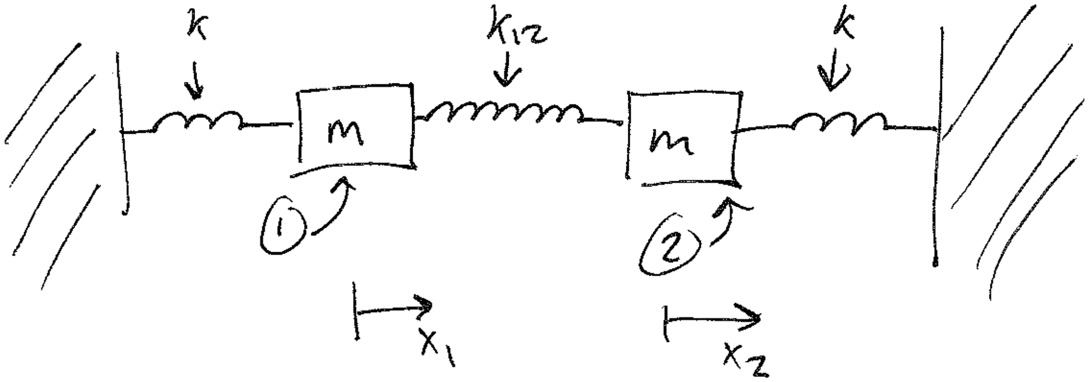
Two Coupled Mechanical Oscillators



3 springs, but just 2 spring constants.

x_1: displacement of 1 from equilibrium
x_2: " 2 "

Newton's 2nd Law

Force on 1: $F_1 = -kx_1 - k_{12}(x_1 - x_2) = m\ddot{x}_1$
Force on 2: $F_2 = -kx_2 + k_{12}(x_1 - x_2) = m\ddot{x}_2$

Equations of Motion:

$$\boxed{\begin{aligned} m\ddot{x}_1 + (k+k_{12})x_1 - k_{12}x_2 &= 0 \\ m\ddot{x}_2 + (k+k_{12})x_2 - k_{12}x_1 &= 0 \end{aligned}}$$

Coupled Differential Equations

x_1 and x_2 appear in both equations.
we must solve for $x_1(t)$ & $x_2(t)$ ~~at the~~ simultaneously.

~~Math~~ Our strategy: Let's look for solutions where both masses execute harmonic motion at the same frequency.

Guessed Solution:

$$x_1 = A_1 e^{i(\omega t + \delta_1)}$$

$$x_2 = A_2 e^{i(\omega t + \delta_2)}$$

ω is unknown. But the key feature of our guess is that both x_1 AND x_2 oscillate at the same frequency.

$\boxed{\text{Normal Mode}}$ * If all the masses in a system oscillate at the same frequency, then we call the motion a "Normal Mode".

Normal Frequency: The frequency of a normal mode.

Let's Simplify Notation: re-write the guessed solution:

$$x_1 = A_1 e^{i(\omega t + \delta_1)} = \underbrace{\left(A_1 e^{i\delta_1}\right)}_{\equiv B_1} e^{i\omega t} \equiv B_1 e^{i\omega t}$$

B_1 is complex, it has two free parameters (real & imaginary parts).

Also: $x_2 = B_2 e^{i\omega t}$

complex, 2 free parameters

Substitute the guessed solution into the equation of motion:

$$\boxed{\begin{aligned} m(-\omega^2 B_1) + (k+k_{12})B_1 - k_{12}B_2 &= 0 \\ m(-\omega^2 B_2) + (k+k_{12})B_2 - k_{12}B_1 &= 0 \end{aligned}}$$

$e^{i\omega t}$ has cancelled everywhere.

Gather Terms:

$$\boxed{\begin{aligned} (k+k_{12}-m\omega^2)B_1 - k_{12}B_2 &= 0 \quad \text{Eq. 1} \\ -k_{12}B_1 + (k+k_{12}-m\omega^2)B_2 &= 0 \quad \text{Eq. 2} \end{aligned}}$$

To have a solution, the determinant must equal zero:

$$(k+k_{12}-m\omega^2)^2 - k_{12}^2 = 0$$

$$k+k_{12}-m\omega^2 = \pm k_{12}$$

2 possibilities.

$$\boxed{\omega = \sqrt{\frac{k+k_{12} \pm k_{12}}{m}}}$$

We have found 2 normal mode frequencies:

Let's call them:

$$\boxed{\begin{aligned} \omega_s &= \text{smaller frequency} = \sqrt{\frac{k}{m}} \\ \omega_L &= \text{larger frequency} = \sqrt{\frac{k+2k_{12}}{m}} \end{aligned}}$$

So the small frequency solution is:

$$x_1 = B_1 e^{i\omega_s t}$$
$$x_2 = B_2 e^{i\omega_s t} , \quad \omega_s = \sqrt{\frac{k}{m}}$$

But we are not done. We can show that for this solution, we must have $B_1 = B_2$. To see this, substitute $\omega_s = \sqrt{k/m}$ into Eq. 1 & Eq. 2:

$$\boxed{\begin{aligned} (k + k_{12} - k)B_1 - k_{12} B_2 &= \emptyset \\ -k_{12} B_1 + (k + k_{12} - k) B_2 &= \emptyset \end{aligned}}$$

or

$$\boxed{\begin{aligned} k_{12}(B_1 - B_2) &= \emptyset \\ -k_{12}(B_1 - B_2) &= \emptyset \end{aligned}}$$

or $\boxed{B_1 = B_2}$ for the small frequency solution.

Let's call it $B_1 = B_2 \equiv B_s$ = "small frequency case.

Then our solution is

$$\boxed{\begin{aligned} X_1 &= B_s e^{i\omega_s t} \\ X_2 &= B_s e^{i\omega_s t} \end{aligned}}$$

B_s = complex = 2 free parameters, (real & imaginary parts)

This is called the "symmetric mode", because both oscillators have exactly the same motion. ⇒ Amplitude, phase, and frequency are identical.

Large Frequency mode: Exactly the same methods leads to

$$B_1 = -B_2 \quad \text{for } \omega_L = \sqrt{\frac{k+2k_{12}}{m}}$$

(−) sign means that x_1 & x_2 are out-of-phase by 180°.

Call $B_1 = B_L$. Then the large frequency solution is

$$\boxed{\begin{aligned} x_1 &= B_L e^{i\omega_L t} \\ x_2 &= -B_L e^{i\omega_L t} \end{aligned}}$$

B_L has 2 free parameters: real & imaginary parts.

We call this the "anti-symmetric mode" because the two oscillators are 180° out-of-phase with each other.

General Solution

The 2 normal mode solution are the simplest type of motion that the system may execute. But we can find a general solution by adding the normal mode solutions. This works because the equations of motion are linear.

$$\boxed{\begin{aligned} x_1 &= B_S e^{i\omega_S t} + B_L e^{i\omega_L t} \\ x_2 &= B_S e^{i\omega_S t} - B_L e^{i\omega_L t} \end{aligned}}$$

The most general solution.

Note that we have 4 free parameters: the real & imaginary parts of B_S & B_L. We need 4 initial conditions to specify them:

position and velocity of ① at $t=0$
& ② at $t=0$.

Let's take the real part and apply one particular set of initial conditions:

$$x_1 = b_S \cos(\omega_S t + \delta_S) + b_L \cos(\omega_L t + \delta_L)$$

$$x_2 = b_S \cos(\omega_S t + \delta_S) - b_L \cos(\omega_L t + \delta_L)$$

b_S, δ_S, b_L, δ_L are free parameters.

Suppose that

$$x_1(t=0) = a$$
$$\dot{x}_1(t=0) = 0$$
$$x_2(t=0) = 0$$
$$\dot{x}_2(t=0) = 0$$

Then the $\dot{x}_1$ & $\dot{x}_2$ requirements are:

$$\boxed{\begin{aligned} \dot{x}_1 &= -\omega_s b_s \sin(\delta_s) - \omega_L b_L \sin(\delta_L) = 0 \\ \dot{x}_2 &= -\omega_s b_s \sin(\delta_s) + \omega_L b_L \sin(\delta_L) = 0 \end{aligned}}$$

Add these equations: $-2\omega_s b_s \sin(\delta_s) = 0 \Rightarrow \boxed{\delta_s = 0}$

Subtract these equations: $-2\omega_L b_L \sin(\delta_L) = 0 \Rightarrow \boxed{\delta_L = 0}$

And the $x_1 = a$ and $x_2 = 0$ requirements are

$$x_1 = b_s \cos(\delta_s) + b_L \cos(\delta_L) = a$$

$$\updownarrow 0 \qquad\qquad \updownarrow 0$$

$$x_2 = b_s \cos(\delta_s) - b_L \cos(\delta_L) = 0$$

or $\boxed{\begin{aligned} b_s + b_L &= a \\ b_s - b_L &= 0 \end{aligned}} \Rightarrow \boxed{\begin{aligned} b_s &= \frac{a}{2} \\ b_L &= \frac{a}{2} \end{aligned}}$

So the complete solution for these initial conditions is

$$\boxed{\begin{aligned} x_1 &= \tfrac{a}{2}\cos(\omega_s t) + \tfrac{a}{2}\cos(\omega_L t) \\ x_2 &= \tfrac{a}{2}\cos(\omega_s t) - \tfrac{a}{2}\cos(\omega_L t) \end{aligned}}$$

What does it look like?: