Mechanical Impedance.

Recall the definition of impedance from our study of oscillating electrical circuits:

$$\vec{V} \equiv \vec{I} Z$$

Z = impedance = a complex number which tells us the ratio of peak voltage to peak current:

$$|Z| = \frac{|V|}{|I|}$$

If Z is complex, then there is also a phase shift between $\vec{V}$ and $\vec{I}$.

Roughly speaking, a large impedance means that there will be a small current produced by a given voltage. Conversely, a if the impedance is small, then the same voltage will produce a large current.

We can define an analogous quantity for mechanical systems like a continuous string. We define

$$Z_{mechanical} \equiv \frac{|\text{transverse force}|}{|\text{transverse velocity}|} = \frac{|F|}{|V_T|}$$

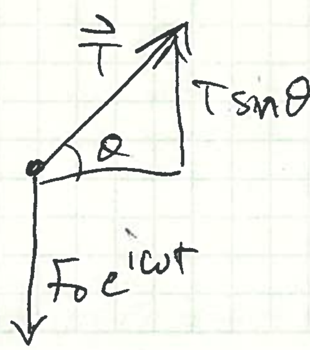
or $|F| \equiv |V_T| Z = |\dot{y}| Z$

With this definition, a large impedance implies a small ~~traus~~ transverse velocity, and a small impedance implies a large transverse velocity (for the same transverse force.)

Calculation of the impedance of a continuous string.

Consider ~~an~~ a long string being driven up and down by an oscillating force at some point on its length. This generates a travelling wave:

At the location where the force is applied, the force must be responsible for maintaining the tension in the string (otherwise there can be no travelling wave.) For example when the tension is positive, the force is negative:



$$F_0 e^{i\omega t} = -T\sin\theta \approx -T\tan\theta = -T\underbrace{\left(\frac{\partial y}{\partial x}\right)}$$

slope of The string.

The travelling wave means that the slope of the string behaves sinusoidally:

$$y(x,t) = Ae^{-i(kx-\omega t)}$$

so $$\frac{\partial y}{\partial x} = -ikAe^{-i(kx-\omega t)}$$

$$\therefore \quad F_0 e^{i\omega t} = -T\frac{\partial y}{\partial x} = (-T)\left(-ikAe^{-i(kx-\omega t)}\right)$$
$$= ikTAe^{-i(kx-\omega t)}$$

$x=\emptyset$ when the force is applied

$$F_0 e^{i\omega t} = ikTAe^{i\omega t}$$

$$A = \frac{F_0}{ikT} = \frac{F_0}{i\left(\frac{\omega}{V_p}\right)T} = \frac{F_0 V_p}{i\omega T}$$

phase velocity

Therefore, $y(x,t) = Ae^{-i(kx-\omega t)} = \dfrac{F_0 V_p}{i\omega T}e^{-i(kx-\omega t)}$

The transverse velocity is $\frac{\partial y}{\partial t}$:

$$\dot{y}(x,t) = \frac{F_0 v_p}{i\omega T}(i\omega)\, e^{-i(kx-\omega t)}$$

$$= \frac{F_0}{(T/v_p)} e^{-i(kx-\omega t)}$$

Then the mechanical impedance is

$$Z = \frac{|\vec{F}|}{|\dot{y}|} = \frac{T}{v_p}$$

Tension

phase velocity

Also, $\sqrt{\frac{T}{\rho}} = v_p$, so $v_p^2 \rho = T$.

mass density

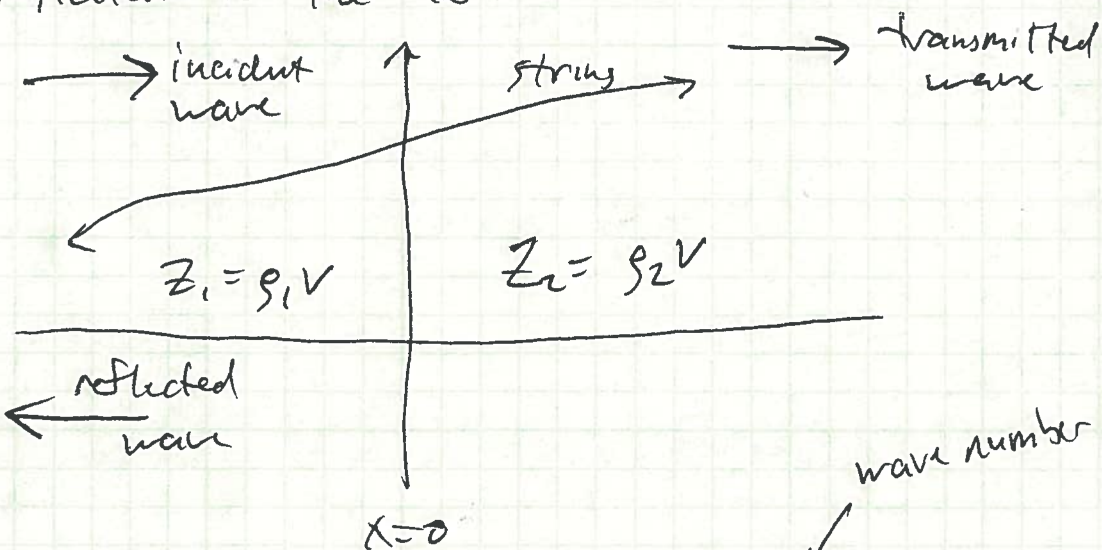
Therefore $\boxed{Z = \frac{T}{v_p} = \frac{v_p^2 \rho}{v_p} = \rho v_p}$

mechanical impedance of string is the phase velocity times the mass density

Reflection & Transmission at a boundary.

Consider a string where the mass density suddenly changes:

What will happen as a travelling wave approaches this discontinuity from the left? In general we can expect that part of the wave will be transmitted to the right, and part will be reflected to the left.



wave number k_1

incident wave : $y_i(x,t) = A_1 e^{-i(k_1x-\omega t)}$

reflected wave : $y_r(x,t) = B_1 e^{-i(-k_1x-\omega t)}$

travelling to the left

transmitted wave: $y_t(x,t) = A_2 e^{-i(k_2 x - \omega t)}$

k_2 ↑ wave number in medium 2

Boundary Conditions:

① At $x = 0$, the rope is continuous, so

$$y_i + y_r = y_t \quad (\text{at } x = 0)$$

② The slope $\frac{\partial y}{\partial x}$ must be finite ~~at~~ and continuous at $x = 0$

Condition ① gives

$$A_1 e^{-i(k_1 x - \omega t)} + B_1 e^{-i(-k_1 x - \omega t)} = A_2 e^{-i(k_2 x - \omega t)}$$

At $x = 0$:

$$A_1 e^{i\omega t} + B_1 e^{i\omega t} = A_2 e^{i\omega t}$$

$$\boxed{A_1 + B_1 = A_2} \quad ①$$

Condition ② gives

$$\frac{\partial}{\partial x}(y_i + y_r) = \frac{\partial}{\partial x}(y_t) \quad \text{at } x = 0:$$

$$-ik_1 A_1 e^{-i(k_1 x - \omega t)} + ik_1 B_1 e^{-i(-k_1 x - \omega t)}$$

$$= -ik_2 A_2 e^{-i(k_2 x - \omega t)}$$

$$-ik_1 A_1 e^{-i(k_1 x - \omega t)} + ik_1 B_1 e^{-i(-k_1 x - \omega t)} = -ik_2 A_2 e^{-i(k_2 x - \omega t}$$

At $x = \emptyset$ this is

$$-k_1 A_1 + k_1 B_1 = -k_2 A_2$$

We can re-write this in terms of the mechanical impedance:

$$k_1 = \frac{\omega}{V_{p1}} \quad , \quad k_2 = \frac{\omega}{V_{p2}}$$

V_{p1}: phase velocity in medium 1

V_{p2}: phase velocity in medium 2

Note that ω is the same in both ~~media~~ regions 1 & 2 - otherwise the ~~wave~~ string would get out-of-phase with itself and be discontinuous at $x = \emptyset$.

Continuing, substitute for k_1 & k_2:

$$-\frac{\omega}{V_{p1}} A_1 + \frac{\omega}{V_{p1}} B_1 = -\frac{\omega}{V_{p2}} A_2$$

Now multiply by the tension (T): (and multiply by -1)

$$\omega \underbrace{\left(\frac{T}{V_{p1}}\right)}_{Z_1} (A_1 - B_1) = \omega \underbrace{\left(\frac{T}{V_{p2}}\right)}_{Z_2} (A_2)$$

$$\boxed{A_1 - B_1 = \frac{Z_2}{Z_1} A_2} \quad ②$$

Also recall Eq. ①:

$$\boxed{A_1 + B_1 = A_2} \quad ①$$

These are the two boundary conditions.

Now we can determine the amplitude of the reflected wave (B_1) in terms of the amplitude of the incoming wave (A_1):

Subract ② from ①:

$$2B_1 = A_2\left(1 - \frac{Z_2}{Z_1}\right) = A_2\left(\frac{Z_1 - Z_2}{Z_1}\right)$$

$$B_1 = \tfrac{1}{2}A_2\left(\frac{Z_1 - Z_2}{Z_1}\right)$$

What is A_2? It's the amplitude of the transmitted wave. We can find it in terms of A_1:

Add ① & ②:

$$2A_1 = A_2\left(1 + \frac{Z_2}{Z_1}\right) = A_2\left(\frac{Z_1 + Z_2}{Z_1}\right)$$

$$\therefore \boxed{\frac{A_2}{A_1} = \frac{2Z_1}{Z_1 + Z_2}}$$

This tells us how large the transmitted wave will be, in terms of the impedances and the amplitude of the incoming wave.

Now we can find B_1:

$$B_1 = \tfrac{1}{2}A_2\left(\frac{Z_1 - Z_2}{Z_1}\right) = \tfrac{1}{2}\underbrace{\left(\frac{2Z_1A_1}{Z_1+Z_2}\right)}_{A_2}\left(\frac{Z_1 - Z_2}{Z_1}\right)$$

$$\boxed{\frac{B_1}{A_1} = \frac{Z_1 - Z_2}{Z_1 + Z_2}}$$

This tells us how large the reflected wave will be in terms of the impedances and the amplitude of the incoming wave.

So our final results are

$$\boxed{\frac{A_2}{A_1} = \text{"transmission coefficient of amplitude"} = \frac{2Z_1}{Z_1+Z_2}}$$

$$\boxed{\frac{B_1}{A_1} = \text{"reflection coefficient of amplitude"} = \frac{Z_1 - Z_2}{Z_1+Z_2}}$$

Two extreme examples

① Suppose that $Z_1 = Z_2$, so that there is no boundary. We call this situation "impedance matching". Then

$$\frac{A_2}{A_1} = \frac{2Z_1}{\underset{\substack{\uparrow \\ Z_2 = Z_1}}{Z_1 + Z_2}} = \frac{2Z_1}{2Z_1} = 1 \leftarrow \text{100\% transmitted amplitude}$$

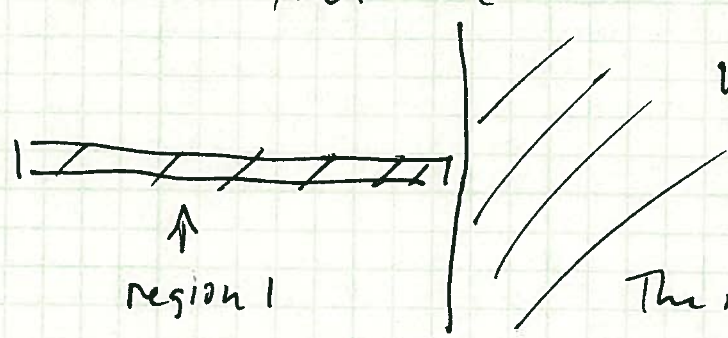
and

$$\frac{B_1}{A_1} = \frac{Z_1 - Z_2}{Z_1 + Z_2} = \emptyset \text{ when } Z_1 = Z_2 \leftarrow \text{Ø\% reflected amplitude}$$

This makes sense: if there is no boundary, then there should be no reflection.

② Suppose that the mass density of region 2 is very, very large. For example, suppose region 2 becomes infinitely heavy, like a brick wall. Then we have

region 2: wall



$Z_2 = \rho v \to \infty$

The impedance of the wall is infinite.

Then $Z_2 \to \infty$, so

$$\frac{A_2}{A_1} = \frac{2Z_1}{Z_1+Z_2} \to \phi \text{ as } Z_2 \to \infty.$$

There is 0% transmission of amplitude into the wall.

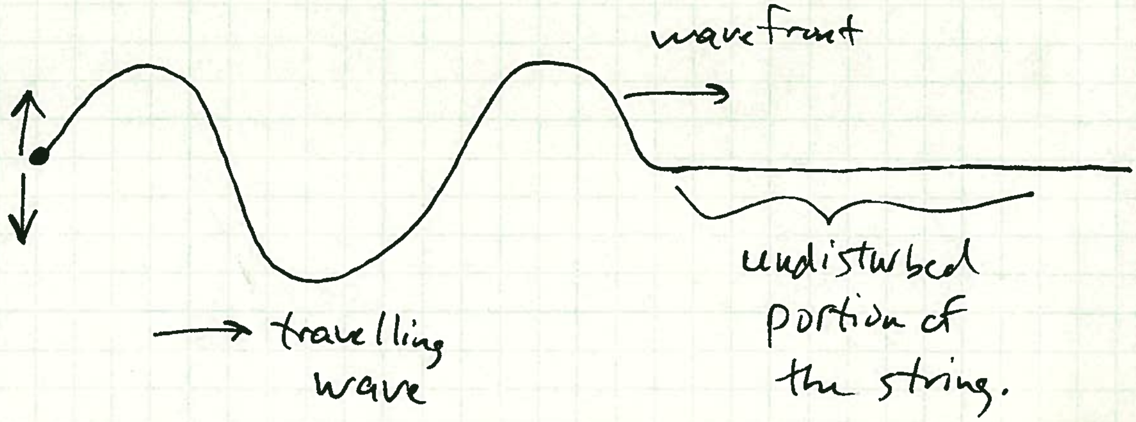
Conversely,

$$\frac{B_1}{A_1} = \frac{Z_1 - Z_2}{Z_1+Z_2} \longrightarrow -1 \text{ when } Z_2 \to \infty.$$

We get 100% reflection, but the amplitude changes sign, when the incoming wave strikes a wall.

Transport of Energy by a mechanical wave

If we grab the end of a string and oscillate it up and down, we set up a traveling wave which propagates down the string:



The part of the string which is oscillating clearly possesses kinetic energy. It also possesses potential energy due to the stretching of the string.

As time goes forward, new sections of the string are brought into motion as the wave propagates, so the total energy contained in the string is increasing, and the energy travels with the wave down the string. This energy comes from the <u>work done</u> by the <u>driving force</u> which makes the string oscillate.

We can calculate the rate of energy transport by calculating the <u>work done</u> by the driving force per unit time. Remember, in SI units

$$\text{Rate of Energy transmission} = \frac{\text{Joules}}{\text{sec}} = \text{Power} = \text{Watts}$$

So consider a sinusoidal traveling wave:

$$y(x,t) = A\sin(kx - \omega t)$$

The driving force is applied at $x = \varnothing$, and the transverse motion of the string at that location is

$$y(x=\varnothing, t) = A\sin(-\omega t) = -A\sin(\omega t)$$

The differential work done in a small displacement (dy)

$$dW = F_t\,dy = F_t\,d(-A\sin(\omega t)) = -F_t A\omega\cos(\omega t)dt$$

↑ transverse component of driving force

The transverse force is

$$F_t = -T\frac{\partial y}{\partial x}\Big|_{x=0} = -T\left[\frac{\partial}{\partial x}\left(A\sin(kx-\omega t)\right)\right]\Big|_{x=0}$$

$$= -TAk\cos(kx-\omega t)\Big|_{x=0}$$

$$= -TAk\cos(-\omega t)$$

$$F_t = -TAk\cos(\omega t)$$

~~The total~~ $\therefore$ In a short time period (dt) the work done is

$$dW = -F_T A\omega\cos(\omega t)\,dt = (-TAk\cos(\omega t))(A\omega\cos(\omega t)\,dt)$$

$$dW = TA^2k\omega\cos^2(\omega t)\,dt$$

The total work done in <u>one</u> <u>complete</u> cycle is

$$\underset{\uparrow \text{ one cycle}}{W_1} = \int_0^{2\pi/\omega} TA^2k\omega\cos^2(\omega t)\,dt$$

$$= TA^2k\omega \underbrace{\int_0^{2\pi/\omega} \cos^2(\omega t)\,dt}_{\frac{\pi}{\omega} = \frac{\pi}{2\pi f} = \frac{1}{2f}}$$

$$W_1 = \frac{\frac{1}{2}TA^2k\omega}{f}$$ ← This is the work done per cycle.

The work done per second is the work done per cycle times the number of cycles per second, which is (f).

$\therefore$ Work per second = Power = $W_1 f = \frac{1}{2}TA^2k\omega$

We can re-write this: $T = \rho v^2 = \rho v\left(\frac{\omega}{k}\right)$

mass density → ρ ; v ← phase velocity

$\therefore$ Power = Energy transmission $= \frac{1}{2}\left(\rho v \frac{\omega}{k}\right)A^2k\omega$

$$\boxed{P = \frac{1}{2}\rho v\omega^2 A^2}$$

Another way to write it: $Z = \rho v$, so

$$\boxed{P = \tfrac{1}{2} Z \omega^2 A^2}$$

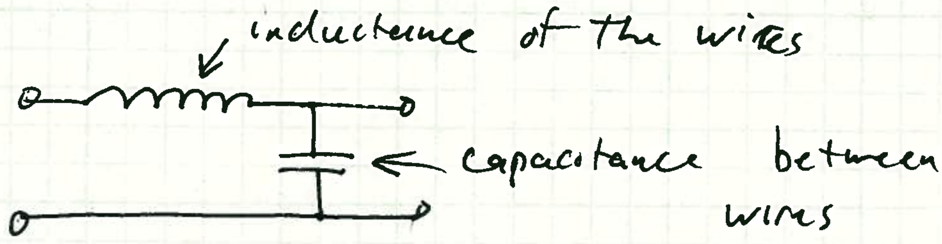
Transmission Lines.

1. Any two conductors that are seperated by a finite distance have some capacitance.

2. Any length of wire, even a straight wire, has some inductance.

So consider two parallel straight wires:

We can model a small section of these wires as



Let L_0 = inductance per unit length

and C_0 = capacitance per unit length.

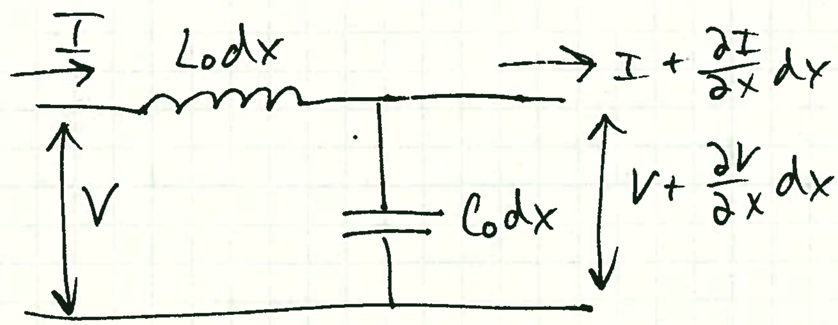
Then a short section (dx) has $L = L_0 dx$
and $C = C_0 dx$.

Let the input current and voltage by I & V.

Then the output current and voltage will be

$$I_{output} = I + \underbrace{\frac{\partial I}{\partial x} dx}_{\text{change in current}}$$

$$V_{output} = V + \underbrace{\frac{\partial V}{\partial x} dx}_{\text{change in voltage}}$$



The change in voltage must be due to the inductor:

$$\frac{\partial V}{\partial x} dx = \underbrace{-(L_0 dx)\frac{\partial I}{\partial t}}_{\substack{\text{Voltage drop} \\ \text{across an inductor,} \\ \text{where } L = L_0 dx}}$$

or $$\frac{\partial V}{\partial x} = -L_0 \frac{\partial I}{\partial t} \quad (1)$$

The change in current must mean that the capacitor has charged up a little bit:

$$dI = \frac{dQ}{dt} = \frac{\partial}{\partial t}(cv) = \frac{\partial}{\partial t}(C_0 dx\, v)$$

$$-\frac{\partial I}{\partial x} dx = C_0 dx \frac{\partial v}{\partial t}$$

or $$-\frac{\partial I}{\partial x} = C_0 \frac{\partial v}{\partial t} \quad (2)$$

Now take $\frac{\partial}{\partial x}$ of (1) and $\frac{\partial}{\partial t}$ of (2):

$$\frac{\partial}{\partial x}(1): \quad \frac{\partial^2 v}{\partial x^2} = -L_0 \frac{\partial I}{\partial x \partial t}$$

$$\frac{\partial}{\partial t}(2): \quad -\frac{\partial^2 I}{\partial t \partial x} = C_0 \frac{\partial^2 v}{\partial t^2}$$

Since $\frac{\partial^2 I}{\partial t \partial x} = \frac{\partial^2 I}{\partial x \partial t}$, we have

$$\boxed{\frac{\partial^2 v}{\partial x^2} = L_0 C_0 \frac{\partial^2 v}{\partial t^2}}$$

These are the wave equations again.

Also, we have

$$\boxed{\frac{\partial^2 I}{\partial x^2} = L_0 C_0 \frac{\partial^2 I}{\partial t^2}}$$

We can immediately infer that voltage & current waves can propagate down the pair of conductors.

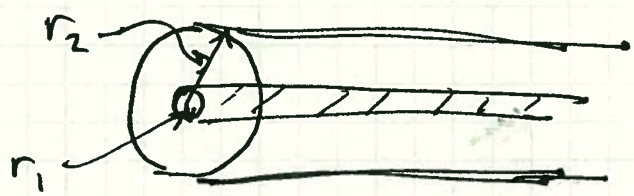
What's the phase velocity of propagation? We simply read it from the wave equation:

$$\frac{\partial^2 V}{\partial x^2} = L_o C_o \frac{\partial^2 V}{\partial t^2}$$

$$\therefore \quad \frac{1}{v^2} = L_o C_o$$

$$\boxed{v = \frac{1}{\sqrt{L_o C_o}}}$$

Coaxial Cable



Has inductance per unit length

$$L_o = \frac{\mu}{2\pi} \ln\left(\frac{r_2}{r_1}\right)$$

where μ = magnetic permeability of the material between the conductors.

Also, it has capacitance per unit length

$$C_0 = \frac{2\pi\epsilon}{\ln(r_2/r_1)} \quad \text{where } \epsilon = \text{permittivity of the dielectric}$$

What's the phase velocity of an electrical wave travelling in this cable?

$$v = \frac{1}{\sqrt{C_0 L_0}} = \frac{1}{\sqrt{\left(\frac{2\pi\epsilon}{\ln(r_2/r_1)}\right)\frac{\mu}{2\pi}\ln(r_2/r_1)}}$$

$$= \frac{1}{\sqrt{\epsilon\mu}}$$

For a typical coaxial cable, with polyethylene as the dielectric,

$$\epsilon \approx 2.25\,\epsilon_0$$

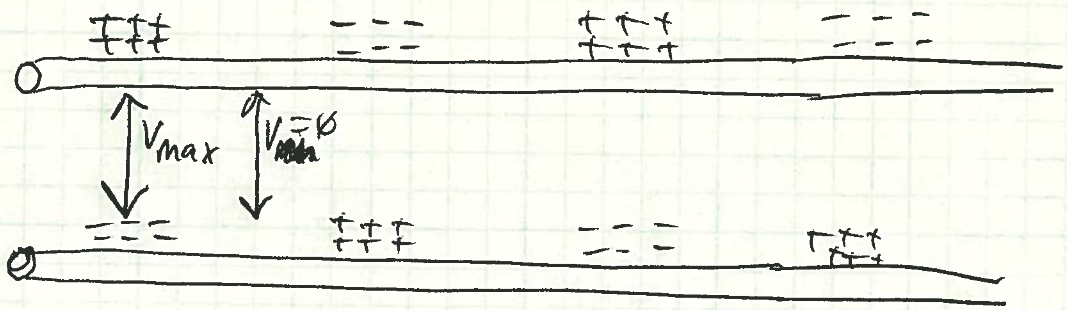
$$\mu \approx \mu_0$$

$$\text{so} \quad v \approx \frac{1}{\sqrt{2.25}} \underbrace{\frac{1}{\sqrt{\epsilon_0\mu_0}}}_{\text{speed of light } = c = 3.0\times10^8 \text{ m/s}}$$

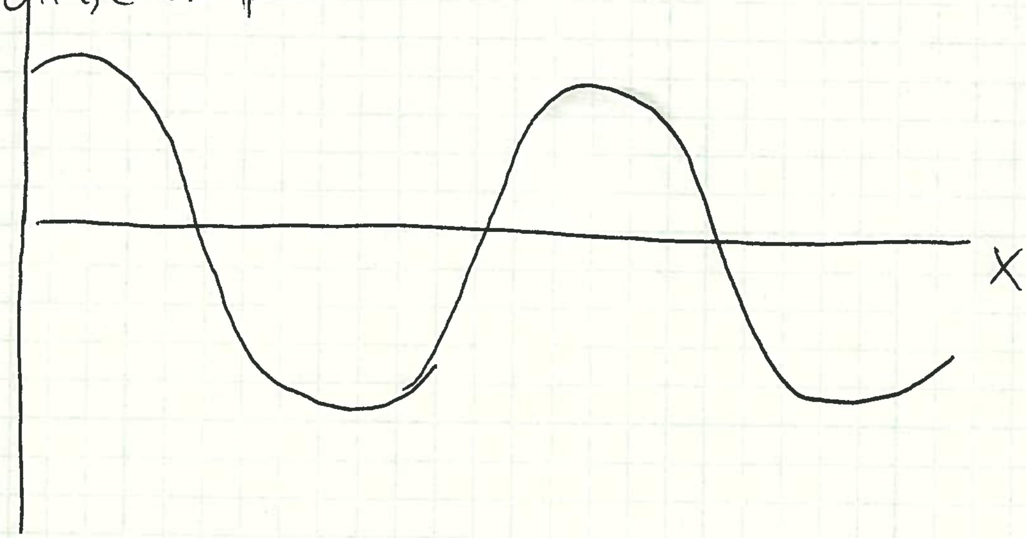
$$v \approx \frac{2}{3}c \approx 2.0\times10^8 \text{ m/s}$$

Characteristic Impedance of a Transmission Line

Transmission Lines support voltage & current waves:



Voltage vs. position



Voltage satisfies the wave Equation:

$$\frac{\partial^2 V}{\partial x^2} = L_o C_o \frac{\partial^2 V}{\partial t^2}$$

, where L_o = inductance per unit length

C_o = Capacitance per unit length

Wave phase velocity is $\frac{1}{\sqrt{L_o C_o}}$

For a coaxial cable, $\frac{1}{\sqrt{L_o C_o}} = \frac{1}{\sqrt{\mu \epsilon}} = v_p$

Coaxial Cable

So the voltage wave in a ~~wave~~ travels at a speed equal to an electromagnetic wave in that dielectric.

The current also satisfies a wave equation:

$$\frac{\partial^2 I}{\partial x^2} = L_0 C_0 \frac{\partial^2 I}{\partial t^2}$$

The Voltage ~~a~~ and Current are related by two equations:

$$\frac{\partial V}{\partial x} = -L_0 \frac{\partial I}{\partial t} \quad \text{and} \quad \frac{\partial I}{\partial x} = -C_0 \frac{\partial V}{\partial t}.$$

We can use these to determine the ratio of <u>peak voltage</u> to <u>peak current</u>, which is called the <u>characteristic impedance</u> of the transmission line.

~~The~~ A solution to the voltage wave Equation is

$$V_+(x,t) = V_{0+} \sin(kx - \omega t)$$

plus sign means this wave travels in the positive x direction

The current associated with this voltage is

$$I_+(x,t) = I_{0+} \sin(kx - \omega t) \ .$$

We can confirm that this is correct by substituting into the differential equations:

$$\frac{\partial V}{\partial x} = V_{0+} k \cos(kx - \omega t)$$

$$\frac{\partial I}{\partial t} = I_{0+} (-\omega) \cos(kx - \omega t)$$

We require that $\frac{\partial V}{\partial x} = -L_0 \frac{\partial I}{\partial t}$

i.e. $V_{0+} k \cos(kx - \omega t) = -L_0 I_{0+} (-\omega) \cos(kx - \omega t)$

$$\frac{V_{0+}}{I_{0+}} = L_0 \underbrace{\frac{\omega}{k}}_{\text{phase velocity.}} = L_0 V_{phase}$$

Therefore there is a fixed relationship between the magnitude of the peak voltage and the peak current (these magnitudes cannot be chosen independently.) We call that ratio the <u>characteristic impedance</u> of the transmission line.

$Z_0 =$ "characteristic Impedance" $\equiv \dfrac{V_{0+}}{I_{0+}}$

$$\therefore \quad Z_0 = L_0 v = L_0 \underbrace{\left(\frac{1}{\sqrt{L_0 C_0}}\right)}_{v} = \sqrt{\frac{L_0}{C_0}}$$

$$\boxed{Z_0 = \sqrt{\frac{L_0}{C_0}}} = \text{units of } \Omega \text{ (Ohms).}$$

For a coaxial cable, $L_0 = \dfrac{\mu}{2\pi} \ln\left(\dfrac{r_2}{r_1}\right)$

$$C_0 = \frac{2\pi\varepsilon}{\ln(r_2/r_1)}$$

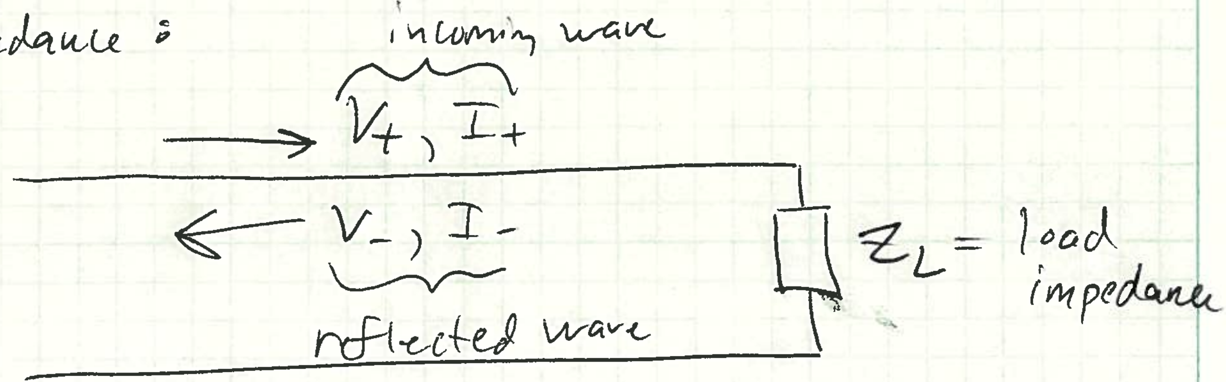
$$\text{so} \quad Z_0 = \sqrt{\frac{\frac{\mu}{2\pi}\ln\left(\frac{r_2}{r_1}\right)}{\frac{2\pi\varepsilon}{\ln(r_2/r_1)}}} = \underbrace{\frac{1}{2\pi}\sqrt{\frac{\mu}{\varepsilon}}\ \ln\frac{r_2}{r_1}}_{\text{coaxial cable characteristic impedance}}$$

Typical Coaxial cables have μ, ε, r_2 & r_1 such that $Z_0 = 50\,\Omega$. But sometimes you find cables that have $Z_0 = 75\,\Omega$, $100\,\Omega$, etc.

Note that $Z_0 = \sqrt{\dfrac{L_0}{C_0}}$ = purely real $\Rightarrow$ Voltage and Current are 100% in phase.

Load Impedance & Reflections

At the end of a transmission line we can connect an electrical device with some impedance:



We can expect to have reflections from this load impedance, just like for mechanical waves. Since the ~~4a~~ mathematics is the same as before (just the wave equation with boundary conditions), the results will be the same as with mechanical waves:

$$\frac{\cancel{\text{Ratio of}}\ \text{Reflected Amplitude}}{\text{Incoming Amplitude}} = \frac{V_-}{V_+} = \frac{Z_L - Z_0}{Z_L + Z_0}$$

$$\frac{\text{Transmitted Amplitude}}{\text{Incoming Amplitude}} = \frac{V_L}{V_+} = \frac{2 Z_L}{Z_L + Z_0}.$$

Notice that there is one important distinction between the reflected amplitude of a voltage wave on a transmission line and the same thing in a mechanical wave: If ~~the~~ $Z_L \to \infty$, we get 100% reflection, but without a phase shift.

$$\frac{V_-}{V_+} = \frac{Z_L - Z_0}{Z_L + Z_0} \to 1 \text{ as } Z_L \to \infty$$

(open circuit)

In the case of a mechanical wave, $Z_2 \to \infty$ means that the last particle is not allowed to move (boundary condition). For a voltage wave, however, the end of the transmission line is allowed to have a non-zero voltage even when $Z_L \to \infty$. So the reflected wave is not forced to cancel the incoming wave.

For the current wave, on the other hand, the expressions are

$$\frac{I_-}{I_+} = \frac{Z_0 - Z_L}{Z_0 + Z_L}$$

and

$$\frac{I_L}{I_+} = \frac{2 Z_0}{Z_L + Z_0}$$

As $Z_L \to \infty$, the reflected current wave does have a 180° phase shift, (unlike the voltage wave), because no current can flow through an infinite impedance.