

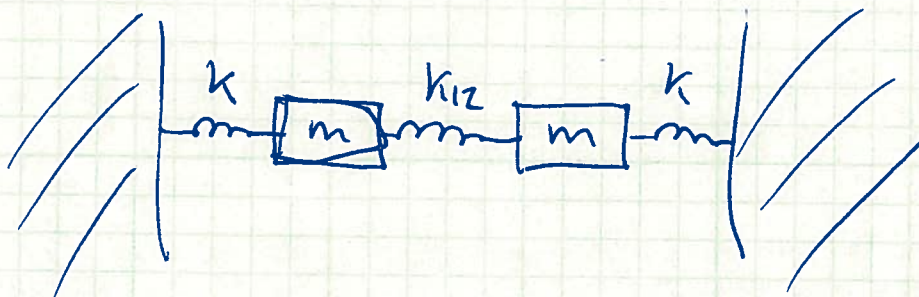
What is a normal mode?

A normal mode is a special type of motion for a multi-particle harmonic system. Its defining characteristic is that the time evolution is very simple: every particle oscillates at the same frequency.

For a system to satisfy this condition, the amplitudes <sup>& phases</sup> of the various particles must be related to each other. So to find a normal mode, we must do 2 things:

- ① determine which frequencies are normal-mode frequencies.
- ② determine the relationship between the amplitudes of motion for ~~that~~ <sup>each</sup> normal mode.

Example Coupled Mechanical Oscillator  
(2 particle system)



We solved the equation of motion and found 2 normal frequencies:

$$\omega_S = \sqrt{\frac{k}{m}} = \text{"small frequency"}$$

$$\omega_L = \sqrt{\frac{k+2k_{12}}{m}} = \text{"large frequency"}$$

We also found the amplitude relationship:

For  $\omega_S$ :

$$\begin{aligned}
 X_1 &= B_S e^{i\omega_S t} \\
 &\quad \updownarrow \text{same amplitude and phase} \\
 X_2 &= B_S e^{i\omega_S t}
 \end{aligned}$$

For  $\omega_L$ :

$$\begin{aligned}
 X_1 &= B_L e^{i\omega_L t} \\
 &\quad \updownarrow \text{same amplitude, opposite phase} \\
 X_2 &= -B_L e^{i\omega_L t}
 \end{aligned}$$

↑ opposite phase

(Recall that  $B_S$  &  $B_L$  are complex, so the phase at  $t=0$  is absorbed into  $B_S$  &  $B_L$ .)

Notice that our 2-particle system has 2 normal modes. In general, an N-particle system will have N-normal modes.



We determined  $B_L$  &  $B_S$  for a particular set of initial conditions:

$$\begin{aligned}
 x_1(t=0) &= a & , & \quad \dot{x}_1(t=0) = 0 \\
 x_2(t=0) &= 0 & , & \quad \dot{x}_2(t=0) = 0
 \end{aligned}$$

Complete solution  $\phi$  (Real part)

$$x_1(t) = \frac{a}{2} \cos(\omega_S t) + \frac{a}{2} \cos(\omega_L t)$$

$$x_2(t) = \frac{a}{2} \cos(\omega_S t) - \frac{a}{2} \cos(\omega_L t)$$

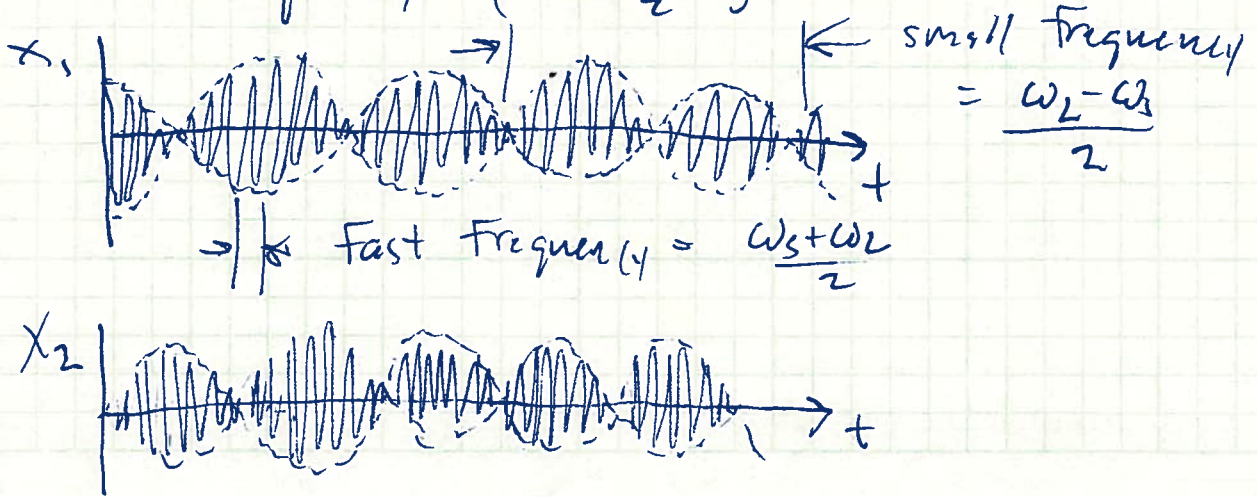
Question: What does this solution look like?

Answer: Re-write it using a (dreaded) trig identity

$$x_1(t) = a \cos\left(\frac{(\omega_L - \omega_S)}{2} t\right) \cos\left(\frac{(\omega_L + \omega_S)}{2} t\right)$$

$$x_2(t) = a \sin\left(\frac{(\omega_L - \omega_S)}{2} t\right) \sin\left(\frac{(\omega_L + \omega_S)}{2} t\right)$$

So we have two harmonic functions multiplied together. There is a "fast oscillation" whose frequency is the average of  $\omega_L$  &  $\omega_S$ . But the amplitude goes up and down with a slow frequency  $\left(\frac{(\omega_L - \omega_S)}{2}\right)$ .



## Question

4

Why do we care about normal modes?

Answer: Two reasons

① The general solution, valid for any initial conditions, can be written as a sum over normal modes:

$$x_1 = B_S e^{i\omega_S t} + B_L e^{i\omega_L t}$$

$$x_2 = B_S e^{i\omega_S t} - B_L e^{i\omega_L t}$$

} sum over normal mode solutions

Any valid motion of the system can be described ~~also~~ by specifying four initial conditions = Real & Imag. parts of  $B_S$  & real & imaginary parts of  $B_L$ .

② The time-evolution of each normal mode is extremely simple: ~~it~~ simply multiply by  $e^{i\omega t}$  for each mode.

This is easier to see if we simplify our notation. Let  $\vec{x} \equiv (x_1, x_2)$  be a vector which describes the current position of  $m_1$  &  $m_2$ .

Rename:  $B_S = a_1$ ,  $\omega_S = \omega_1$

$B_L = a_2$ ,  $\omega_L = \omega_2$

Then

$$\begin{aligned} x_1(t) &= a_1 e^{i\omega_1 t} + a_2 e^{i\omega_2 t} \\ x_2(t) &= a_1 e^{i\omega_1 t} - a_2 e^{i\omega_2 t} \end{aligned}$$



With vector notation I can combine two equations into one:

$$(x_1(t), x_2(t)) = a_1 \underbrace{(1, 1)}_{\text{constant vector}} e^{i\omega_1 t} + a_2 \underbrace{(1, -1)}_{\text{constant vector}} e^{i\omega_2 t}$$

Annotations: "initial conditions" points to  $a_1$  and  $a_2$ ; "time evolution" points to the exponential terms; "constant vector" labels the vectors  $(1, 1)$  and  $(1, -1)$ .

I can simplify the notation further if I define

$$\vec{q}_1 \equiv \text{constant vector} \equiv (1, 1)$$

$$\vec{q}_2 \equiv \text{constant vector} \equiv (1, -1)$$

Then

$$\vec{x}(t) = a_1 \vec{q}_1 e^{i\omega_1 t} + a_2 \vec{q}_2 e^{i\omega_2 t}$$

Should we make it even simpler? Use summation notation:

$$\vec{x}(t) = \sum_{n=1}^2 a_n \vec{q}_n e^{i\omega_n t}$$

This equation says exactly the same thing as our original general solution, but it is written more compactly and elegantly.

For example, we still need 4 initial conditions

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to specify  $Re(a_1), Im(a_1), Re(a_2), Im(a_2)$ .

The vectors  $\vec{q}_1$  &  $\vec{q}_2$  are the "normal mode eigenvectors". They are fixed, constant

vectors which describe the fixed relationship between the amplitudes of  $x_1$  &  $x_2$  in each normal mode.

$\vec{q}_1 = (1, 1) =$  "  $x_1$  &  $x_2$  have the same amplitude and phase in this mode " (symmetric mode)

$\vec{q}_2 = (1, -1) =$  "  $x_1$  &  $x_2$  have the same amplitude, but a phase difference of  $180^\circ$ , in this mode " (antisymmetric mode)

In general, the system is ~~not~~ not in a single normal mode, but is in a sum, or superposition, of normal modes. The fixed relationship between amplitudes of  $x_1$  &  $x_2$  will only occur when the system happens to be in a pure normal mode.

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$a_1$  &  $a_2$ , which are determined by the initial conditions, are called the "normal coordinates". They describe "how much of each normal mode" is in the ~~total~~ motion.



If we want to simplify further, we could absorb the time evolution factor into  $a_1$  &  $a_2$ :

$$\vec{x}(t) = \sum_{n=1}^2 a_n \vec{q}_n e^{i\omega_n t} = \sum_{n=1}^2 \underbrace{(a_n e^{i\omega_n t})}_{a_n(t)} \vec{q}_n$$

$$a_n(t) \equiv a_n e^{i\omega_n t}$$

$$= \sum_{n=1}^2 a_n(t) \vec{q}_n = a_1(t) \vec{q}_1 + a_2(t) \vec{q}_2$$

Since  $a_n(t) \equiv a_n e^{i\omega_n t}$ , we see that each normal mode evolves in time by picking up a phase factor of  $e^{i\omega_n t}$ . Notice that the magnitude of each normal mode component does not change, only its phase:

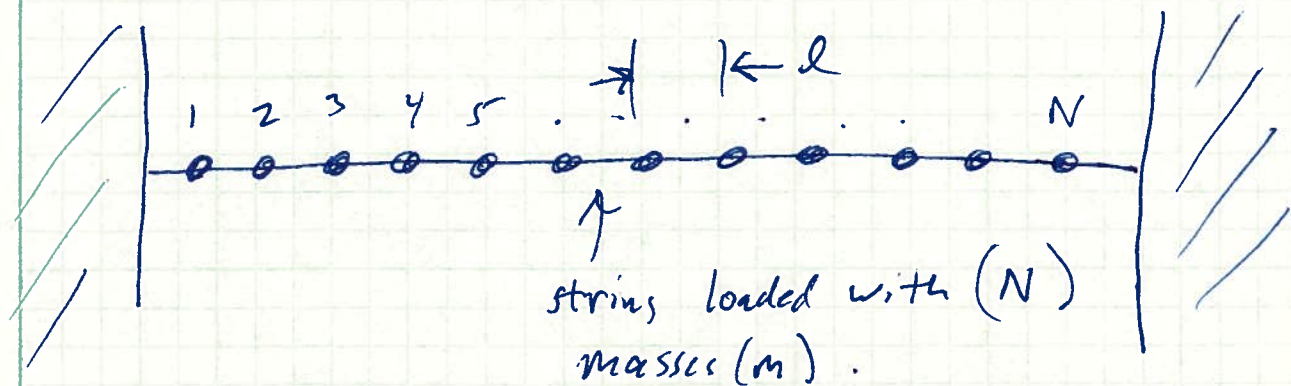
$$|a_n(t)| = |a_n e^{i\omega_n t}| = |a_n| \underbrace{|e^{i\omega_n t}|}_{\text{magnitude} = 1} = |a_n| = \text{constant.}$$

Therefore, whatever normal modes we have at  $t=0$ , we will have forever. Normal modes do not appear or disappear as time goes forward. (This is because we assumed no drag forces and no driving forces either. Drag forces would cause the amplitudes to decay, driving forces would cause them to grow (transient effect).)

# The loaded string & N coupled oscillators and transverse motion.

For a many particle system, the normal modes are easier to visualize if the motion is transverse to the direction of the springs (instead of in the same direction.) So let's switch to transverse motion.

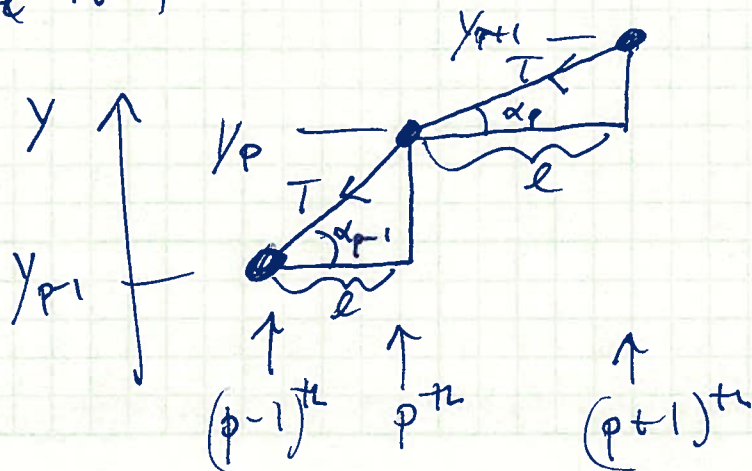
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Evenly spaced. (distance =  $l$ )

String Tension =  $T$

Consider ~~the~~ ~~with~~ one particular ~~with~~ mass, let it be the  $p^{th}$  mass (something between 1 & N.)



$l$  = distance between masses



The neighboring masses exert a restoring force in the  $y$  direction through the string tension

$$F_y^{(p)} = -T \sin(\alpha_{p-1}) + T \sin(\alpha_{p+1})$$

For small  $\alpha$  ( $F_x^{(p)}$  will equal zero, otherwise the string will move left or right)

For small displacements  $y$  in the  $y$  direction, we can approximate the sine function:

$$\sin(\alpha_{p-1}) \approx \frac{y_p - y_{p-1}}{l} \quad (\text{because } \sin \theta \approx \tan \theta \text{ for small } \theta)$$

$$\sin(\alpha_p) \approx \frac{y_{p+1} - y_p}{l}$$

$$\text{So } F_y^p \approx -\frac{T}{l} (y_p - y_{p-1}) + \frac{T}{l} (y_{p+1} - y_p)$$

$$\text{By Newton's 2nd Law, } F_y^p = m \ddot{y}_p$$

$$\therefore \ddot{y}_p + \frac{T}{ml} (2y_p) - \frac{T}{ml} y_{p-1} - \frac{T}{ml} y_{p+1} = 0$$

$$\text{Define } \omega_0^2 \equiv \frac{T}{ml}$$

$$\ddot{y}_p + 2\omega_0^2 y_p - \omega_0^2 (y_{p+1} + y_{p-1}) = 0$$

Equation of motion for the  $p^{\text{th}}$  mass.

Depends on  $y_{p+1}$  &  $y_{p-1}$   $\leftarrow$  Coupled

We will look for normal mode solutions:

$$y_p = A_p e^{i\omega t}$$

normal mode: all masses go at the same frequency.

stopped here  
3/6/12

Our job is to determine:

- ① What frequencies  $\omega$  is this a valid solution? (Normal frequencies.)
- ② For each normal frequency, what are the relationships between the amplitudes  $A_p$ ?

Substitute the guess into the equation of motion:

$$-\omega^2 A_p e^{i\omega t} + 2\omega_0^2 A_p e^{i\omega t} - \omega_0^2 (A_{p+1} e^{i\omega t} + A_{p-1} e^{i\omega t}) = 0$$

$$\text{or } (-\omega^2 + 2\omega_0^2) A_p - \omega_0^2 (A_{p-1} + A_{p+1}) = 0$$

$$\frac{A_{p-1} + A_{p+1}}{A_p} = \frac{-\omega^2 + 2\omega_0^2}{\omega_0^2}$$

↑ constant, independent of  $p$  (same constant for all masses.)



Make the following guess:

$$A_p = C \sin(p\theta) \quad , \quad \text{where } \theta \text{ is some constant that we must determine.}$$

Does this guess work? Try it:

insert guess

$$\frac{A_{p-1} + A_{p+1}}{A_p} \stackrel{?}{=} \text{constant, independent of } p$$

$$\frac{C \sin((p-1)\theta) + C \sin((p+1)\theta)}{C \sin(p\theta)} \stackrel{?}{=} \text{constant}$$

trig identity

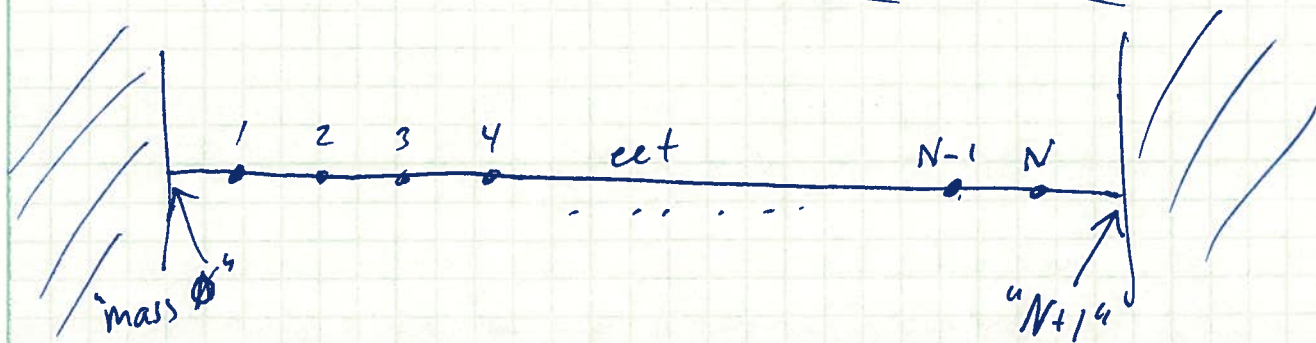
$$\frac{2C \sin(p\theta) \cos(\theta)}{C \sin(p\theta)} \stackrel{?}{=} \text{constant, independent of } p$$

$$\boxed{2 \cos(\theta) = \text{constant, independent of } p}$$

✓ yes

So our guess is viable. But we must determine  $\theta$  and  $\omega^2$ .

To fix  $\theta$ , use the boundary condition:



Because the ends of the string are fixed, the amplitude ~~the~~  $A_p$  should go to zero for  $p=0$  &  $p=N+1$ . Let's see if that can be made to work

$$A_p = C \sin(p\theta)$$

$$A_0 = C \sin(0\theta) = 0 \quad \checkmark \quad \text{yes}$$

And

$$A_{N+1} = C \sin((N+1)\theta) = 0$$

$$\boxed{\begin{array}{l} (N+1)\theta = n\pi, \quad n = 1, 2, 3, 4, \dots \\ \theta = \frac{n\pi}{N+1}, \quad n = 1, 2, 3 \end{array}}$$

Therefore our solution for  $A_p$  is

$$\boxed{A_p = C \sin\left(\frac{pn\pi}{N+1}\right)}$$

What about the normal frequencies?

$$\frac{A_{p-1} + A_{p+1}}{A_p} = \frac{-\omega^2 + 2\omega_0^2}{\omega_0^2}$$



$$2 \cos(\theta) = \frac{-\omega^2 + 2\omega_0^2}{\omega_0^2}$$

$$\uparrow \\ \frac{n\pi}{N+1}$$



$$2 \cos\left(\frac{n\pi}{N+1}\right) = \frac{-\omega^2 + 2\omega_0^2}{\omega_0^2}$$

$$\omega^2 = 2\omega_0^2 \left(1 - \cos\left(\frac{n\pi}{N+1}\right)\right)$$

$$\omega^2 = 4\omega_0^2 \sin^2\left(\frac{n\pi}{2(N+1)}\right)$$

trig identity

$$\omega_n = 2\omega_0 \sin\left(\frac{n\pi}{2(N+1)}\right)$$

Normal frequencies.

Add an (n) subscript because RHS depends on (n)

So we have found the normal mode solutions. The amplitude relationship is

$$A_{p_n} = C \sin\left(\frac{p n \pi}{N+1}\right)$$
$$\omega_n = 2\omega_0 \sin\left(\frac{n\pi}{2(N+1)}\right)$$

Normal Modes for a string loaded with N masses.

and the normal frequencies are:

In this expression:

- p is an integer which tells us which mass we are talking about
- N is the number of masses (p=1, 2, ..., N)
- n tells us which ~~of the~~ normal mode we are considering.
- $\omega_0 = T/ml$

# Properties of <sup>The</sup> Normal Modes of the loaded string

Recall that the displacement of the  $p$ th mass for a particular normal mode is

$$y_p = A_{pn} e^{i\omega t} = C \sin\left(\frac{pn\pi}{N+1}\right) e^{i\omega t}$$

Here we have assumed that the phase at  $t=0$  is zero. If we want to allow a non-zero phase we could write

$$y_p = A_{pn} e^{i(\omega t + \delta)} \quad \text{or} \quad \cancel{y_p = A_{pn} e^{i\delta} e^{i\omega t}}$$

$$\text{or} \quad y_p = \underbrace{(A_{pn} e^{i\delta})}_{B_{pn}} e^{i\omega t}$$

$B_{pn}$  where  $B_{pn}$  is complex.

Also, the allowed frequencies are

$$\omega_n = 2\omega_0 \sin\left(\frac{n\pi}{2(N+1)}\right)$$

Question: How many normal modes are there?

Answer: For a system of  $N$  masses, there are  $N$  normal modes

We can see this as follows:

$$\omega_{N+2} = 2\omega_0 \sin\left(\frac{(N+2)\pi}{2(N+1)}\right) = 2\omega_0 \sin\left[\frac{(2(N+1) - N)\pi}{2(N+1)}\right]$$



$$= 2\omega_0 \sin \left[ \pi - \frac{N\pi}{2(N+1)} \right]$$

trig identity

$$= 2\omega_0 \sin \left( \frac{N\pi}{2(N+1)} \right)$$

$= \omega_N$   $\leftarrow$   $\omega_{N+2}$  just duplicates  $\omega_N \dots$   
it is not an independent solution.

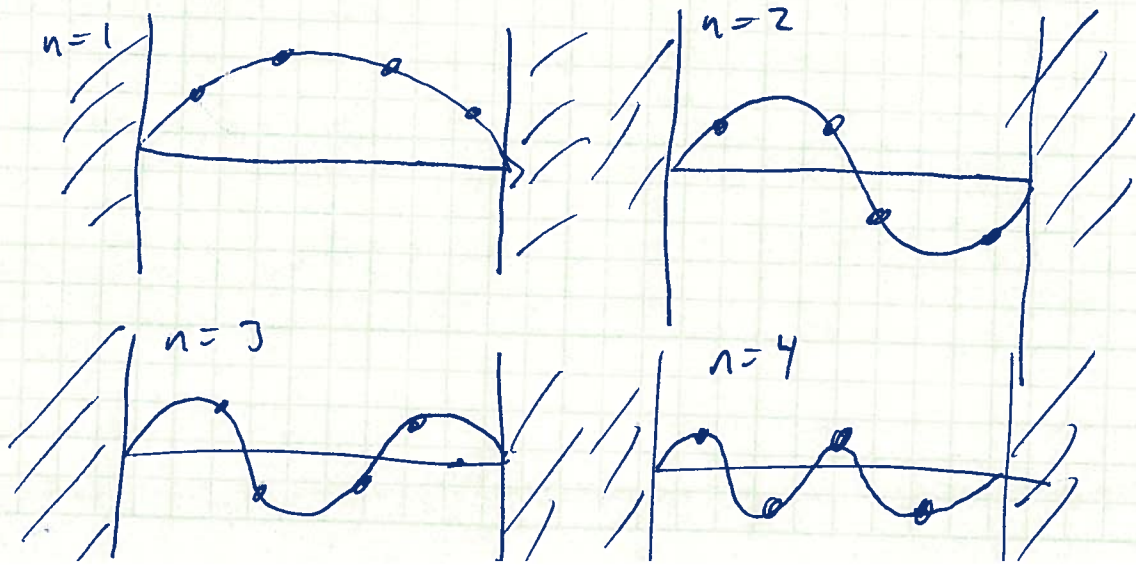
Similarly,  $\omega_{N+3} = \omega_{N-1}$

It can also be shown that the amplitude relationship ( $A_n$ ) repeats itself when  $n > N$ .

Conclusion: There are  $N$  independent normal modes of a system of  $N$  masses on a string.

What do the modes look like?

For  $N=4$ :



The general solution is a superposition of normal modes:

$$y_p = \sum_{n=1}^N a_n \sin\left(\frac{pn\pi}{N+1}\right) e^{i\omega_n t}$$

amplitude factor.

time evolution

complex coefficient,  
2 free parameters each

The number of free parameters is  $2N$ : real & imaginary parts of  $a_n$ , where  $n=1, 2, \dots, N$ . These  $2N$  free parameters will be fixed by the  $2N$  initial conditions: the position & velocity of every particle at  $t=0$ .

We can also switch to vector notation (if we like).

Let

$$\vec{y} = (y_1, y_2, y_3, \dots, y_N)$$

$$\vec{q}_n = \left( \sin\left(\frac{n\pi}{N+1}\right), \sin\left(\frac{2n\pi}{N+1}\right), \sin\left(\frac{3n\pi}{N+1}\right), \dots, \sin\left(\frac{Nn\pi}{N+1}\right) \right)$$

Then

$$\vec{y} = \sum_{n=1}^N a_n \vec{q}_n e^{i\omega_n t}$$

coefficient of the  $n$ th mode
normal mode vector
time evolution

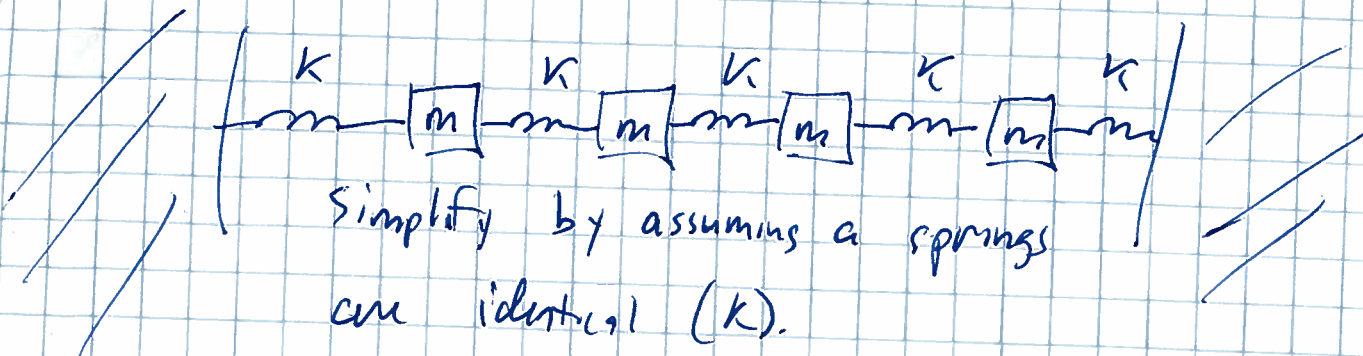
(amplitude relationship)



# Longitudinal Oscillations

(17)

Consider again masses ~~on~~ connected by springs:



Eq. of Motion for the  $p^{\text{th}}$  particle:

$$m \ddot{x}_p = \underset{\substack{\uparrow \\ m\omega_0^2}}{k} (x_{p+1} - x_p) - \underset{\substack{\uparrow \\ m\omega_0^2}}{k} (x_p - x_{p-1})$$

$$\ddot{x}_p + 2\omega_0^2 x_p - \omega_0^2 (x_{p+1} + x_{p-1}) = 0$$

This is identical in form to the Eq. of Motion for transverse oscillations:

$$\ddot{y}_p + 2\omega_0^2 y_p - \omega_0^2 (y_{p+1} + y_{p-1}) = 0$$

So the solution must be identical:

Normal Modes:

$$x_{pn}(t) = C_n \sin\left(\frac{pn\pi}{N+1}\right) e^{i\omega_n t}, \quad \omega_n = 2\omega_0 \sin\left(\frac{n\pi}{2(N+1)}\right)$$

General Solution: 
$$x_p = \sum_{n=1}^N a_n \sin\left(\frac{pn\pi}{N+1}\right) e^{i\omega_n t}$$