac{pn\pi}{N+1}\right)$$

which mass \uparrow
 which normal mode

eg The solution is the same for longitudinal oscillations, except the motion is in the x direction, rather than the y direction

Continuous Systems - string fixed at $x=0$ & $x=L$



Equation of Motion:

$$\left[\frac{\partial^2 y}{\partial x^2} = \frac{\rho}{T} \frac{\partial^2 y}{\partial t^2} \right]$$

classical Wave Equation.

Normal Mode Solutions

$$y_n(x) = C_n \sin\left(\frac{n\pi x}{L}\right)$$

$$\omega_n = \sqrt{\frac{T}{\rho}} \frac{n\pi}{L}, \quad n = 1, 2, 3, \dots, \infty.$$

General Solution:

$$y(x,t) = \sum_{n=1}^{\infty} C_n \sin\left(\frac{n\pi x}{L}\right) e^{i\omega_n t}$$

Fourier's Trick

Once the normal modes and normal frequencies of a system are known, the only thing that remains is to find the expansion coefficients to describe the system at $t = \phi$. We use Fourier's Trick to do this.

For the loaded string, Fourier's Trick says

$$a_i = \frac{\vec{y}_0 \cdot \vec{q}_i}{|\vec{q}_i|^2}$$

where \vec{y}_0 is the set of initial positions $(y_1(t=\phi), y_2(t=\phi), \dots)$

For the continuous string fixed at $x=0$ and $x=L$, Fourier's Trick says

$$a_n = \frac{2}{L} \int_0^L y(x, t=0) \sin\left(\frac{n\pi x}{L}\right) dx$$

Fourier's Trick depends upon the fact that the eigen vectors are orthogonal:

$$\int_0^L \sin\left(\frac{n\pi x}{L}\right) \sin\left(\frac{m\pi x}{L}\right) dx = \frac{L}{2} \delta_{nm} \quad \text{for the continuous string}$$

and $\sum_{j=1}^N \sin\left(\frac{j\pi x}{N+1}\right) \sin\left(\frac{j\pi x}{N+1}\right) = \left(\frac{N+1}{2}\right) \delta_{nm}$ for the discrete loaded string.

Mathematics of Fourier Series & Fourier Transform

Any periodic function with period $2L$ can be written

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right) \right]$$

where $a_n = \frac{1}{L} \int_{-L}^L f(x) \cos\left(\frac{n\pi x}{L}\right) dx$

$$b_n = \frac{1}{L} \int_{-L}^L f(x) \sin\left(\frac{n\pi x}{L}\right) dx.$$

This same thing can be written in complex notation:

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{in\pi x/L}$$

where $c_n = \frac{1}{2L} \int_{-L}^L f(x) e^{-in\pi x/L} dx \leftarrow \text{Fourier's Trick}$

The two forms can be converted into each other:

$$a_n = c_n + c_{-n}$$

$$b_n = i(c_n - c_{-n})$$

$$a_0 = c_0 + c_0 = 2c_0$$

and
$$c_n = \begin{cases} \frac{1}{2}(a_{-n} + ib_n) & , \text{ for } n < 0 \\ \frac{1}{2}a_0 & , \text{ for } n = 0 \\ \frac{1}{2}(a_n - ib_n) & , \text{ for } n > 0 \end{cases}$$

The complex form is more compact and elegant. It also generalizes to the case where $f(x)$ is no longer periodic (period $L \rightarrow \infty$):

$$F(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dk A(k) e^{ikx} \quad (\text{non-periodic } F(x))$$

$$A(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(x) e^{-ikx} dx \leftarrow \text{"Plancherel's Theorem"}$$

But it's really just another example of Fourier's Trick.

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Travelling waves - Continuous string with no boundaries

If we have a continuous string with no boundary conditions (no walls), then we can have travelling wave solutions

$$y(x,t) = A \sin(kx - \omega t)$$

$$k = \frac{2\pi}{\lambda}, \quad \lambda = \text{wavelength.}$$

$$\omega = 2\pi f = \frac{2\pi}{T}$$

The peaks and troughs move forward at the "phase velocity"

$$v_{\text{phase}} = v = \frac{\omega}{k} = \lambda f$$

The equation of motion is still the wave equation:

$$\frac{\partial^2 y}{\partial x^2} = \frac{\rho}{T} \frac{\partial^2 y}{\partial t^2}$$

And the travelling waves satisfy this as long as

$$v = \sqrt{\frac{T}{\rho}} = \text{phase velocity.}$$

So we could write the Eq. of Motion as

$$\frac{\partial^2 y}{\partial x^2} = \frac{1}{v^2} \frac{\partial^2 y}{\partial t^2}$$

(8)

The general solution for a continuous string with no boundaries is a sum over all travelling waves. But since any wavelength is allowed, we have to sum over a continuum of k -values:

$$y(x, t=0) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dk A(k) e^{ikx}$$

where $A(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dx F(x) e^{-ikx}$

Then, as time goes forward, each normal mode (e^{ikx}) gets its own phase factor ($e^{i\omega t}$)

$$y(x, t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dk A(k) e^{i(kx + \omega t)}$$