Recap of Fourier's Trick

For the loaded string, we have orthogonal eigenvectors:

$$\vec{q}_n \cdot \vec{q}_m = \underbrace{\sum_{p=1}^{N} \sin\left(\frac{pn\pi}{N+1}\right) \sin\left(\frac{pm\pi}{N+1}\right)}_{\text{a dot product of discrete vectors}} = \left(\frac{N+1}{2}\right) \delta_{nm}$$

Kronecker Delta:

$$\delta_{nm} = \begin{cases} 1, & n=m \\ \emptyset, & n \neq m \end{cases}$$

For the continuous string, we also have orthogonal eigenvectors:

$$\underbrace{\int_{\emptyset}^{L} \underbrace{\sin\left(\frac{n\pi x}{L}\right)}_{\text{eigenvector}} \underbrace{\sin\left(\frac{m\pi x}{L}\right)}_{\text{eigenvector}} dx}_{\text{a dot product of continuous vectors}} = \frac{L}{2} \delta_{nm}$$

they are orthogonal

~~Thus~~ The property of orthogonality makes Fourier's Trick work:

For the loaded string: Let $\vec{y}_0$ be the vector of initial conditions. Then

$$\vec{y}_0 \cdot \vec{q}_m = \left(\sum_{n=1}^{N} (a_n \vec{q}_n)\right) \cdot \vec{q}_m = \sum_{n=1}^{N} a_n \underbrace{\vec{q}_n \cdot \vec{q}_m}_{\left(\frac{N+1}{2}\right)\delta_{nm}} = \sum_{n=1}^{N} a_n \left(\frac{N+1}{2}\right) \delta_{nm}$$

δ_{nm} kills all terms in the sum except $n=m$.

$$= a_m\left(\frac{N+1}{2}\right) = a_m\left(\vec{q}_m\cdot\vec{q}_m\right) = a_m|\vec{q}_m|^2$$

$$\therefore \quad \boxed{a_m = \frac{\vec{y}_0\cdot\vec{q}_m}{|\vec{q}_m|^2}} \quad \text{Fourier's Trick}$$

expansion coefficient

initial conditions

normalization

For the continuous string the same reasoning applies.
Let $y(x, t=0) = y(x)$ be the initial condition.
Then

$$\underbrace{\int_0^L y(x)\sin\left(\frac{m\pi x}{L}\right)dx}_{\text{a dot product}} = \int_0^L\left(\sum_{n=1}^{\infty} a_n \sin\left(\frac{n\pi x}{L}\right)\right)\sin\left(\frac{m\pi x}{L}\right)dx$$

$$= \sum_{n=1}^{\infty} a_n \underbrace{\int_0^L \sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{m\pi x}{L}\right)dx}_{\frac{L}{2}\delta_{nm}}$$

$$= \sum_{n=1}^{\infty} a_n\left(\frac{L}{2}\delta_{nm}\right)$$

kills all terms in the sum except $n=m$

$$= a_m \frac{L}{2}$$

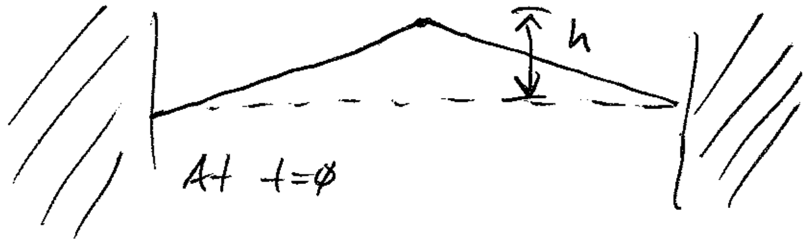
$$\therefore \quad \boxed{a_m = \left(\frac{2}{L}\right)\int_0^L y(x)\sin\left(\frac{m\pi x}{L}\right)dx} \quad \underline{\text{Fourier's Trick}}$$

expansion coefficient

normalization

initial conditions

We did an example: the triangular string:



How does the string evolve in time? Answer:

$$y(x,t) = \sum_{n=1}^{\infty} a_n \sin\left(\frac{n\pi x}{L}\right) e^{i\omega_n t} \quad , \quad \omega_n = \sqrt{\frac{T}{\rho}}\,\frac{n\pi}{L}$$

And what are the $\{a_n\}$ for the triangular initial conditions? Answer

$$a_n = \frac{2}{L}\int_0^L y(x,t=0)\,\sin\left(\frac{n\pi x}{L}\right)dx$$

We calculated this integral on Thursday and found

$$a_n = \begin{cases} \dfrac{8h}{(n\pi)^2}(-1)^{(n-1)/2} & , \text{ for } n = \text{odd} \\ 0 & , \text{ For } n = \text{even} \end{cases}$$

Therefore, at $t = 0$,

$$y(x,t=0) = \sum_{\substack{n=1,\\ \text{odd} \\ (n)\text{ only!}}}^{\infty} \underbrace{\frac{8h}{(n\pi)^2}(-1)^{(n-1)/2}}_{a_n} \sin\left(\frac{n\pi x}{L}\right) = \text{a triangle.}$$

Here we have succeeded in writing a very simple function, a triangle, as an extremely complicated <u>infinite</u> sum of sine functions. Why did we do that?

Answer: Because once we've written the initial condition as a sum over normal modes, now we can get the time dependence in a trivial way: Just tack-on the exponential phase factor to each term in the sum:

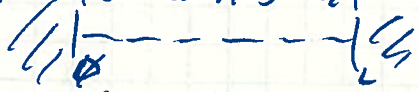
$$\boxed{y(x,t) = \sum_{\substack{n=1,\\ \text{odd}(n)\\ \text{only}}}^{\infty} \frac{8h}{(n\pi)^2}(-1)^{(n-1)/2} \sin\left(\frac{n\pi x}{L}\right) e^{i\omega_n t}}$$

Generalized Fourier Series

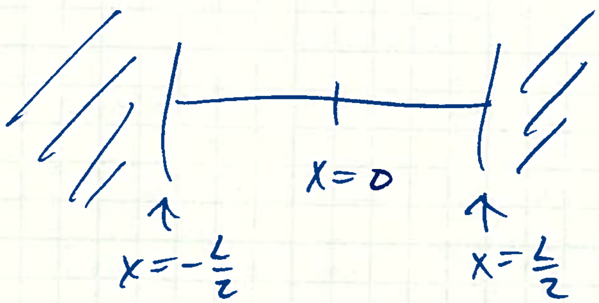
We've been studying a special type of Fourier series called a "Fourier Sine Series"

$$y(x,t=\emptyset) = \sum_{n=1}^{\infty} a \sin\left(\frac{n\pi x}{L}\right)$$

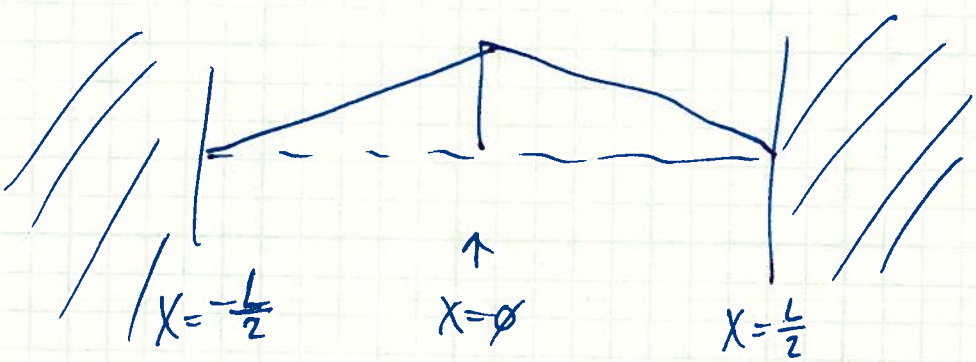
We like this series because it describes a string attached to walls at $x=\emptyset$ & $x=L$.



In general, however, we may choose to attach our string at other locations, like:



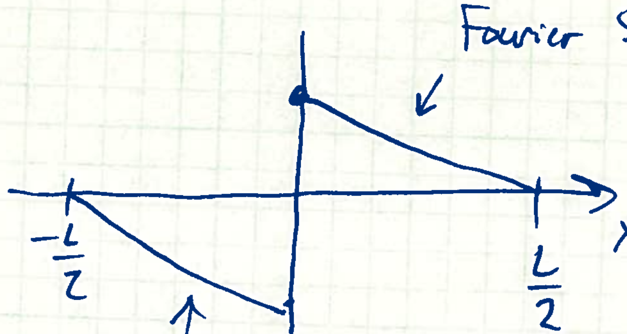
Suppose, for example, that the initial shape of our string is a triangle, and our coordinate system is centered on the middle of the string:



Can we represent this shape as a sum of sine functions?

Answer: No, because sine functions are odd and this function is even. If we tried, we would get

Fourier Sine Series is OK here



but it gives the wrong sign here

It's because every term in the series is odd:

$$f(x) = \sum_{n=1}^{\infty} a_n \sin\left(\frac{n\pi x}{L}\right)$$

$$\therefore f(-x) = \sum_{n=1}^{\infty} a_n \sin\left(\frac{n\pi(-x)}{L}\right) = -\sum_{n=1}^{\infty} a_n \sin\left(\frac{n\pi x}{L}\right)$$

$$= -f(x) \leftarrow \text{odd function}$$

To represent an even function, we'll need a ~~cosine~~ "Fourier Cosine Series"

$$f(x) = \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right) \leftarrow \text{"Fourier Cosine Series" for even functions.}$$

How do we determine the expansion coefficients $\{a_n\}$ for this series?

<u>Answer</u>: The cosine functions are <u>orthogonal</u>:

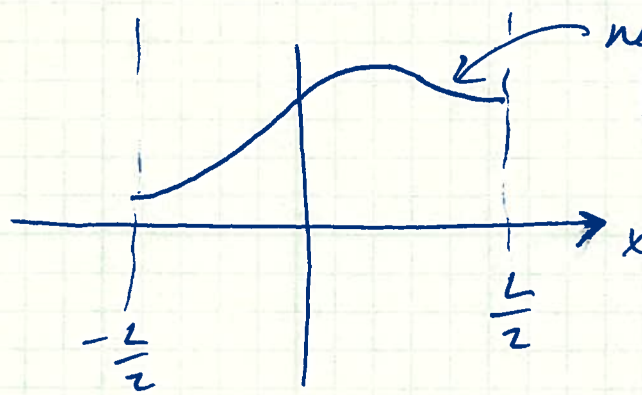
$$\int_{-\frac{L}{2}}^{\frac{L}{2}} \cos\left(\frac{n\pi x}{L}\right)\cos\left(\frac{m\pi x}{L}\right) dx = \frac{L}{2}\,\delta_{nm}$$

<u>Therefore</u> <u>Fourier's Trick</u> works for them also:

$$a_n = \frac{2}{L}\int_{-\frac{L}{2}}^{\frac{L}{2}} \cos\left(\frac{n\pi x}{L}\right) f(x)\, dx$$

In <u>general</u>, an arbitrary function is neither even or odd, but is a <u>sum</u> of <u>even</u> and odd parts:

$$f(x) = f_{odd}(x) + f_{even}(x)$$



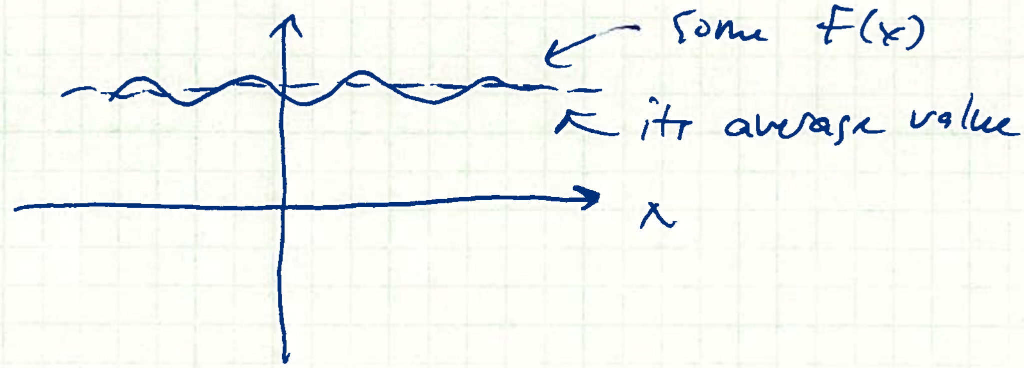
neither even or odd, but a sum of even & odd parts

To represent a function like this, we need both Sine & Cosine Terms:

$$f(x) = \sum_n \left[a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right)\right]$$

But there's still <u>one</u> <u>thing</u> <u>missing</u>:

If the average value of the function is zero, then sines & cosines are fine. But if the function has a y-offset, then we have to add a constant:

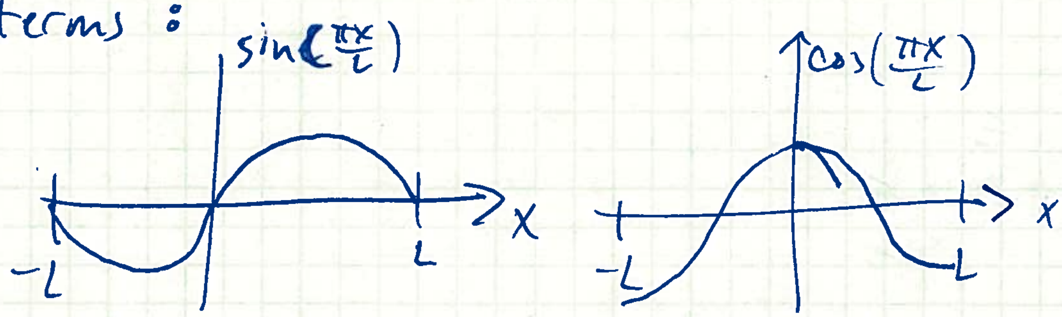


Finally, a complete, general Fourier Series is given by

$$\boxed{f(x) = \frac{a_0}{2} + \sum_n \left[a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right)\right]}$$

average value of $f(x)$

This series can represent almost any <u>periodic</u> function. But what is the period? Look at the $n=1$ terms:



The full period is $\boxed{2L}$.

Generalized Fourier Series:

$f(x)$ is ① periodic with period $2L$

② "square integrable" from $-L$ to L.

Then $f(x)$ can be written as

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left[a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right)\right]$$

where

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right)dx$$

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right)dx$$

Note that normalization factor has changed because now we integrate over a distance of $2L$ instead of L.

Also, note that

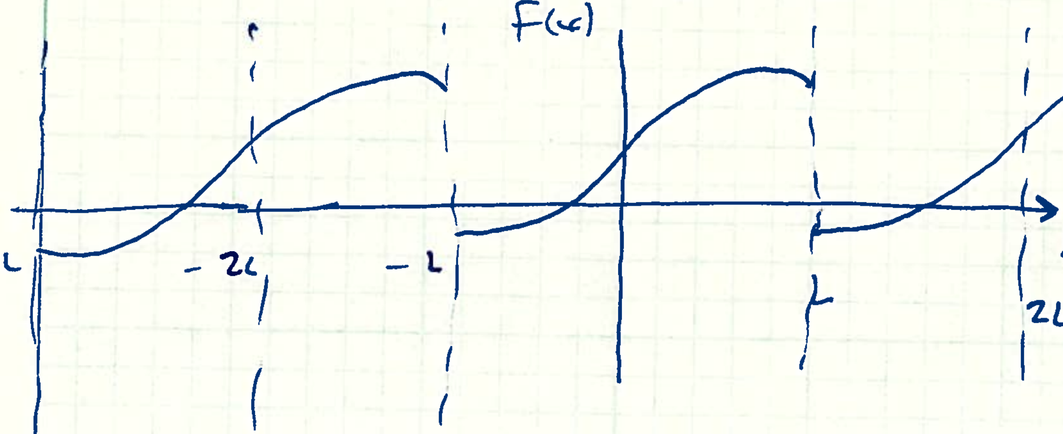
$$a_0 = \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{(0)\pi x}{L}\right)dx$$

$$= \frac{1}{L}\int_{-L}^{L} f(x)\,dx$$

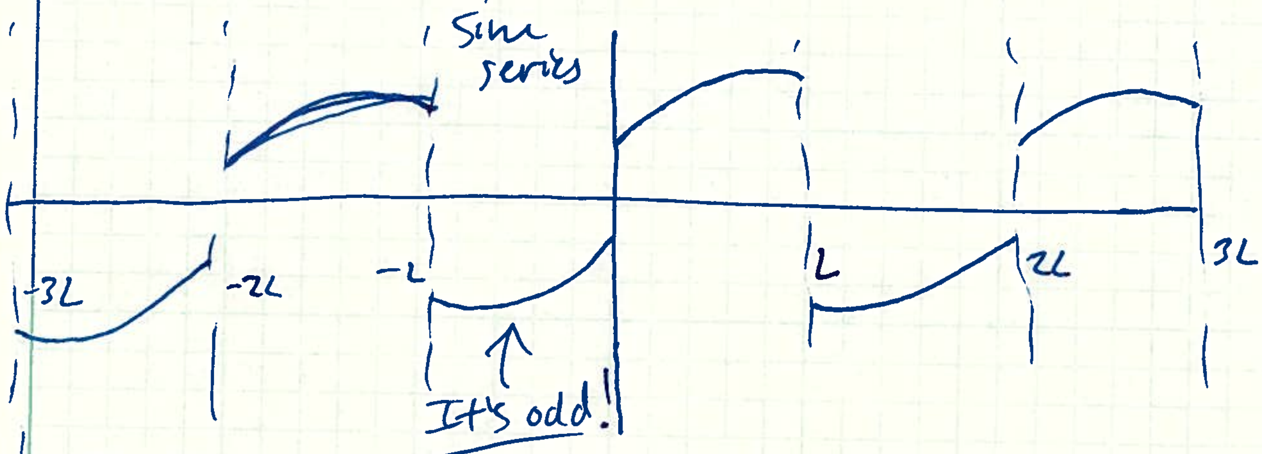
$= 2\times$ average value of $f(x)$ between $-L$ & L

Picture it:

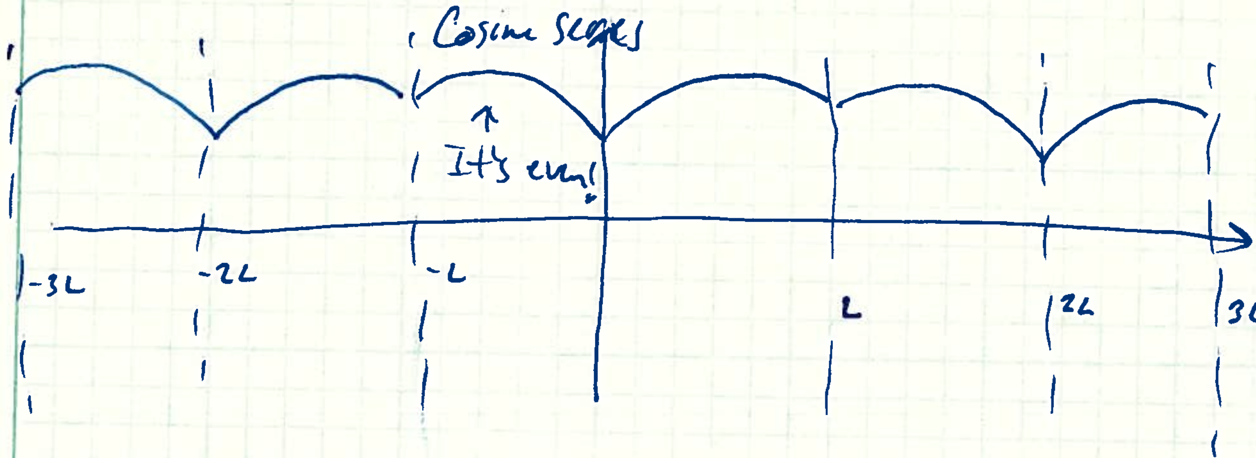
Suppose $F(x)$ looks like this:



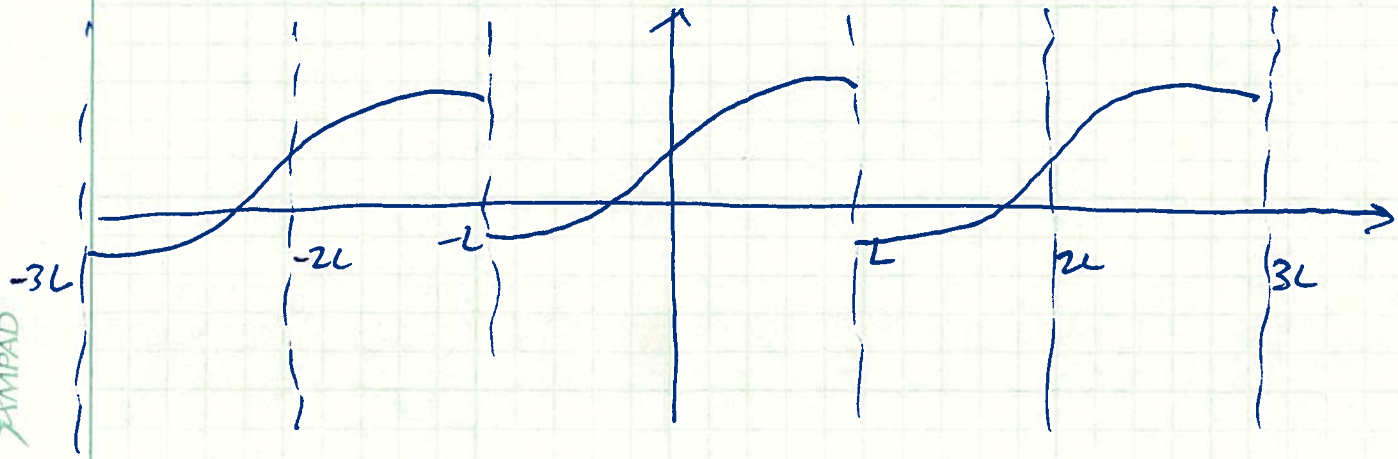
The sine series looks like this:



The cosine series looks like this

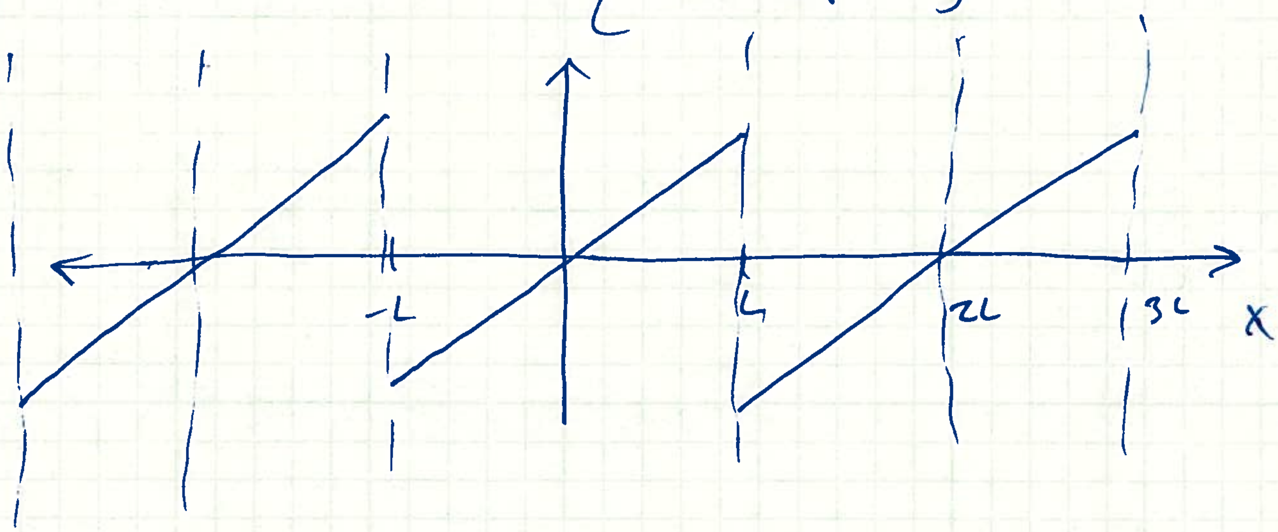


The complete, generalized Fourier Series looks exactly like $f(x)$



Example Calculation of a complete Fourier Series

Sawtooth: $f(x) = \begin{cases} x, & \text{for } -L < x < L \\ \text{and repeating} & \end{cases}$



Fourier coefficients:

$$a_0 = \frac{1}{L}\int_{-L}^{L} x\,dx = \emptyset \leftarrow \text{average value is zero}$$

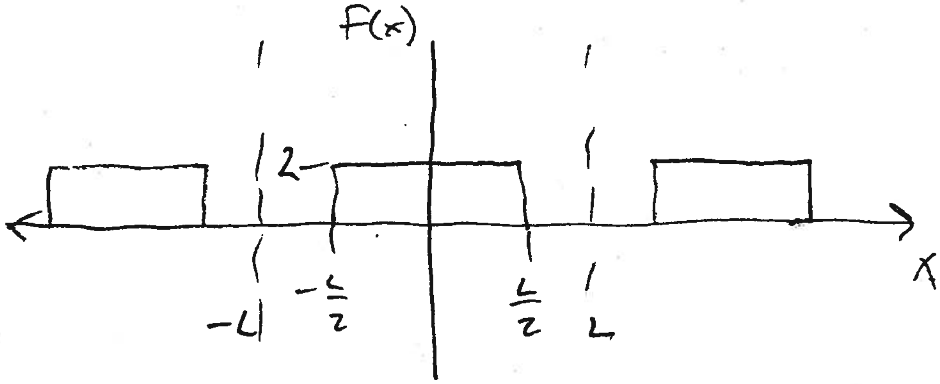
$$a_n = \frac{1}{L}\int_{-L}^{L} \underbrace{x\cos\left(\frac{n\pi x}{L}\right)}_{\text{integrand is odd}} dx = \emptyset \leftarrow \text{no cosine terms! (function is odd)}$$

$$b_n = \underbrace{\frac{1}{L}\int_{-L}^{L} x \sin\left(\frac{n\pi x}{L}\right) dx}_{\text{look up this integral or integrate by parts}} = -2\left(\frac{L}{n\pi}\right)\cos(n\pi) + 2\left(\frac{L}{n\pi}\right)^2 \cancel{\sin(n\pi)} \to \emptyset$$

$$\therefore \quad \boxed{b_n = \frac{2L}{n\pi}(-1)^{n+1}}$$

$$\therefore \quad f(x) = \frac{2L}{\pi}\sin\left(\frac{\pi x}{L}\right) - \frac{2L}{2\pi}\sin\left(\frac{2\pi x}{L}\right) + \frac{2L}{3\pi}\sin\left(\frac{3\pi x}{L}\right) + \ldots$$

Example: Another type of square wave:



$$f(x) = \begin{cases} 0, & -L < x < -\frac{L}{2} \\ 2, & -\frac{L}{2} < x < \frac{L}{2} \\ 0, & \frac{L}{2} < x < L \end{cases}$$ & repeating with period $2L$.

It's an even function of x, so the sine terms will be zero:

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right)dx = \emptyset$$

$$a_0 = \frac{1}{L}\int_{-L}^{L} f(x)\,dx = \frac{1}{L}\int_{-\frac{L}{2}}^{\frac{L}{2}} 2\,dx = 2 = 2 \times \text{average value of } f(x)$$

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right)dx$$

$$= \frac{1}{L}\int_{-\frac{L}{2}}^{\frac{L}{2}} 2\cos\left(\frac{n\pi x}{L}\right)dx$$

$$= \left(\frac{2}{L}\right)\left(\frac{L}{n\pi}\right)\sin\left(\frac{n\pi x}{L}\right)\Bigg|_{-\frac{L}{2}}^{\frac{L}{2}}$$

$$= \left(\frac{2}{n\pi}\right)\left(\sin\frac{n\pi}{2} - \sin\left(-\frac{n\pi}{2}\right)\right) = \left(\frac{4}{n\pi}\right)\sin\left(\frac{n\pi}{2}\right) = \begin{cases} \emptyset, & n \text{ even} \\ \frac{4}{n\pi}(-1)^{(n-1)/2}, & n \text{ odd} \end{cases}$$

$$a_n = \left(\frac{4}{n\pi}\right)(-1)^{(n-1)/2} \quad \text{for } n \text{ odd only}$$

$$\therefore \quad f(x) = \frac{a_0}{2} + a_1 \cos\left(\frac{\pi x}{L}\right) + a_2 \cos\left(\frac{2\pi x}{L}\right) + a_3 \cos\left(\frac{3\pi x}{L}\right) + \cdots$$

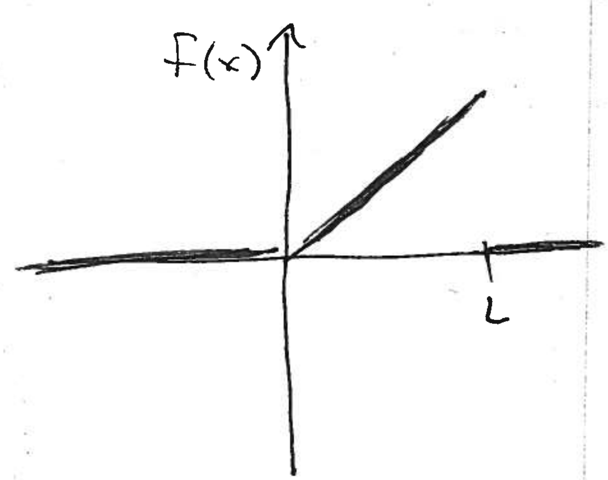
$$\frac{a_0}{2} \to 1, \quad a_1 \to \frac{4}{\pi}, \quad a_2 \to 0, \quad a_3 \to \frac{-4}{3\pi}$$

$$= 1 + \frac{4}{\pi}\cos\left(\frac{\pi x}{L}\right) - \frac{4}{3\pi}\cos\left(\frac{3\pi x}{L}\right) + \cdots$$

$$= 1 + \sum_{n=\text{odd}}^{\infty} (-1)^{(n-1)/2}\left(\frac{4}{n\pi}\right)\cos\left(\frac{n\pi x}{L}\right)$$

Consider this function

$$f(x) = \begin{cases} 0, & x < 0 \\ x, & 0 \le x \le L \\ 0, & x > L \end{cases}$$

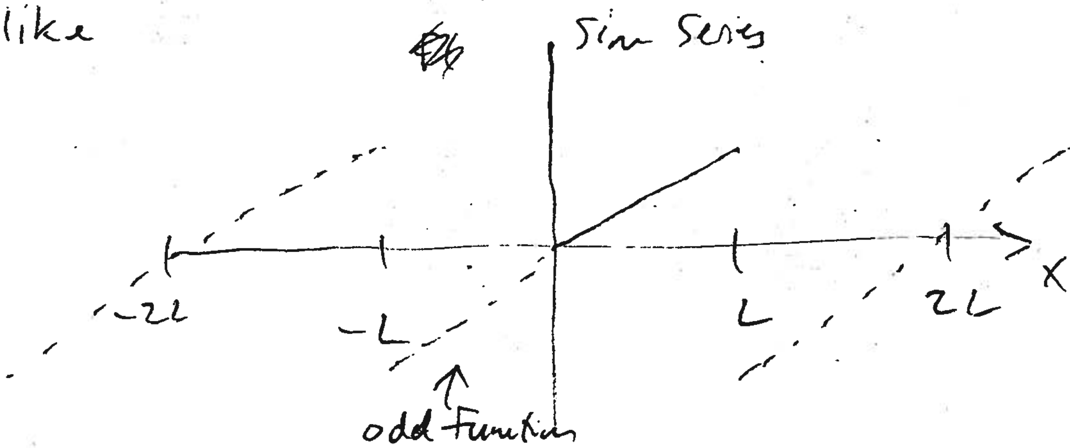


Suppose we wish to represent this function as a Fourier series in the interval $[0, L]$, and we don't care if the Fourier series gives 0 outside the interval. So we would like to write

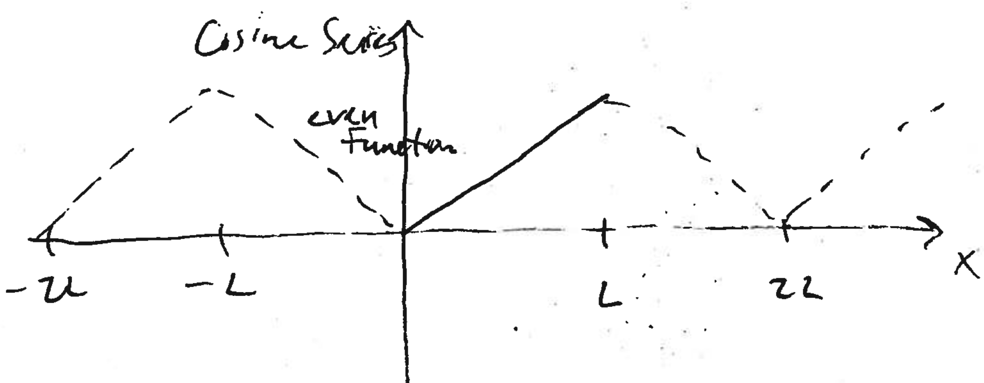
$$f(x) = \begin{cases} 0, & x < 0 \\ \text{A Fourier Series}, & 0 \le x \le L \\ 0, & x > L \end{cases}$$

Will we need sin terms, cosine terms, or both?

<u>Answer</u>: we can use <u>either</u> a Sine Series or a Cosine series. The Sine series will look like



The Cosine Series will look like



Between $x=0$ and $x=L$, either one can represent $f(x)$.

<u>Sine Series Representation</u>:

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right)$$

$$b_n = \frac{2}{L}\int_0^L x \sin\left(\frac{n\pi x}{L}\right)dx = \frac{2}{L}\left[\left(\frac{L}{n\pi}\right)^2 \sin\left(\frac{n\pi x}{L}\right) - \left(\frac{L}{n\pi}\right) x \cos\left(\frac{n\pi x}{L}\right)\right]\Bigg|$$

$$= \frac{2}{L}\left[-\left(\frac{L^2}{n\pi}\right)\underbrace{\cos(n\pi)}_{(-1)^n}\right] = \boxed{\frac{2L}{(n\pi)}(-1)^{n+1}}$$

$$\boxed{f(x) = \sum_{n=1}^{\infty}(-1)^{n+1}\frac{2L}{(n\pi)^2}\sin\left(\frac{n\pi x}{L}\right)}$$

Cosine Series Representation

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right)$$

$$a_0 = \frac{2}{L}\int_0^L x\,dx = \left(\frac{2}{L}\right)\left(\frac{1}{2}L^2\right) = L = 2 \times \text{average value of } f(x)$$

$$a_n = \frac{2}{L}\int_0^L x\cos\left(\frac{n\pi x}{L}\right)dx = \frac{2}{L}\left[\left(\frac{L}{n\pi}\right)^2\cos\left(\frac{n\pi x}{L}\right) + \left(\frac{L}{n\pi}\right)x\sin\left(\frac{n\pi x}{L}\right)\right]\Bigg|_0^L$$

$$= \frac{2}{L}\left[\left(\frac{L}{n\pi}\right)^2\cos(n\pi) - \left(\frac{L^2}{n\pi}\right)\right]$$

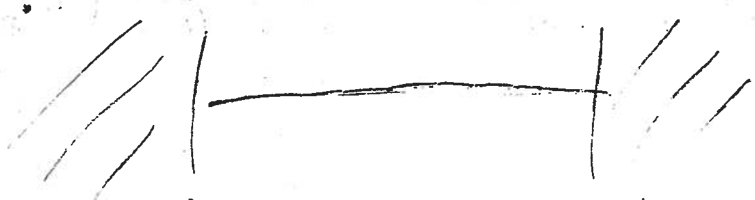
$$= \frac{2L}{(n\pi)^2}\underbrace{\left(\cos(n\pi) - 1\right)}_{-2,\ 0,\ -2,\ 0,\ \ldots}$$

$$\boxed{= \frac{-4L}{(n\pi)^2}}, \quad n = \text{odd only.}$$

$$\boxed{f(x) = \frac{L}{2} + \sum_{n=\text{odd}}^{\infty}\frac{-4L}{(n\pi)^2}\cos\left(\frac{n\pi x}{L}\right)}$$

So mathematically we can represent our function as either a Sine Series or Cosine Series, as long as we only care about the result between $x = 0$ and $x = L$.

But which series is physically relevant for a string connected to walls at $x=0$ and $x=L$?



$x=0$ $\quad$ $x=L$

Answer: For this physical system, only the Sine Series is relevant, because the sin functions are the normal modes for this system. That means we can write the time evolution in a trivial way for the Sine Series:

$$\boxed{y(x,t) = \sum_{n=} (-1)^{n+1} \frac{2L}{(n\pi)^2} \sin\left(\frac{n\pi x}{L}\right) e^{i\omega_n t}}$$ correct

The Cosine Series is of no use to use for this system because the cosine terms do not satisfy the equation of motion and boundary condition:

$$y(x,t) \neq \underbrace{\cancel{\frac{L}{2} + \sum_{n=odd}^{\infty} \frac{4L}{(n\pi)^2} (-1)^{(n+1)/2} \cos\left(\frac{n\pi x}{L}\right) e^{i\omega_n t}}}_{\text{Wrong!}}$$

Cosine terms are not normal modes!

Fourier Series in complex notation.

We've been using the trigonometric form of the Fourier Series:

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left[a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right)\right] \qquad (1)$$

This is a long, complicated expression, and you have to do at least three integrals to find the expansion coefficients:

$$a_0 = \frac{1}{L}\int_{-L}^{L} f(x)\,dx$$

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right)dx$$

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right)dx$$

So it would be nice to have a simpler, more compact way to write a Fourier Series. We can do that using complex exponentials:

$$\boxed{f(x) = \sum_{n=-\infty}^{\infty} c_n e^{in\pi x/L}} \qquad (2)$$

← Completely equivalent to (1)!

!!

The first thing to note about the complex form is that the sum over (n) goes from $-\infty$ to $+\infty$, while the trigonometric form ~~goes from~~ has a sum that starts at 1 and goes to $+\infty$.

The relationship between 1 and 2 is the following:

To convert from the complex form to the trig form:

$$\boxed{\begin{aligned} a_n &= c_n + c_{(-n)} \\ b_n &= i\left(c_n - c_{(-n)}\right) \\ a_0 &= c_\emptyset + c_\emptyset = 2c_\emptyset \end{aligned}}$$

To convert from the trig form to the complex form:

Note that the $\{c_n\}$ are complex. real & imaginary parts.

$$\boxed{c_n = \begin{cases} +\frac{1}{2}\left(a_{(-n)} + i b_{(-n)}\right), & \text{for } n < \emptyset \\ \frac{1}{2} a_0, & \text{for } n = \emptyset \\ \frac{1}{2}\left(a_n - i b_n\right), & \text{for } n > \emptyset \end{cases}}$$

We can explicitly show that the two forms are equivalent using the <u>above relations</u> and <u>Euler's Formula.</u>

Start with the trig form:

$$F(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left[a_n \underbrace{\cos\left(\frac{n\pi x}{L}\right)}_{\boxed{\frac{1}{2}\left(e^{in\pi x/L} + e^{-in\pi x/L}\right)}} + b_n \overbrace{\sin\left(\frac{n\pi x}{L}\right)}^{\boxed{\frac{1}{2i}\left(e^{in\pi x/L} - e^{-in\pi x/L}\right)}}\right]$$

$$= \frac{a_0}{2} + \sum_{n=1}^{\infty} \frac{a_n}{2} e^{in\pi x/L} + \underbrace{\sum_{n=1}^{\infty} \frac{a_n}{2} e^{-in\pi x/L}}_{\text{call this term (A)}} + \sum_{n=1}^{\infty} \frac{b_n}{2i} e^{in\pi x/L} + \underbrace{\sum_{n=1}^{\infty} \frac{-b_n}{2i} e^{-in\pi x}}_{\text{call this term (B)}}$$

We can use a trick to re-write term (A) & term (B)

$$\text{(A)}: \sum_{n=1}^{\infty} \frac{a_n}{2} e^{-in\pi x/L} \quad \leftarrow \text{Let } \boxed{m \equiv -n}$$

$$= \sum_{m=-1}^{-\infty} \frac{a_{(-m)}}{2} e^{im\pi x/L}$$

But m is just a dummy varicble, and we can re-name it if we like. So let's just convert m back to n: $(m \to n)$

$$\text{(A)}: \sum_{n=-1}^{-\infty} \frac{a_{(-n)}}{2} e^{in\pi x/L}$$

Similiarly, term (B) can be writtn:

$$(B): \sum_{n=-1}^{-\infty} \frac{-b_{(-n)}}{2i} e^{in\pi x/L}$$

So we have

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \frac{a_n}{2} e^{in\pi x/L} + \sum_{n=-1}^{-\infty} \frac{a_{(-n)}}{2} e^{in\pi x/L}$$

$$+ \sum_{n=1}^{\infty} \frac{b_n}{2i} e^{in\pi x/L} + \sum_{n=-1}^{-\infty} \frac{-b_{(-n)}}{2i} e^{in\pi x/L}$$

Now substitute our conversion relations:

$$a_0 = 2c_0$$

$$a_n = c_n + c_{(-n)}$$

$$a_{(-n)} = c_{(-n)} + c_n = a_n$$

$$b_n = i\left(c_n - c_{(-n)}\right)$$

$$b_{(-n)} = -i\left(c_n - c_{(-n)}\right) = -b_n$$

$$f(x) = c_0 + \sum_{n=1}^{\infty} \left(\frac{c_n + c_{(-n)}}{2}\right) e^{in\pi x/L} + \sum_{n=1}^{-\infty} \left(\frac{c_n + c_{(-n)}}{2}\right) e^{in\pi x/L}$$

add $\updownarrow$ $\updownarrow$ cancel add $\updownarrow$ $\updownarrow$ cancel

$$+ \sum_{n=1}^{\infty} \left(\frac{c_n - c_{(-n)}}{2}\right) e^{in\pi x/L} + \sum_{n=-1}^{-\infty} \left(\frac{c_n - c_{(-n)}}{2}\right) e^{in\pi x/L}$$

$$f(x) = C_0 + \sum_{n=1}^{\infty} c_n e^{in\pi x/L} + \sum_{n=-1}^{-\infty} c_n e^{in\pi x/L}$$

$$\downarrow$$

$$C_0 = C_0 \underbrace{e^{i(\emptyset)\pi x/L}}_{1}$$

So we have a term $c_n e^{in\pi x/L}$ for all n from $-\infty$ to $+\infty$, including $n = \emptyset$.

$$\boxed{f(x) = \sum_{n=-\infty}^{\infty} c_n e^{in\pi x/L}}$$

If we want to use this series to represent a particular function $f(x)$, we'll need to calculate the coefficients $\{c_n\}$. How can we do that?

Answer: The basis functions $\{e^{in\pi x/L}\}$ are orthogonal:

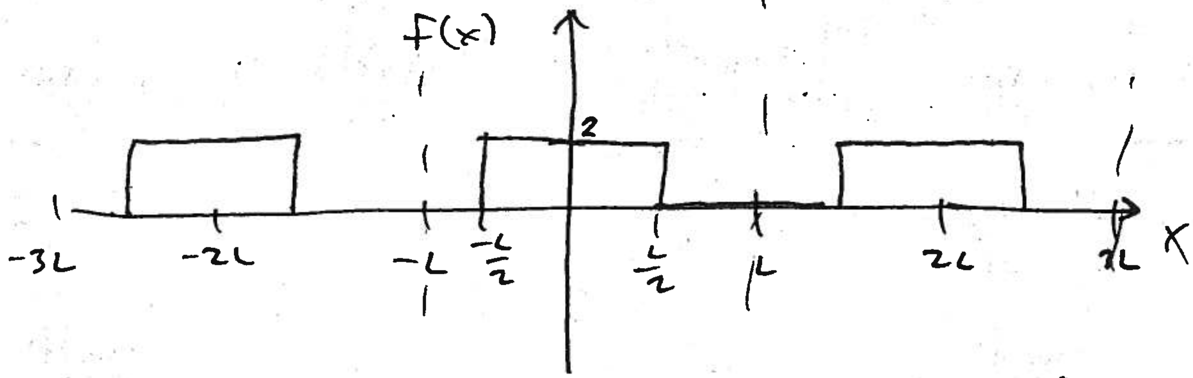
$$\int_{-L}^{L} \left(e^{in\pi x/L}\right)\left(e^{-im\pi x/L}\right) dx = \begin{cases} 2L, & n = m \\ \emptyset, & n \neq m \end{cases}$$

$$= 2L\,\delta_{nm}$$

Therefore Fourier's Trick will work:

$$\boxed{c_n = \frac{1}{2L} \int_{-L}^{L} f(x)\, e^{-in\pi x/L}\, dx}$$

Example Square wave:



We've already calculated the trig series for this function. (In class on Thursday). Result:

$$a_0 = 2$$

$$a_n = \frac{4}{n\pi}(-1)^{(n-1)/2} \quad , \text{ odd } n \text{ only (even } n \text{ is zero).}$$

$$b_n = \emptyset. \leftarrow \text{Sine terms are zero.}$$

The complex calculation is:

$$c_0 = \frac{1}{2L}\int_{-L}^{L} f(x)\,\underbrace{e^{-i(\emptyset)\pi x/L}}_{1}\,dx = \frac{1}{2L}\int_{-\frac{L}{2}}^{\frac{L}{2}} 2\,dx = \boxed{1}$$

$$c_n = \frac{1}{2L}\int_{-L}^{L} f(x) e^{-in\pi x/L}\,dx$$

$$= \frac{1}{2L}\int_{-\frac{L}{2}}^{\frac{L}{2}} 2e^{-in\pi x/L}\,dx$$

$$= \frac{1}{L}\left[\left(\frac{-L}{in\pi}\right) e^{-in\pi x/L}\right]\Bigg|_{-\frac{L}{2}}^{\frac{L}{2}}$$

$$c_n = \left(\frac{-1}{in\pi}\right)\left(e^{-in\pi/2} - e^{in\pi/2}\right)$$

$$= \frac{2}{n\pi}\underbrace{\left(\frac{e^{in\pi/2} - e^{-in\pi/2}}{2i}\right)}_{\sin\left(\frac{n\pi}{2}\right)}$$

$$\boxed{c_n = \frac{2}{n\pi}\sin\left(\frac{n\pi}{2}\right)} \leftarrow \text{for } n \neq 0. \ (n > 0 \ \& \ n < 0).$$

Note that this is not valid for $n = 0$ because we would divide by zero.

Is our result the same as our trig calculation?

Check it:

$$a_n = c_n + c_{(-n)} = \frac{2}{n\pi}\sin\left(\frac{n\pi}{2}\right) + \frac{2}{(-n\pi)}\sin\left(-\frac{n\pi}{2}\right)$$

$$= \frac{4}{n\pi}\underbrace{\sin\frac{n\pi}{2}}_{(-1)^{(n-1)/2} \text{ for odd } n}$$

$$= \underbrace{\frac{4}{n\pi}(-1)^{(n-1)/2}}_{} \quad \text{For odd } n$$

Yes, this is the same result as before.

Also $a_0 = 2c_0$ as expected.

Also check the b_n:

$$b_n = i\left(c_n - c_{(-n)}\right) = i\left(\frac{2}{n\pi}\sin\left(\frac{n\pi}{2}\right) - \frac{2}{(-n\pi)}\sin\left(\frac{-n\pi}{2}\right)\right)$$

$$\boxed{= 0}$$

Yes this is the same as before. There are no sine terms because the function is <u>even</u>.

<u>Summary</u>: Two ways to write this Fourier Series

$$\boxed{f(x) = 1 + \sum_{\substack{n=1,3,5\ldots \\ \text{(odd } n \text{ only)}}}^{\infty} (-1)^{(n-1)/2}\left(\frac{4}{n\pi}\right)\cos\left(\frac{n\pi x}{L}\right)}$$

AND

$$\cancel{f(x) = 1 + \sum_{\substack{n=-\infty \\ (\text{except } n=0)}}^{\infty} \left(\frac{4}{n\pi}\right) (-1)^{(n-1)/2}}$$

$$\boxed{f(x) = 1 + \sum_{n=-\infty}^{\infty} (-1)^{(n-1)/2}\left(\frac{2}{n\pi}\right) e^{in\pi x/L}}$$

the $n=0$ term

except $n=0$ is excluded