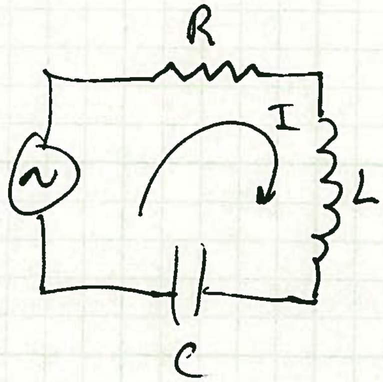
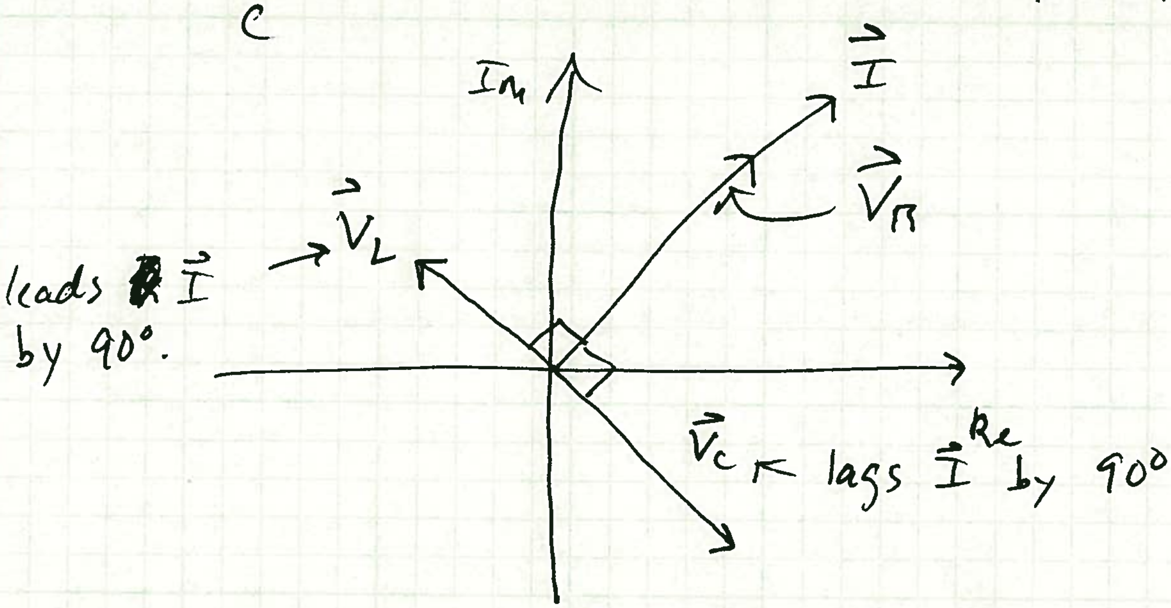
Driven RLC (series) circuit

$V_S = V_0 e^{i\omega t}$

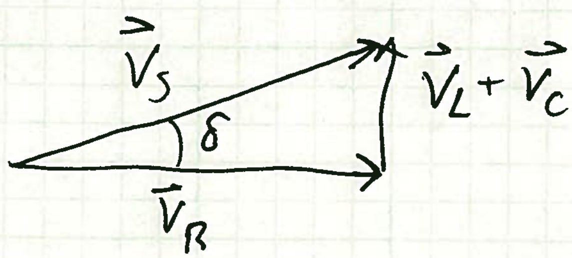


Only one current phasor: $\vec{I}$
(this circuit has only one current).

Voltage phasors can be drawn relative to $\vec{I}$: $\vec{V}_R, \vec{V}_L, \vec{V}_C$



Voltage Loop Rule: $\vec{V}_S = \vec{V}_R + \vec{V}_L + \vec{V}_C$



Phase shift:

$$\delta = \tan^{-1}\left[\frac{|\vec{V}_L + \vec{V}_C|}{|\vec{V}_R|}\right] = \cancel{\tan^{-1}\left[\frac{\left|i\omega L I_0 - \frac{i I_0}{\omega C}\right|}{|R I_0|}\right]}$$

$$= \tan^{-1}\left[\frac{\left|i\omega L I_0 - \frac{iI_0}{\omega C}\right|}{|RI_0|}\right]$$

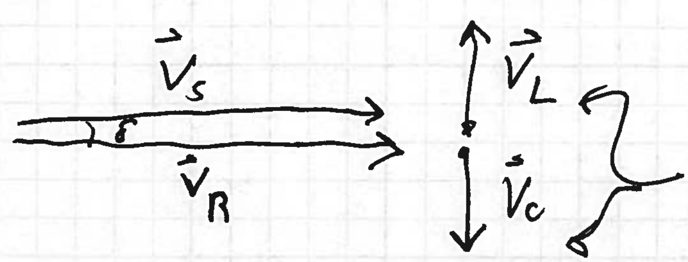
$$\left|i\omega L I_0 - \frac{iI_0}{\omega C}\right| = I_0\left|i\omega L - \frac{i}{\omega C}\right| = I_0\left(\omega L - \frac{1}{\omega C}\right)$$

$$|RI_0| = RI_0$$

$$= \tan^{-1}\left[\frac{I_0\left(\omega L - \frac{1}{\omega C}\right)}{I_0\,(R)}\right],$$ or, using $\omega_0^2 = \frac{1}{LC}$ and $\gamma = \frac{R}{L}$,

$$\boxed{\delta = \tan^{-1}\left[\frac{\omega^2 - \omega_0^2}{\omega\gamma}\right]}$$

What does this mean? Well, if we choose to drive the circuit with frequency $\omega = \omega_0$, then the phase difference between $\vec{V}_S$ and V_R is <u>zero</u>. Then the phasor diagram looks like:



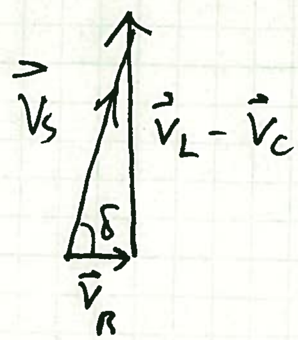
$\vec{V}_L$ & $\vec{V}_C$ are equal and opposite, when $\omega = \omega_0$.

Conversely, suppose we choose ω = very, very large.

Then $\vec{V}_C = \left(\frac{-i}{\omega C}\right)\vec{I} \approx \emptyset$

and $\vec{V}_L = (i\omega L)\vec{I} \approx$ large, and the

phasor diagram is

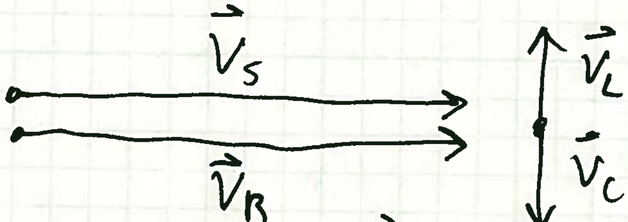


, and $\delta \to 90°$.

~~when does this~~

Question At what driving frequency is the current maximal?

Answer: $\vec{I}$ is maximal when $\vec{V}_R$ is maximal, since $\vec{V}_R = \vec{I}R$. But $|\vec{V}_R|$ can never be larger that $|\vec{V}_S|$; ~~since they have to add to zero in the~~



This happens when $\vec{V}_L + \vec{V}_C = \emptyset$

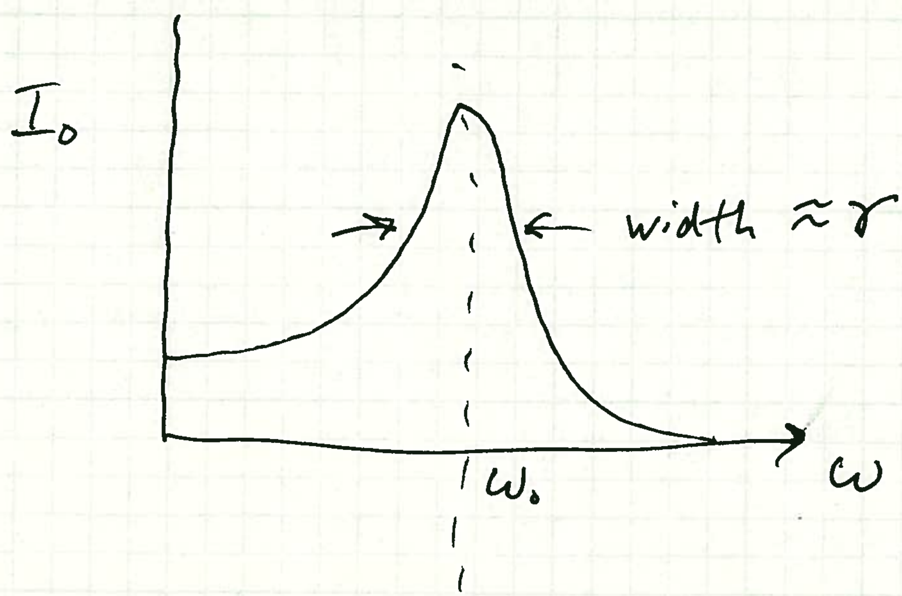
or $(i\omega L)I_0 + \left(\frac{-i}{\omega C}\right)I_0 = \emptyset$

or $\omega^2 = \frac{1}{LC}$

$$\boxed{\omega = \frac{1}{\sqrt{LC}} = \omega_0}$$

← condition for maximal current.

The amplitude of the current displays a resonanance near $\omega = \omega_0$:



Series and Parallel impedance

We have the following rules for impedances in AC circuits:

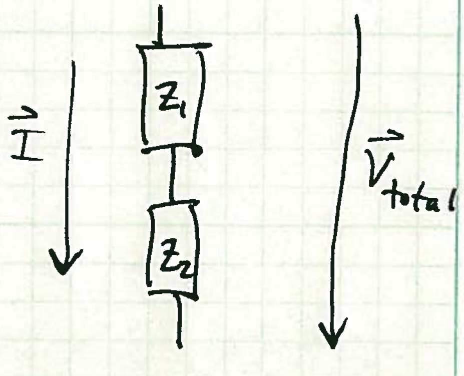
(1) Resistors: $V = I Z_R$, where $Z_R = R$

(2) Capacitors: $V = I Z_C$, where $Z = \frac{-i}{\omega C}$

(3) Inductors: $V = I Z_L$, where $Z_L = i\omega L$

phase shift between V & I

If we combine two elements in series, the the total voltage drop is



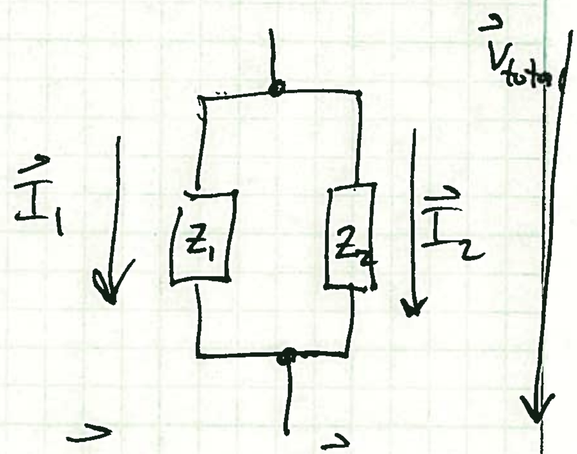
$\vec{V}_{total} = \vec{I} Z_1 + \vec{I} Z_2 \leftarrow$ because Z_1 & Z_2 share the same $\vec{I}$

$= \vec{I}(Z_1 + Z_2)$

$\therefore \vec{V}_{total} = \vec{I}(Z_{\cancel{total}\ series})$ where $\boxed{Z_{series} = Z_1 + Z_2.}$

$\therefore$ The impedance of two elements in series is the simple sum of Z_1 & Z_2.

If we combine two elements in parallel, then the total ~~current~~ voltage drop is



$\vec{V}_{total} = \vec{I}_1 Z_1$ and

$\vec{V}_{total} = \vec{I}_2 Z_2$.

The total current is

$\vec{I}_{total} = \vec{I}_1 \cancel{Z_1} + \vec{I}_2 \cancel{Z_2} = \dfrac{\vec{V}_{total}}{Z_1} + \dfrac{\vec{V}_{total}}{Z_2}$

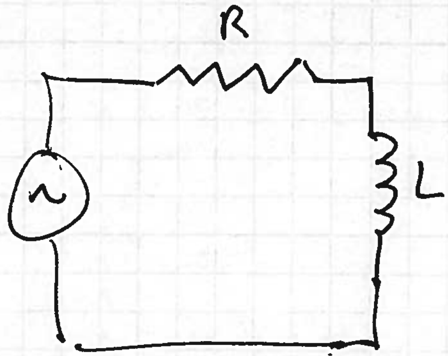
$\therefore \vec{V}_{total} = \vec{I}_{total}\left(\dfrac{1}{Z_1^{-1} + Z_2^{-1}}\right)$

$\vec{V}_{total} = \vec{I}_{total}\, Z_{parallel}$, where $\boxed{Z_{parallel} = \dfrac{1}{Z_1^{-1} + Z_2^{-1}}}$

$$\therefore \quad \boxed{\begin{aligned} Z_{series} &= Z_1 + Z_2 \\ Z_{parallel} &= \frac{1}{Z_1^{-1} + Z_2^{-1}} \end{aligned}}$$

Example RL ~~par~~. Series circuit (driven)

V_s



$$Z_{total} = Z_R + Z_L$$

$$Z_{total} = R + i\omega L$$

$$\therefore \quad \vec{V}_s = \vec{I}_{total}\, Z_{total}$$

$$\vec{V}_s = \vec{I}_{total}\,(R + i\omega L)$$

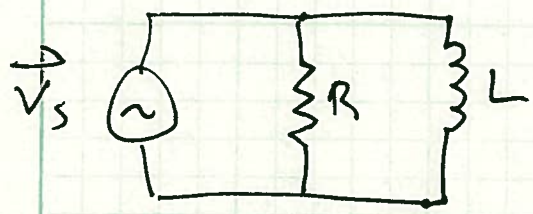
$\therefore$ phase difference between $\vec{V}_s$ & $\vec{I}_{total}$:

$$\delta = \text{phase of } (R + i\omega L) = \tan^{-1}\left(\frac{\omega L}{R}\right)$$

$\therefore$ magnitude of current:

$$|\vec{I}_{total}| = I_0 = \frac{\overbrace{|\vec{V}_s|}^{V_0}}{\underbrace{|R + i\omega L|}_{\sqrt{R^2 + (\omega L)^2}}} = \frac{V_0}{\sqrt{R^2 + (\omega L)^2}}$$

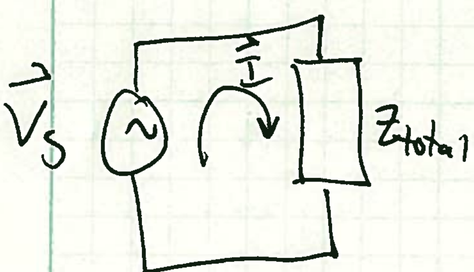
Example RL parallel circuit (driven)



$$Z_{total} = \frac{1}{Z_R^{-1} + Z_L^{-1}}$$

$$= \frac{1}{R^{-1} + (i\omega L)^{-1}}$$

or



$$= \frac{(R\omega L) \times (\omega L + iR)}{(\omega L - iR) \times (\omega L + iR)}$$

$$= \left(\frac{R\omega L^2}{(\omega L)^2 + R^2}\right)\left(\omega + i\frac{R}{L}\right)$$

$$Z_{total} = \left(\frac{R\omega L^2}{(\omega L)^2 + R^2}\right)(\omega + i\gamma)$$

$$\therefore \quad \vec{V}_{s\,\cancel{total}} = \vec{I}_{total}\left[\left(\frac{R\omega L^2}{(\omega L)^2 + R^2}\right)(\omega + i\gamma)\right]$$

where $\gamma \equiv \frac{R}{L}$

Phase difference between $\vec{V}_s$ & $\vec{I}_{total}$:

$$\delta = \tan^{-1}\left(\frac{\gamma}{\omega}\right)$$

← For very high ω, $\delta \to 0$, like a resistor

← For very low ω, $\delta \to 90°$, like an inductor.

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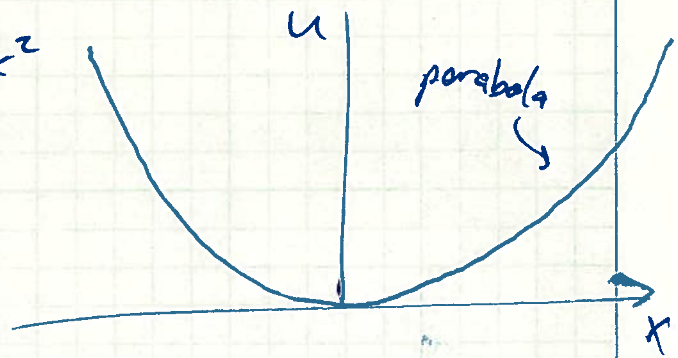
Exam 1 Review

Simple Harmonic Oscillator

(no damping)
(no driving force)

$F = -kx \Rightarrow U = \frac{1}{2}kx^2$



Eq. of motion:

$\ddot{x} + \frac{k}{m}x = \emptyset$

Define $\omega_0^2 = k/m$

Solution: $x = Ae^{i(\omega_0 t + \delta)}$, A, δ determined by initial conditions

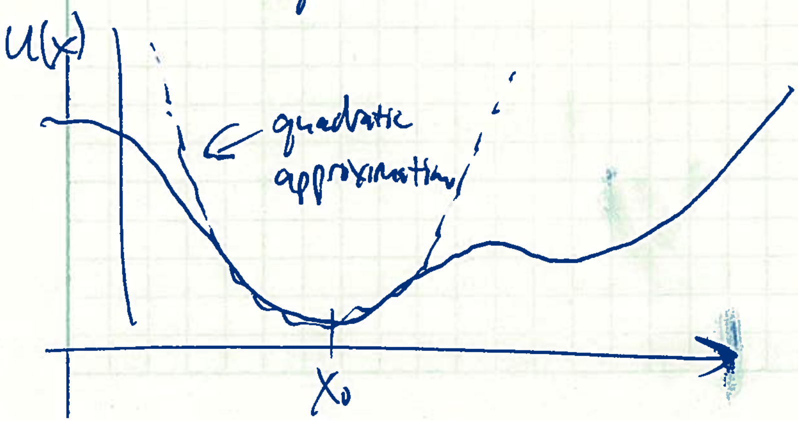
$\omega_0 \uparrow$ natural frequency.

Small Oscillations: Near a stable equilibrium, almost any potential function is approximately quadratic.

$$U(x_0 + a) \approx U(x_0) + U'(x_0)a + \frac{1}{2}U''(x_0)a^2 + \ldots$$

x_0 ↑ equilibrium; a ↑ displacement

$U'(x_0) = \emptyset$ since x_0 is an equilibrium point.



Frequency of small oscillations : $\omega_0 \approx \sqrt{\dfrac{u''(x_0)}{m}}$

near an equilibrium.

Plane Pendulum: $u = mg\ell(1-\cos\theta)$

$$\Rightarrow \omega_0 = \sqrt{g/\ell}$$

Complex Numbers

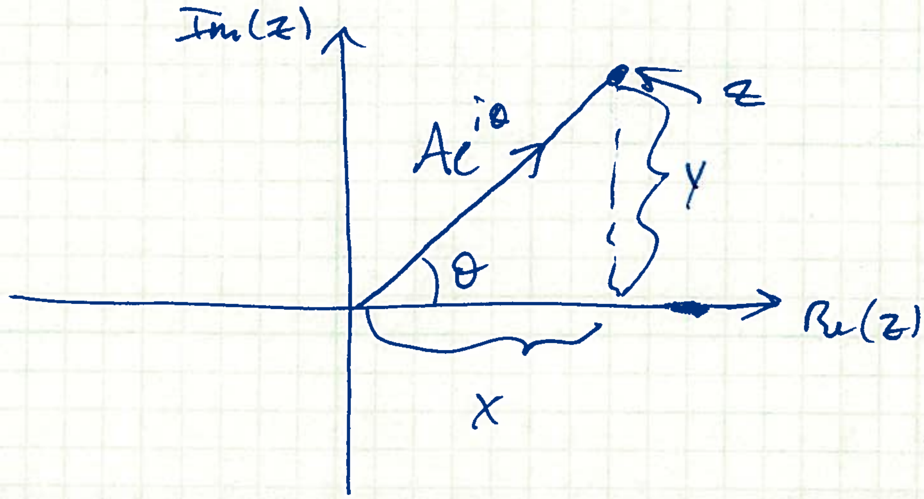
$$z = x+iy = Ae^{i\theta},$$

$$A = \sqrt{x^2+y^2}, \quad \theta = \tan^{-1}(y/x)$$

& $\quad x = A\cos\theta, \quad y = A\sin\theta$

$$\boxed{e^{i\theta} = \cos(\theta) + i\sin(\theta)}$$ Euler Formula

$$x = \frac{e^{i\theta}+e^{-i\theta}}{2}, \quad y = \frac{e^{i\theta}-e^{-i\theta}}{2}$$



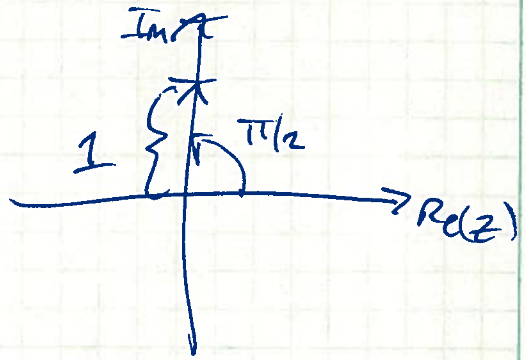
$z^* \equiv$ complex conjugate $= x - iy$ (if $z = x + iy$)

$= Ae^{-i\theta}$ (if $z = Ae^{i\theta}$)

Division: $\dfrac{1}{x+iy} = \dfrac{(x-iy)}{(x+iy)(x-iy)} = \dfrac{x-iy}{x^2+y^2}$

Also: $zz^* = (x+iy)(x-iy) = x^2+y^2 = A^2$.

Also: $e^{i\pi/2} = i$



Also: $e^{i\pi} = -1$

Also: $e^{i\pi} + 1 = 0$ ← nice algebraic equation.

Forced Oscillator

Eq of Motion

$\ddot{x} + \omega_0^2 x = \dfrac{F(t)}{m}$ If $F(t) = F_0 e^{i\omega t}$

$\ddot{x} + \omega_0^2 x = \dfrac{F_0}{m} e^{i\omega t}$

"natural frequency" "driving frequency"

Also: Add damping: $F_{drag} = -bv = -b\dot{x}$

some constant

$$\boxed{\ddot{x} + \gamma\dot{x} + \omega_0^2 x = \frac{F_0}{m} e^{i\omega t}}$$

where $\gamma \equiv \frac{b}{m}$

$\omega_0^2 = k/m$

~~Solution:~~

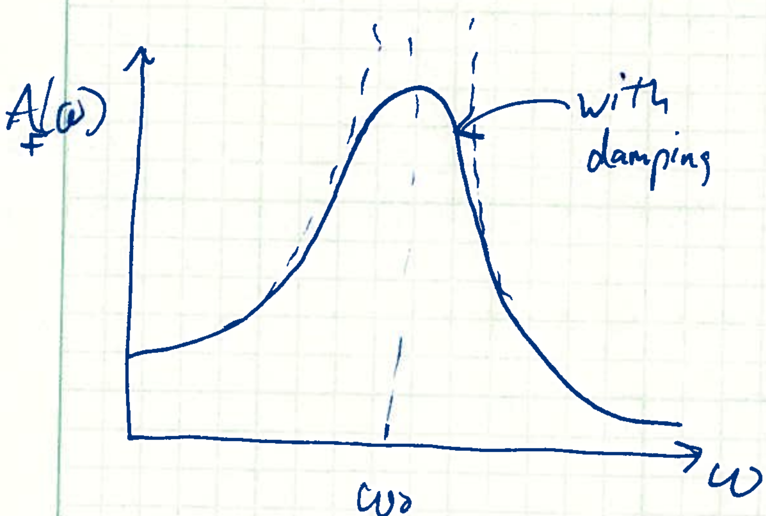
Steady-State (Long-term) Solution:

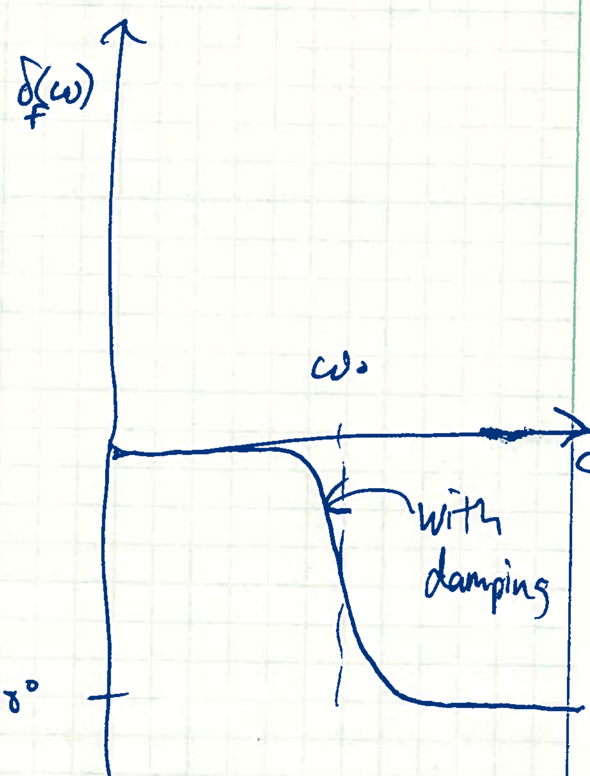
$$x(t) = A_f(\omega_f)\, e^{i(\omega_f t + \delta_f(\omega_f))}$$, $\boxed{\text{no free parameters}}$

where $$A_f(\omega) = \frac{F_0/m}{\sqrt{(\omega_0^2-\omega^2)^2 + (\omega\gamma)^2}}$$

and $$\delta_f(\omega) = -\tan^{-1}\left[\frac{\omega\gamma}{(\omega_0^2-\omega^2)}\right]$$

ω_f = driving frequency





Including the transient Solution (Short-term Behavior):

$$x(t) = A(\omega_f)e^{i(\omega_f t + \delta_f(\omega_f))} + Be^{-\gamma t/2}e^{i(\omega_d t + \delta_d)}$$

ω_d = damped frequency $= \sqrt{\omega_0^2 - \gamma^2/4}$

B & δ_d = Free parameters, determined by initial conditions.

Special Case: Damped oscillator, no forcing function

Then $F_0 = \emptyset$, $A(\omega_f) = \emptyset$,

and $x(t) = B\underbrace{e^{-\gamma t/2}}_{\text{damping factor}}e^{i(\omega_d t + \delta_d)}$

Energy

- Mechanical Oscillators: $KE = \frac{1}{2}m\dot{x}^2$

  $U = \frac{1}{2}kx^2$

- Electrical Oscillators: $U_E = \int_{\text{all space}} \frac{1}{2}\varepsilon_0|\vec{E}|\,dV$

  for capacitors $\rightarrow = \frac{1}{2}QV = \frac{1}{2}CV^2 = \frac{Q^2}{2C}$

  $U_B = \int_{\text{all space}} \frac{1}{2\mu_0}|\vec{B}|^2\,dV$

  for inductors $\longrightarrow = \frac{1}{2}LI^2$

Energy loss $Q \equiv \frac{\omega_0}{\gamma}$ = unitless & very large for lightly damped oscillators.

For lightly damped oscillators

$$Q = \frac{1}{\text{Fraction of energy lost in time } t = \frac{1}{\omega_0}} = \frac{2\pi}{\text{Fraction of energy loss in one period}}$$

Energy loss: $E(t) = E_0 e^{-\gamma t}$ $= KE(t) + U(t)$ for mechanical oscillators

$= U_E(t) + U_B(t)$ for electrical oscillators

AC circuits

Voltage rules:

$|V_C| = \left|\frac{1}{C}Q\right|$ Capacitor

$|V_L| = \left|L\frac{dI}{dt}\right|$ Inductor

$|V_R| = |IR|$ Resistor

Simple LC circuit: $\omega_0 = \frac{1}{\sqrt{LC}}$ (simple harmonic oscillator)

Impedances:

$Z_R = R$

$Z_L = i\omega L$

$Z_C = \frac{-i}{\omega C}$

Series Combination: $Z_{series} = Z_1 + Z_2$

Parallel Combination: $Z_{parallel} = \left[(Z_1)^{-1} + (Z_2)^{-1}\right]^{-1}$

For any element or combination of elements,

$$\vec{V} = \vec{I} Z$$

Z ← impedance, possibly complex.

$\vec{V}$ ← Voltage phasor

$\vec{I}$ ← current phase

In general Z is complex, which means that there is a phase offset between $\vec{V}$ & $\vec{I}$.
The ~~ratio~~ ratio of $\frac{V_0}{I_0} = |Z|$.

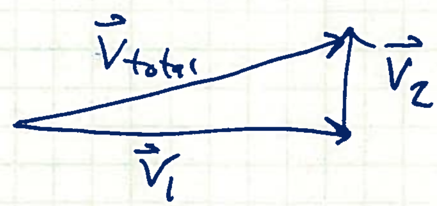
$$\vec{V}_L = \vec{I}_L \underbrace{(i\omega L)}$$

← this means V_L leads I_L by 90°.

$$\vec{V}_C = \vec{I}_C \underbrace{\left(\frac{-i}{\omega C}\right)}$$

← this means V_C lags I_C by 90°

Phasor Diagrams show the geometric relationships between $\vec{V}$ & $\vec{I}$ for a circuit.
This can be very useful when combined with a sum rule like $\vec{V}_{total} = \vec{V}_1 + \vec{V}_2$ or $\vec{I}_{total} = \vec{I}_1 + \vec{I}_2$



$\therefore |V_{total}| = \sqrt{V_1^2 + V_2^2}$