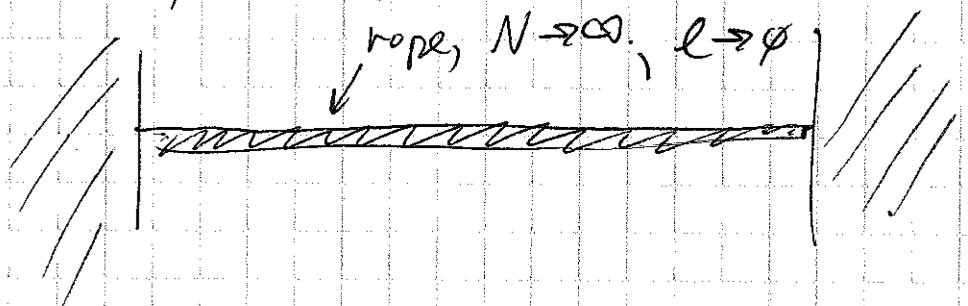
Continuous Systems – Wave Equation

We can model a continuous system, like a rope, as being a limit where the number of particles goes to infinity and l goes to zero.

rope, $N \to \infty$, $l \to 0$



For N masses, our equation of motion was

$$\ddot{y}_p + 2\omega_0^2 y_p - \omega_0^2(y_{p+1} + y_{p-1}) = 0$$

or

$$\ddot{y}_p = \omega_0^2(y_{p+1} - y_p) - \omega_0^2(y_p - y_{p-1})$$

Recall $\omega_0^2 = \dfrac{T}{ml}$

$$\therefore \quad m\ddot{y}_p = \frac{T}{l}(y_{p+1} - y_p) - \frac{T}{l}(y_p - y_{p-1})$$

Divide by l:

$$\frac{m}{l}\ddot{y}_p = \frac{T}{l}\left(\frac{y_{p+1} - y_p}{l}\right) - \frac{T}{l}\left(\frac{y_p - y_{p-1}}{l}\right)$$

As l goes to zero:

$$\lim_{l\to 0}\left(\frac{y_{p+1} - y_p}{l}\right) \Rightarrow \lim_{l\to 0}\left(\frac{y(x+l) - y(x)}{l}\right) = \left.\frac{dy}{dx}\right|_{x+\frac{l}{2}}$$

$$\lim_{l\to 0}\left(\frac{y_p - y_{p-1}}{l}\right) \Rightarrow \lim_{l\to 0}\left(\frac{y(x) - y(x-l)}{l}\right) = \left.\frac{dy}{dx}\right|_{x-\frac{l}{2}}$$

Also, let $\frac{m}{\ell} = \rho$ = mass density per unit length

Then

$$\rho \ddot{y}(x) = \frac{T}{\ell}\left[\left.\frac{dy}{dx}\right|_{x+\frac{\ell}{2}} - \left.\frac{dy}{dx}\right|_{x-\frac{\ell}{2}}\right]$$

$$\rho \frac{d^2y}{dt^2} = T \lim_{\ell \to 0} \underbrace{\left[\frac{\left.\frac{dy}{dx}\right|_{x+\frac{\ell}{2}} - \left.\frac{dy}{dx}\right|_{x-\frac{\ell}{2}}}{\ell}\right]}_{\frac{d^2y}{dx^2}}$$

$$\boxed{\frac{d^2y}{dx^2} = \frac{\rho}{T}\frac{d^2y}{dt^2}}$$ "Wave Equation"

This is the Eq. of Motion for a continuous system of masses. It is Newton's 2nd Law.

Solution: The normal modes we can get by allowing $N \to \infty$ in the N-mass system, while $\ell \to 0$ such that $(N+1)\ell = L$ = total length

For N particles,

$$y_{pn} = C_n \sin\left(\frac{pn\pi}{N+1}\right)$$ Now $p\ell = x$ = distance along rope

i.e. $y_n(x) = C_n \sin\left(\frac{\overbrace{(pl)}^{x}\pi n}{\underbrace{l(N+1)}}\right) = \boxed{C_n \sin\left(\frac{n\pi x}{L}\right)}$

L = total length $\quad n = 1, 2, \ldots \infty$

Amplitude relationship for normal modes of of continuous system

The frequencies are

$$\omega_n = 2\,\omega_0 \sin\left(\frac{n\pi}{2(N+1)}\right)$$

$$\omega_n = 2\omega_0 \sin\left(\frac{n\pi l}{2\underbrace{(N+1)l}_{L}}\right) = 2\omega_0 \sin\left(\frac{n\pi l}{2L}\right)$$

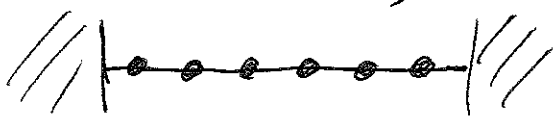
In the limit when $l \to \emptyset$, $\sin\left(\frac{n\pi l}{2L}\right) \to \frac{n\pi l}{2L}$

$$\omega_n = 2\,\omega_0 \left(\frac{n\pi l}{2L}\right)$$

$$\omega_0 = \sqrt{\frac{T}{ml}} = \sqrt{\frac{T/l^2}{m/l}} = \frac{1}{l}\sqrt{\frac{T}{\rho}}, \quad \rho = \frac{m}{l}$$

$$\boxed{\omega_n = \sqrt{\frac{T}{\rho}}\;\frac{n\pi}{L}, \quad n = 1, 2, 3, \ldots \infty}$$

Comparison of the loaded string with the continuous string.

	Loaded String	Continuous String
Diagram:		
Eq. of Motion:	$\ddot{y}_p + 2\omega_0^2 y_p - \omega_0^2 (y_{p+1} + y_{p-1}) = 0$	$\frac{\partial^2 y}{\partial x^2} = \frac{\rho}{T}\frac{\partial^2 y}{\partial t^2}$
Nature:	Discrete (N masses)	Continuous (infinite # of masses)
Number of Normal Modes:	N	infinite
Normal Mode Amplitude Relationship:	$y_{pn} = C_n \sin\left(\frac{pn\pi}{N+1}\right)$	$y_n(x) = C_n \sin\left(\frac{n\pi x}{L}\right)$
Normal Frequencies:	$\omega_n = 2\omega_0 \sin\left(\frac{n\pi}{2(N+1)}\right)$	$\omega_n = \sqrt{\frac{T}{\rho}}\ \frac{n\pi}{L}$
General Solution:	$y_p = \sum_{n=1}^{\infty} C_n \sin\left(\frac{pn\pi}{N+1}\right) e^{i\omega_n t}$	$y_n(x) = \sum_{n=1}^{\infty} C_n \sin\left(\frac{n\pi x}{L}\right) e^{i\omega_n t}$
	initial conditions	initial conditions.

How to determine the $\{C_n\}$: Initial Conditions and Fourier's Trick.

For any set of initial conditions, we must find the coefficients $\{C_n\}$ which satisfy those initial conditions. Once the $\{C_n\}$ have been determined, then the time development for all time is given by the general solution.

In general c_n will be complex, so it has real and imaginary parts:

$$\underbrace{c_n}_{} = \underset{\substack{\uparrow \\ \text{real} \\ \text{part}}}{a_n} + i\underset{\substack{\uparrow \\ \text{imag.} \\ \text{part}}}{b_n}$$

If there are N normal modes, then there are $2N$ free parameters to determine: the real & imag. part of

For the loaded string, at $t=\emptyset$, the general solution is

$$y_p = \sum_{n=1}^{N} c_n \sin\left(\frac{pn\pi}{N+1}\right) \quad \leftarrow \text{position at } t=\emptyset$$

~~The velocity is~~ ↑ But only the real part matters!

$$\cancel{\dot{y}_p = \sum_{n=1}^{N} \underbrace{(i\omega_n) c_n}_{(-b_n + i a_n)\omega_n} \sin\left(\frac{pn\pi}{N+1}\right) \leftarrow \text{velocity at } t=\emptyset}$$

$$\boxed{\mathrm{Re}[y_p] = \sum_{n=1}^{N} a_n \sin\left(\frac{pn\pi}{N+1}\right) \leftarrow \text{real position at } t=\emptyset}$$

The initial velocity is

$$\dot{y}_p = \sum_{n=1}^{N} \underbrace{(i\omega_n) c_n}_{} \sin\left(\frac{pn\pi}{N+1}\right)$$

$$i\omega_n c_n = i\omega_n(a + ib_n) = (-b_n + i a_n)\omega_n$$

But only the real part matters!

$$\mathrm{Re}\left[\dot{Y}_p\right] = \sum_{n=1}^{N} \left(-b_n \omega_n\right) \sin\left(\frac{pn\pi}{N+1}\right) \leftarrow \text{real velocity at } t=\emptyset.$$

$\therefore$ The initial position determines the $\{a_n\}$ (the real part of the $\{c_n\}$), while the initial velocity determines the $\{b_n\}$ (the imaginary part of the $\{c_n\}$)

Simplifying Assumption:

For the time being, let's assume that the system is released from rest at $t=\emptyset$, so the initial velocity is zero.

$$\boxed{\text{Then } b_n = \emptyset \text{ for all } n.}$$
$\leftarrow$ special case where the initial velocity is zero.

In this case, our job is to determine the $\{a_n\}$ coefficients, given the initial positions.

Note: This will also be true for the continuous string (that an initial velocity of zero eliminates the imaginary part of the $\{c_n\}$.)

Initial Conditions and Fourier's Trick

"Fourier's Trick" (terminology from David Griffith's Quantum Mechanics book), is a way to determine the expansion coefficients $\{a_n\}$ while doing very little work. It allows you to get the answer right away, given the initial conditions. It relies on the following observation:

> The eigenvectors which describe the normal modes of the loaded string are orthogonal to each other.

Recall: $q_n = \left(\underset{\substack{\uparrow \\ \text{mass \#1}}}{\sin\left(\frac{n\pi}{N+1}\right)}, \underset{\substack{\uparrow \\ \text{mass \#2}}}{\sin\left(\frac{2n\pi}{N+1}\right)}, \sin\left(\frac{3n\pi}{N+1}\right), \ldots, \underset{\substack{\uparrow \\ \text{mass \#N}}}{\sin\left(\frac{Nn\pi}{N+1}\right)} \right)$

Illustration: Consider $N=2$.

$$q_1 = \left(\underset{\substack{\uparrow \\ p=1}}{\sin\frac{\pi}{3}}, \underset{\substack{\uparrow \\ p=2}}{\sin\frac{2\pi}{3}}\right) = (0.866, 0.866) \leftarrow \text{symmetric mode}$$

$$q_2 = \left(\underset{\substack{p=1 \\ \downarrow}}{\sin\frac{2\pi}{3}}, \underset{\substack{p=2 \\ \downarrow}}{\sin\frac{4\pi}{3}}\right) = (0.866, -0.866)$$

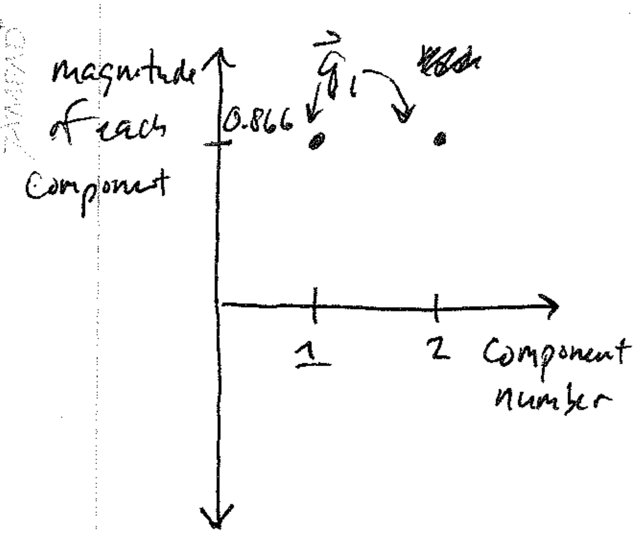
To show that they are orthogonal, take the dot product:

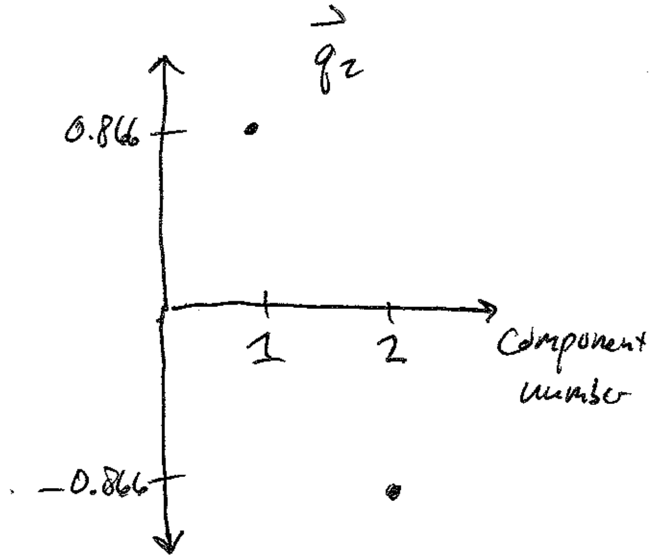
$$\vec{q}_1 \cdot \vec{q}_2 = \left((0.866)(0.866) + (0.866)(-0.866) \right) \boxed{= \phi}$$

Orthogonal.

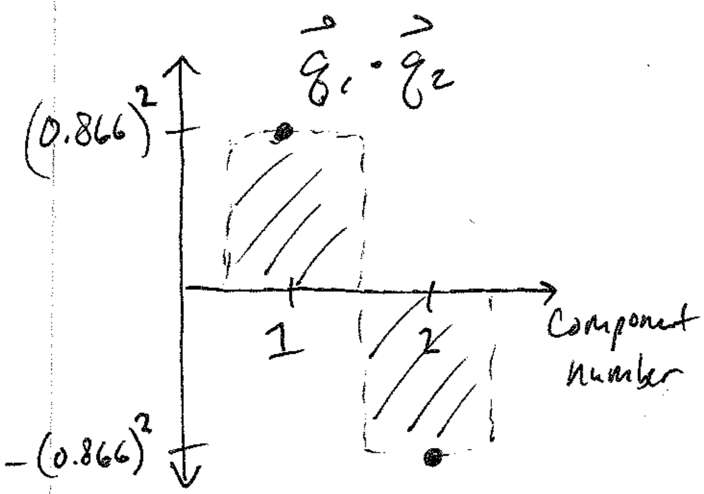
When the dot product is zero, the vectors are orthogonal.

Let's draw this:





To visualize the dot product, multiply the two graphs component-by-component and add them all up:



Visualize the sum by adding the areas under the multiplied components.

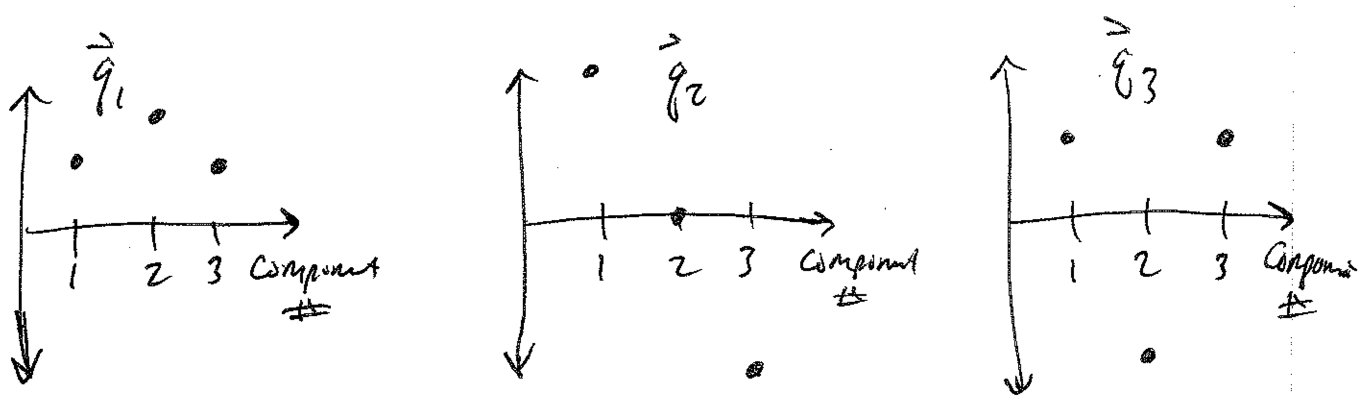
These add to zero, and the area is zero (because the 2nd component is negative).

Illustration: Consider $N=3$

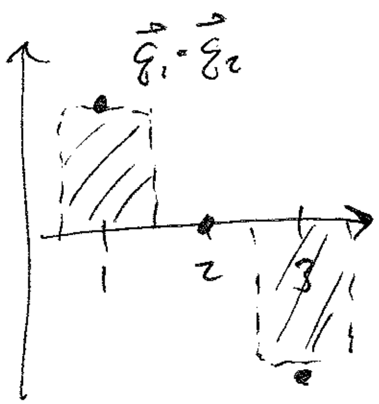
$$\vec{q}_1 = \left(\sin\frac{\pi}{4},\ \sin\frac{2\pi}{4},\ \sin\frac{3\pi}{4}\right) = (0.707,\ 1,\ 0.707)$$

$$\vec{q}_2 = \left(\sin\frac{2\pi}{4},\ \sin\frac{4\pi}{4},\ \sin\frac{6\pi}{4}\right) = (1,\ 0,\ -1)$$

$$\vec{q}_3 = \left(\sin\frac{3\pi}{4},\ \sin\frac{6\pi}{4},\ \sin\frac{9\pi}{4}\right) = (0.707,\ -1,\ 0.707)$$



Try $\vec{q}_1 \cdot \vec{q}_2$:

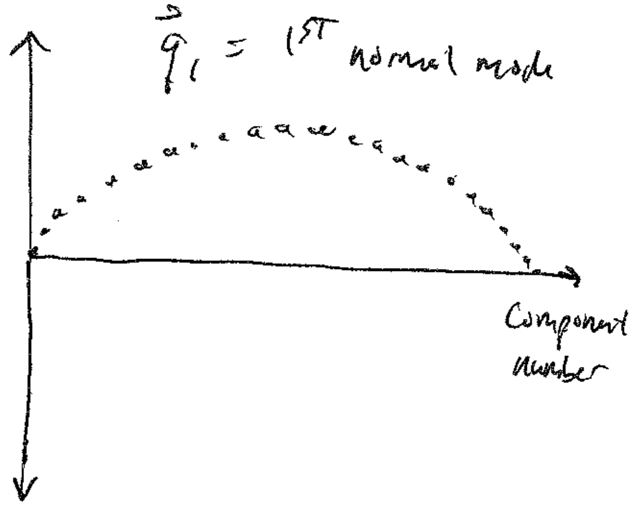


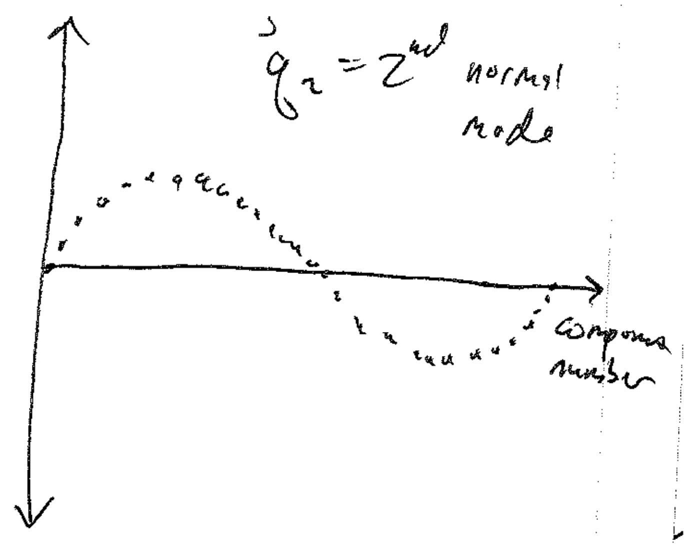
← Areas sum to zero, so $\vec{q}_1 \cdot \vec{q}_2 = 0$.

The same thing happens for $\vec{q}_2 \cdot \vec{q}_3 = 0$,
$\vec{q}_1 \cdot \vec{q}_3 = 0$.

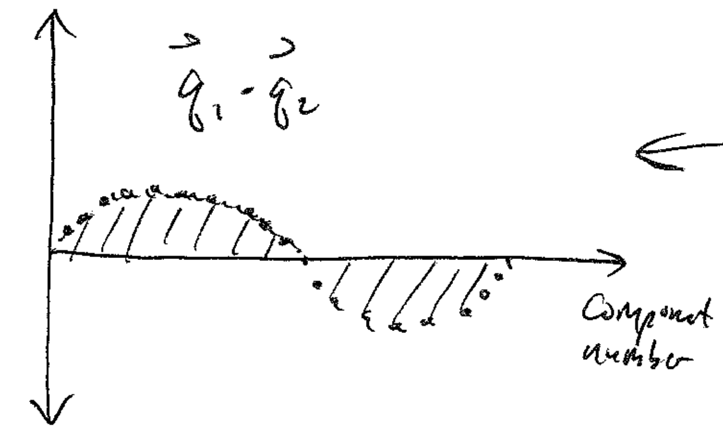
Illustration: Consider the loaded string with a large number of masses: $N =$ large.

Let's draw the 1^{st} and 2^{nd} normal modes:





What does the dot product look like?

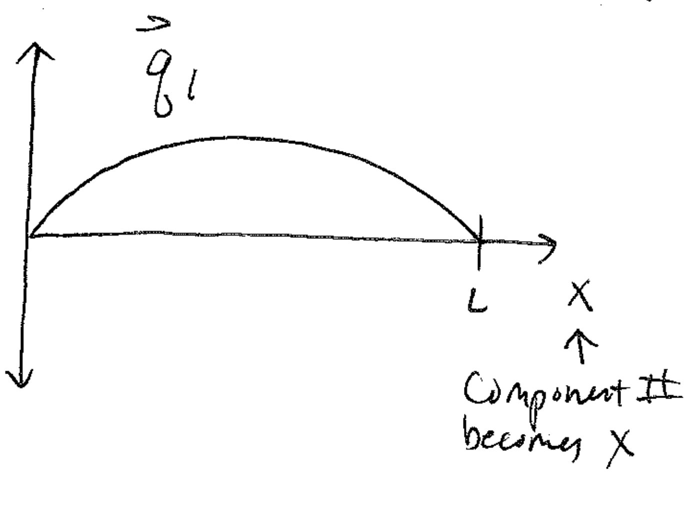


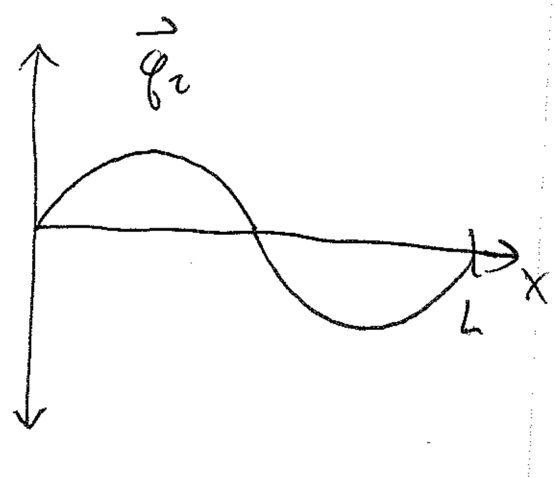
← Area = ϕ, so $\vec{q}_1 \cdot \vec{q}_2 = \phi$.

Illustration: The continuous case: $N \to \infty$.

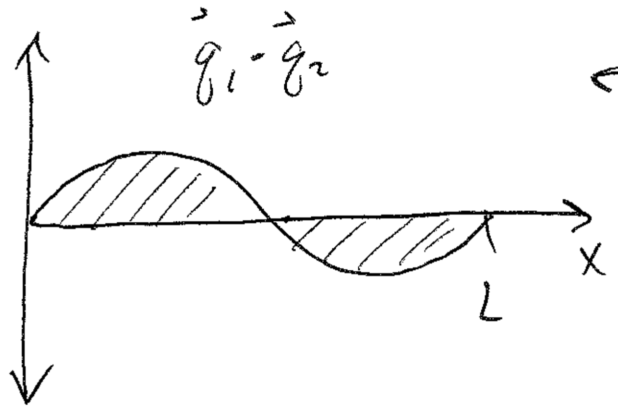
$\vec{q}_1$ = normal mode 1 = $\sin\left(\frac{\pi x}{L}\right)$

$\vec{q}_2$ = normal mode 2 = $\sin\left(\frac{2\pi x}{L}\right)$





The dot product looks like



← Area = ∅, so they are orthogonal.

Mathematical Statements:

Eigenvectors of the loaded string are orthogonal:

$$\vec{q}_n \cdot \vec{q}_m = \left(\sin\left(\frac{n\pi}{N+1}\right), \sin\left(\frac{2n\pi}{N+1}\right), \ldots\right) \cdot \left(\sin\left(\frac{m\pi}{N+1}\right), \sin\left(\frac{2m\pi}{N+1}\right) \ldots\right)$$

$$= \sum_{p=1}^{N} \sin\left(\frac{pn\pi}{N+1}\right) \sin\left(\frac{pm\pi}{N+1}\right) = \begin{cases} \left(\frac{N+1}{2}\right), & \text{if } n=m \\ \emptyset, & \text{if } n \neq m \end{cases}$$

In this last step I'm invoking a known trig identity (I am not proving it here.)

To simplify the notation, let

$$\delta_{nm} \equiv \text{"Kronecker Delta"} \equiv \begin{cases} 1, & n=m \\ \emptyset, & n \neq m \end{cases}$$

Then we can say that

$$\boxed{\vec{q}_n \cdot \vec{q}_m = \sum_{p=1}^{N} \sin\left(\frac{pn\pi}{N+1}\right) \sin\left(\frac{pm\pi}{N+1}\right) = \left(\frac{N+1}{2}\right) \delta_{nm}}$$

or even

$$\boxed{\vec{q}_n \cdot \vec{q}_m = \left(\frac{N+1}{2}\right) \delta_{nm}}$$

This says that $\vec{q}_n$ and $\vec{q}_m$ are orthogonal: their dot product is zero if they are different eigenvectors.

For the continuous case, the mathematical statement is:

eigenvector $n = \sin\left(\frac{n\pi x}{L}\right)$

a continuous vector, a function of a continuous variable x.

Statement of orthogonality:

$$\boxed{\underbrace{\int_0^L \sin\left(\frac{n\pi x}{L}\right) \sin\left(\frac{m\pi x}{L}\right) dx}_{\text{a continuous dot product}} = \frac{L}{2} \delta_{nm}}$$

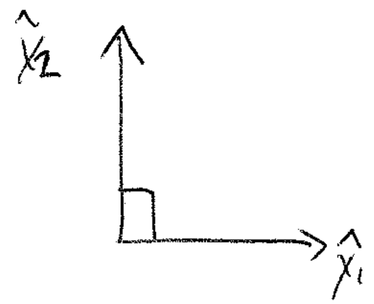
you will prove this on the next homework.

This says that the continuous vectors $\sin\left(\frac{n\pi x}{L}\right)$ and $\sin\left(\frac{m\pi x}{L}\right)$ are orthogonal: their dot product is zero if $n \neq m$.

More on orthogonal vectors & functions & the Kronecker Delta.

Consider a 2 dimensional vector $\vec{y} = (y_1, y_2)$

Suppose that $\hat{y}_1$ and $\hat{y}_2$ are orthogonal unit vectors:



Then

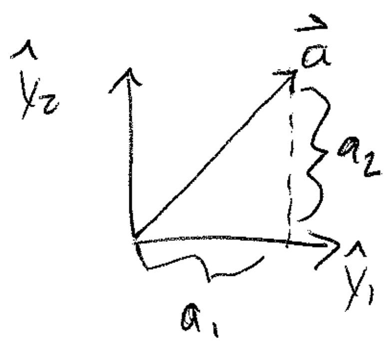
$$\hat{y}_1 \cdot \hat{y}_2 = 0$$
$$\hat{y}_1 \cdot \hat{y}_1 = 1$$
$$\hat{y}_2 \cdot \hat{y}_2 = 1$$

Summarizing: $\boxed{\hat{y}_i \cdot \hat{y}_j = \delta_{ij}}$ for i & $j = 1, 2$

Kronecker Delta: $\delta_{ij} = \begin{cases} 0, & i \neq j \\ 1, & i = j \end{cases}$

The Kronecker Delta describes the orthogonality of these unit vectors.

We can write an arbitrary vector $\vec{a}$ as a linear combination of $\hat{y}_1$ & $\hat{y}_2$:



$$\vec{a} = a_1 \hat{y}_1 + a_2 \hat{y}_2 = \sum_{i=1}^{2} a_i \hat{y}_i$$

If we take the dot product of $\vec{a}$ with $\hat{y}_1$, we "pick out" the component of $\vec{a}$ in the direction of $\hat{y}_1$:

$$\vec{a}\cdot\hat{y}_1 = (a_1\hat{y}_1 + a_2\hat{y}_2)\cdot\hat{y}_1$$

$$= a_1 \underbrace{\hat{y}_1\cdot\hat{y}_1}_{1} + \cancel{a_2\hat{y}_2\cdot\hat{y}_1} \rightarrow \emptyset,$$ because $\hat{y}_1$ & $\hat{y}_2$ are orthogonal.

$$= a_1$$

$$\therefore \boxed{a_1 = \vec{a}\cdot\hat{y}_1}$$

Similarly, $\boxed{a_2 = \vec{a}\cdot\hat{y}_2}$

In general, $\boxed{a_i = \vec{a}\cdot\hat{y}_i}$

A better notation is to use the Kronecker Delta:

$$\cancel{\vec{a}\cdot\hat{y}_1 = \left(\sum_{i=1}^{2} a_i\hat{y}_i\right)\cdot\hat{y}_1}$$

$$\vec{a}\cdot\hat{y}_i = \left(\sum_j a_j\hat{y}_j\right)\cdot\hat{y}_i = \sum_j a_j \underbrace{(\hat{y}_j\cdot\hat{y}_i)}_{\delta_{ij}} = \sum_j a_j\delta_{ij}$$

Kronecker Delta kills all terms in the sum except $j = i$

$$= a_i$$

$$\therefore \boxed{a_i = \vec{a}\cdot\hat{y}_i}$$

Fourier's Trick & Initial conditions

Finally, we can show how to incorporate the initial conditions into the general solution. We will use Fourier's Trick, which relies on the orthogonality of the eigenvectors.

Suppose our system is a loaded string with N masses. ~~The Also Suppose the~~ Then the general solution is

$$\vec{y}(t) = \sum_{n=1}^{N} a_n \vec{q}_n e^{i\omega_n t}$$

a_n are real, as long as the initial velocities are zero.

Now suppose I have a set of initial conditions:

$$\left.\begin{aligned} y_1(t=0) &= Y_1 \\ y_2(t=0) &= Y_2 \\ y_3(t=0) &= Y_3 \\ &\vdots \end{aligned}\right\} \text{Initial positions of the } N \text{ masses.}$$

Let's put them into a vector

$$\vec{Y}_0 = \vec{y}(t=0) = (y_1(t=0),\ y_2(t=0),\ \ldots\ldots)$$

We know that the general solution must give $\vec{Y}_0$ at $t=0$.

$$\vec{y}_0 = \sum_{n=1}^{N} a_n \vec{q}_n \underbrace{e^{i\omega_n(\phi)}}_{1} = \sum_{n=1}^{N} a_n \vec{q}_n$$

Now, how do we determine the $\{a_n\}$ coefficients? Simply by taking the dot product of $\vec{y}_0$ with each eigenvector. For example, consider $\vec{y}_0 \cdot \vec{q}_1$:

$$\vec{y}_0 \cdot \vec{q}_1 = \left(\sum_{n=1}^{N} a_n \vec{q}_n\right) \cdot \vec{q}_1$$

$$= \sum_{n=1}^{N} a_n \underbrace{(\vec{q}_n \cdot \vec{q}_1)}_{\delta_{n1}} = a_1 \vec{q}_1 \cdot \vec{q}_1 = a_1 |\vec{q}_1|^2$$

kills all terms in the sum except $n=1$

$$\therefore \boxed{a_1 = \frac{\vec{y}_0 \cdot \vec{q}_1}{|\vec{q}_1|^2}}$$

This is how to calculate a_1.

Similarly, we can get a_2 by calculating

$$\boxed{a_2 = \frac{\vec{y}_0 \cdot \vec{q}_2}{|\vec{q}_2|^2}}$$

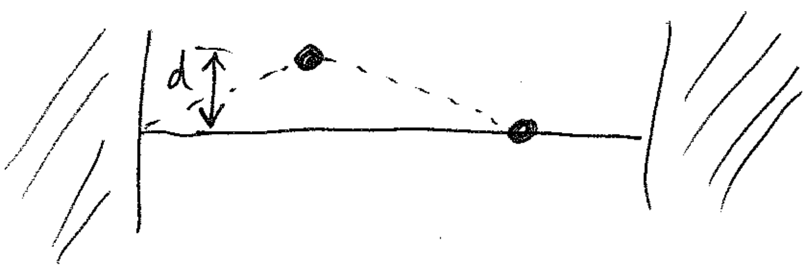
In general,

$$\boxed{a_n = \frac{\vec{y}_0 \cdot \vec{q}_n}{|\vec{q}_n|^2}}$$

Fourier's Trick.

Fourier's Trick is the best way to calculate the expansion coefficients $\{a_n\}$.

<u>Example</u>: Suppose $N=2$, and suppose the initial condition is $y_1(t=\emptyset) = d$
$y_2(t=\emptyset) = \emptyset$



The initial condition vector is

$$\vec{y}_0 = (d, 0)$$

The eigenvectors are $\vec{q}_1 = (1,1)$ and $\vec{q}_2 = (1,-1)$

$$\therefore \quad a_1 = \frac{\vec{y}_0 \cdot \vec{q}_1}{|\vec{q}_1|^2} = \frac{(d,0)\cdot(1,1)}{(1^2+1^2)} = \boxed{\frac{d}{2}}$$

$$a_2 = \frac{\vec{y}_0 \cdot \vec{q}_2}{|\vec{q}_2|^2} = \frac{(d,0)\cdot(1,-1)}{1^2+(-1)^2} = \boxed{\frac{d}{2}}$$

$$\therefore \quad \vec{y}(t) = \sum_{n=1}^{2} a_n \vec{q}_n e^{i\omega_n t} = \frac{d}{2}(1,1)e^{i\omega_1 t} + \frac{d}{2}(1,-1)e^{i\omega_2 t}$$

or

$$y_1(t) = \frac{d}{2}e^{i\omega_1 t} + \frac{d}{2}e^{i\omega_2 t}$$
$$y_2(t) = \frac{d}{2}e^{i\omega_1 t} - \frac{d}{2}e^{i\omega_2 t}$$

Fourier's Trick relies upon the following mathematical identity:

$$\int_0^L \sin\left(\frac{m\pi x}{L}\right)\sin\left(\frac{n\pi x}{L}\right)dx = \begin{cases} 0, & m \neq n \\ \frac{L}{2}, & m = n \end{cases}$$

We like to write this more compactly:

Define $\boxed{\delta_{mn} \equiv \begin{cases} 0, & m \neq n \\ 1, & m = n \end{cases}}$ "Kronecker Delta"

Thus $\delta_{11} = 1$, $\delta_{12} = \emptyset$, $\delta_{13} = \emptyset$, $\delta_{22} = 1$, ect

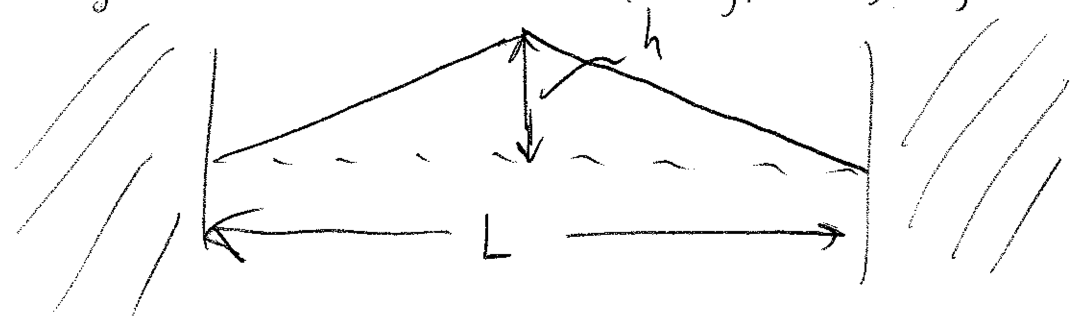
Using the Kronecker Delta we can say

$$\boxed{\int_0^L \sin\left(\frac{m\pi x}{L}\right)\sin\left(\frac{n\pi x}{L}\right)dx = \frac{L}{2}\delta_{mn}}$$

or

$$\boxed{\frac{2}{L}\int_0^L \sin\left(\frac{m\pi x}{L}\right)\sin\left(\frac{n\pi x}{L}\right)dx = \delta_{mn}}$$

Let's go back to our triangular string:



This is the shape at $t=\phi$. The functional form is

$$y(x,t=\phi) = \begin{cases} \left(\frac{2h}{L}\right)x & , \quad 0 \le x \le \frac{L}{2} \\ \left(\frac{2h}{L}\right)(L-x) & , \quad \frac{L}{2} \le x \le L \end{cases}$$

We want to describe this simple function in a much more complicated way: as an infinite sum of normal modes:

$$y(x,t=\phi) = \sum_{n=1}^{\infty} a_n \sin\left(\frac{n\pi x}{L}\right)$$

The question is: what are the $\{a_n\}$?

Fourier's Trick tells us that any particular coefficient, ~~can be calculated~~ for example, the m^{th} coefficient (a_m), can be calculated by evaluating this integral:

$$\boxed{a_m = \frac{2}{L}\int_0^L \sin\left(\frac{m\pi x}{L}\right) y(x)\, dx}$$

For our function $y(x)$, this integral is

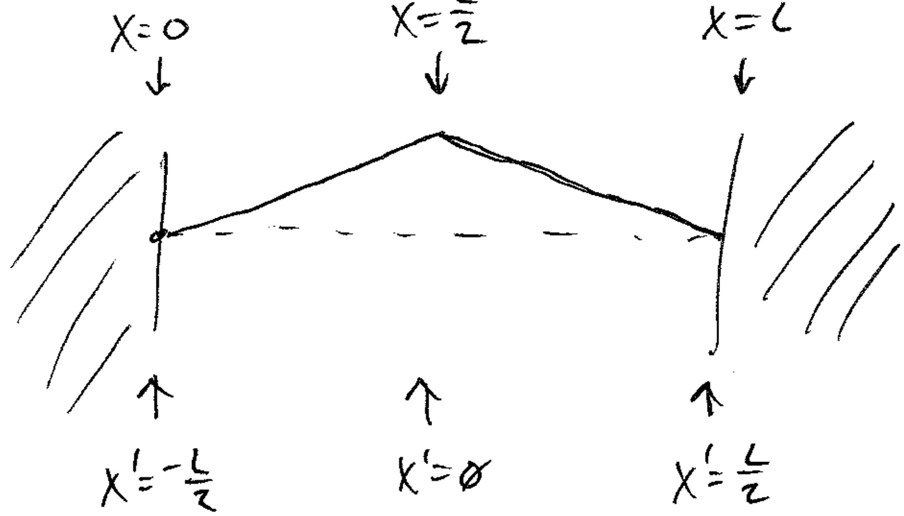
$$a_m = \frac{2}{L}\int_0^{L/2} \sin\left(\frac{m\pi x}{L}\right)\left(\frac{2hx}{L}\right)dx + \frac{2}{L}\int_{\frac{L}{2}}^{L} \sin\left(\frac{m\pi x}{L}\right)\left(\frac{2h(L-x)}{L}\right)dx$$

It turns out that the easiest way to evaluate this integral is to move our coordinate system -

$$\text{Let} \quad x' \equiv x - \frac{L}{2}$$

$$\text{so that} \quad x = x' + \frac{L}{2}$$

This means that $x' = \phi$ is the center of the string



In terms of x', our string position at $t=\phi$ is

$$\boxed{y(x', t=\phi) = \begin{cases} \left(\frac{2h}{L}\right)\left(x' + \frac{L}{2}\right), & -\frac{L}{2} \le x' \le \phi \\ \left(\frac{2h}{L}\right)\left(-x' + \frac{L}{2}\right), & 0 \le x' \le \frac{L}{2} \end{cases}}$$

Note that y is an even function of x'.

Also, we have the following math theorem:

$$\boxed{\begin{aligned} &\text{If} \quad x = x' + \frac{L}{2}, \\ &\text{Then} \quad \sin\left(\frac{m\pi x}{L}\right) = \begin{cases} (-1)^{(m-1)/2} \cos\left(\frac{m\pi x'}{L}\right), & \text{for } m = \text{odd} \\ (-1)^{m/2} \sin\left(\frac{m\pi x'}{L}\right), & \text{for } m = \text{even} \end{cases} \end{aligned}}$$

Now our integral has 2 cases:

$$a_m = \frac{2}{L}\int_{-\frac{L}{2}}^{\frac{L}{2}} y(x')\,(-1)^{(m-1)/2} \cos\left(\frac{m\pi x'}{L}\right) dx' \quad \text{for } m=\text{odd}$$

AND

$$a_m = \frac{2}{L}\int_{-\frac{L}{2}}^{\frac{L}{2}} \underbrace{y(x')\,(-1)^{(m)/2} \sin\left(\frac{m\pi x'}{L}\right)} dx' \quad \text{for } m=\text{even.}$$

This integrand is an ~~even~~ <u>odd</u> function of x', because $y(x')$ is even, and $\sin\left(\frac{m\pi x'}{L}\right)$ is odd.

Therefore the integral is <u>zero</u> because we integrate from $-\frac{L}{2}$ to $\frac{L}{2}$.

So we only need to evaluate the case for $m=$ odd:

$$a_m = \frac{2}{L}\int_{-\frac{L}{2}}^{\frac{L}{2}} \underbrace{y(x')\,(-1)^{(m-1)/2} \cos\left(\frac{m\pi x'}{L}\right)} dx' \quad , \; m=\text{odd.}$$

This integrand is <u>even</u> because $y(x')$ and $\cos\left(\frac{m\pi x'}{L}\right)$ are both even functions of x'. Since we integrate from $-\frac{L}{2}$ to $\frac{L}{2}$, we can just integrate from zero to $\frac{L}{2}$ and multiply by 2:

$$a_m = (2)\frac{2}{L}\int_{0}^{\frac{L}{2}} y(x')\,(-1)^{(m-1)/2} \cos\left(\frac{m\pi x'}{L}\right) dx'$$

$$a_m = (2)\left(\frac{2}{L}\right)(-1)^{(m-1)/2}\left(\frac{2h}{L}\right)\int_0^{\frac{L}{2}}\left(-x' + \frac{L}{2}\right)\cos\left(\frac{m\pi x'}{L}\right)dx'$$

$$= \left(\frac{8h}{L^2}\right)(-1)^{(m-1)/2}\left[\left(-\left(\frac{L}{m\pi}\right)^2\cos\left(\frac{m\pi x'}{L}\right) - \frac{x'L}{m\pi}\sin\left(\frac{m\pi x'}{L}\right) + \left(\frac{L}{2}\right)\left(\frac{L}{m\pi}\right)\sin\frac{m\pi x'}{L}\right)\Bigg|_0^{\frac{L}{2}}\right]$$

zero for m=odd

cancel

$$= \left(\frac{8h}{L^2}\right)(-1)^{(m-1)/2}\left[-\left(\frac{L}{m\pi}\right)^2 \cos\left(\frac{m\pi}{2}\right) - \frac{L^2}{2m\pi}\sin\left(\frac{m\pi}{2}\right) + \left(\frac{L^2}{2m\pi}\right)\sin\left(\frac{m\pi}{2}\right) - \left(-\left(\frac{L}{m\pi}\right)^2\right)\right]$$

$$\boxed{a_m = \frac{8h}{(m\pi)^2}(-1)^{(m-1)/2}, \quad m = \text{odd}} \quad \text{and} \quad \boxed{a_m = \emptyset \text{ for } m\text{ even}}$$

Therefore

$$a_1 = \frac{8h}{\pi^2}$$

$$a_2 = \emptyset$$

$$a_3 = \frac{-8h}{9\pi^2}$$

$$a_4 = \emptyset$$

$$a_5 = \frac{8h}{25\pi^2}$$

$$\vdots$$

Or we can write

$$\cancel{}\ y(x,t=\phi) = \frac{8h}{\pi^2}\sin\left(\frac{\pi x}{L}\right) - \frac{8h}{9\pi^2}\sin\left(\frac{3\pi x}{L}\right) + \frac{8h}{25\pi^2}\sin\left(\frac{5\pi x}{L}\right)$$
$$+ \ldots$$

Why did we do this?

Recall our motivation: The general solution to the wave equation is a sum over normal modes:

$$\frac{\partial^2 y}{\partial x^2} = \frac{\rho}{T}\frac{\partial^2 y}{\partial t^2} \implies y(x,t) = \sum_{n=1}^{\infty} c_n \sin\left(\frac{n\pi x}{L}\right) e^{i\omega_n t}$$

where c_n = real and imaginary
$\equiv a_n + i b_n$

Given the initial condition:

$$y(x,t=\phi) = \text{a triangle} = \begin{cases} \left(\frac{2h}{L}\right)x & , 0 \le x \le \frac{L}{2} \\ \frac{2h}{L}(L-x) & , \frac{L}{2} \le x \le L \end{cases}$$

We found the a_n:

$$a_n = \begin{cases} \left(\frac{8h}{(n\pi)^2}\right)(-1)^{(n-1)/2} & \text{for } n = \text{odd} \\ \phi & \text{for } n = \text{even} \end{cases}$$

What about the imaginary part, $\{b_n\}$?
It is determined by the initial velocity:

$$\dot{y}(x, t=0) = \sum_{n=1}^{\infty} -\omega_n b_n \sin\left(\frac{n\pi x}{L}\right)$$

If we release the string from rest, then we must have $\dot{y}(x, t=0) = 0 \Rightarrow \boxed{b_n = 0 \text{ for all } n}$

So our final, time-dependent solution is

$$\boxed{y(x,t) = \sum_{\substack{n=1 \\ (\text{only odd}(n))}}^{\infty} \frac{8h}{(n\pi)^2} (-1)^{\left(\frac{n-1}{2}\right)} \sin\left(\frac{n\pi x}{L}\right) e^{i\omega_n t}} \quad \boxed{\text{odd } n \text{ only!}}$$

where $\omega_n = \sqrt{\frac{T}{\rho}} \frac{n\pi}{L}$, $n = 1, 2, 3, \ldots$

We write the initial condition function $y(x, t=0)$ as a sum over normal modes because then the time development is extremely simple: each normal mode goes forward in time with its own harmonic factor $\left(e^{i\omega_n t}\right)$.