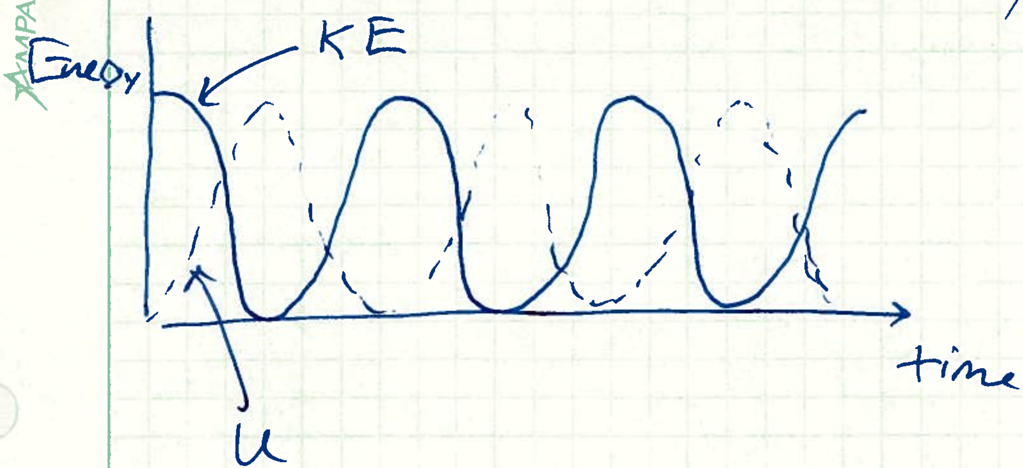
Phys 273 | Lecture 10 | 1

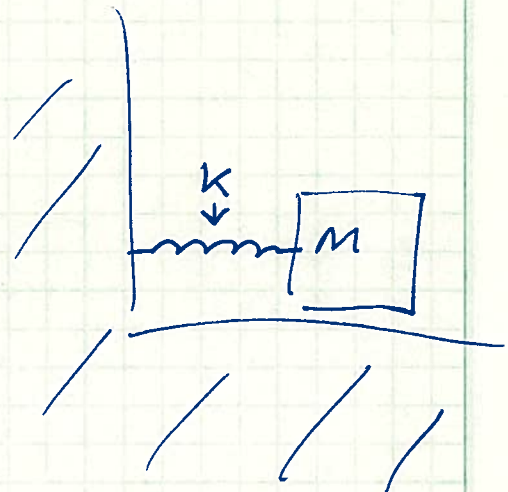
Electrical Oscillators

Mechanical Oscillator: energy is converted between kinetic and elastic potential

$KE = \frac{1}{2}mv^2 = \frac{1}{2}m\dot{x}^2$

$U = \frac{1}{2}kx^2$

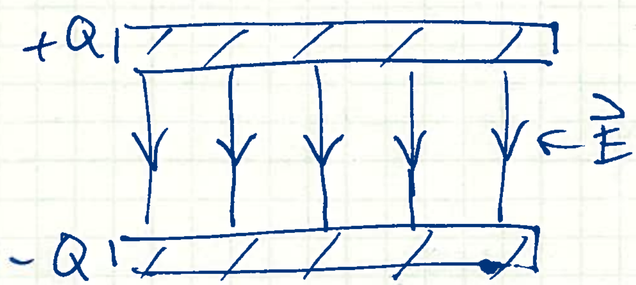




Electrical Oscillator: Energy is converted between electric ($\vec{E}$ field) and magnetic ($\vec{B}$ Field).

1. Capacitor: Device for storing energy in an electric field

Ex: Parallel Plate Capacitor



Each small volume of space (dV) with an electric field $\vec{E}$ stores a small amount of electric energy (dU_E):

$$dU_E = \underbrace{\tfrac{1}{2}\varepsilon_0 |\vec{E}|^2}_{u_E} dV \quad , \quad u_E \equiv \tfrac{1}{2}\varepsilon_0 |\vec{E}|^2$$

Energy, big U

energy density, little u

$=$ "electric energy density of free space"

$= \dfrac{\text{Joules}}{\text{meter}^3}$

The total energy stored is

"big U" $\rightarrow U_E = \displaystyle\int_{\text{all space}} \tfrac{1}{2}\varepsilon_0 |\vec{E}|^2 dV$

If the electric field is created by a capacitor with charges $+Q$ and $-Q$ and voltage difference V, then the total energy can be written

$$U_E = \tfrac{1}{2}QV = \tfrac{1}{2}CV^2 = \frac{Q^2}{2C}$$

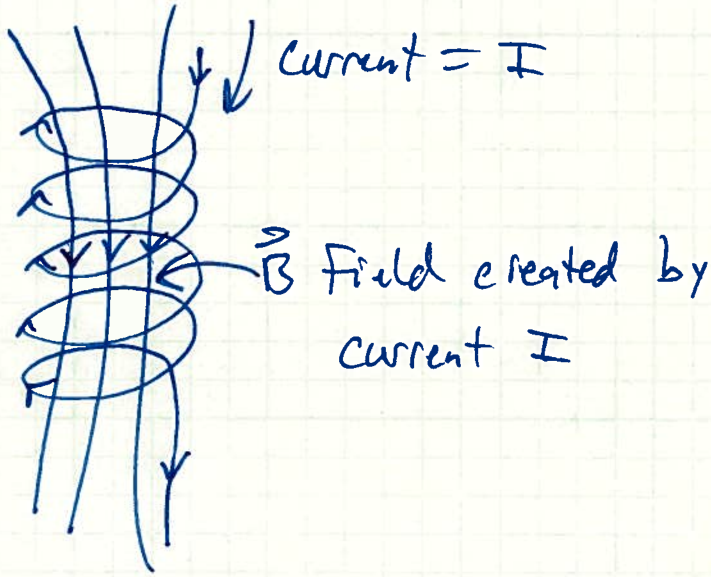
where C = capacitance = $\dfrac{Q}{V}$

Capacitance is a constant that only depends on the shape and material of the capacitor.

"$Q = CV$" says "charge on the capacitor is proportional to the voltage across it. C is the proportionality constant."

② Inductor: Device for storing energy in a magnetic field.

A simple solenoid:



The energy density of a magnetic field is

$$dU_B = \frac{1}{2\mu_0}|\vec{B}|^2 dV \quad , \quad u_B = \frac{1}{2\mu_0}|\vec{B}|^2$$

"little u" = "magnetic energy density of free space" = $\frac{\text{Joules}}{\text{meter}^3}$

The total magnetic energy is

"big U" $\rightarrow U_B = \int_{\text{all space}} \frac{1}{2\mu_0}|\vec{B}|^2 dV$

For an inductor, the total energy can be written as

$$U_B = \frac{1}{2}LI^2 \quad , \text{ where } L = \text{"self-inductance"}$$

L is determined by the shape and material of the inductor. It is the proportionality constant between magnetic flux and current:

Φ_B = magnetic flux through the inductor $= L\,I$

L ← proportionality constant. I ← current

Similarities between C & L

Device	Circuit Symbol	MKS unit	Stores energy in:	Proportionality Constant:	Determined by
C	capacitor symbol	Farad	$\vec{E}$	$Q = CV$	shape and material
L	inductor symbol	Henry	$\vec{B}$	$\Phi_B = LI$	shape and material

If you know the shape & material of your $\left\{\begin{array}{l}\text{capacitor} \\ \text{inductor}\end{array}\right\}$, then you can calculate $\left\{\begin{array}{c} C \\ L \end{array}\right\}$.

Neither C nor L depends on: $\left.\begin{array}{c} Q \\ V \\ I \end{array}\right\}$ These things depend on time, but C & L are constant.

Voltage Rules:

Capacitor: $|V_c| = \left|\frac{1}{C} Q\right|$ or $\boxed{V_C = \frac{1}{C} Q}$ (ignoring any sign)

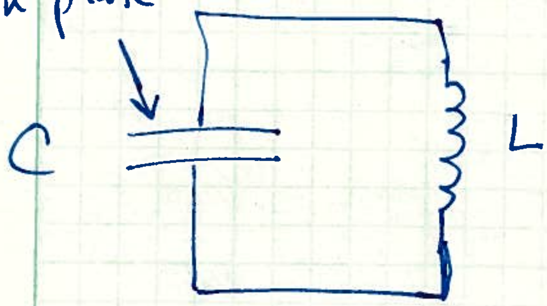
Inductor: $|V_L| = \left|-\frac{d\Phi_B}{dt}\right| = \left|-\frac{d}{dt}(LI)\right| = \left|L\frac{dI}{dt}\right|$

$\therefore \boxed{V_L = L\frac{dI}{dt}}$ (ignoring any sign)

LC oscillator

Simplest electrical oscillator. Energy exchanges between electric & magnetic.

q = charge on plate



Voltage Loop rule:

$$V_C + V_L = \emptyset$$

$$\left(\frac{1}{C}q\right) \quad \left(L\frac{dI}{dt}\right) = L\frac{d^2q}{dt^2}$$

since $I = \dfrac{dq}{dt}$

$$\therefore \quad \frac{1}{C}q + L\frac{d^2q}{dt^2} = \emptyset$$

$$\boxed{\ddot{q} + \frac{1}{LC}q = \emptyset}$$

Simple harmonic oscillator equation

Solution: $q(t) = q_0 e^{i(\omega_0 t + \delta)}$, where $\omega_0 = \dfrac{1}{\sqrt{LC}}$ = "natural freq."

q_0 & δ are determined by the initial conditions

$$\dot{q}(t) = I(t) = (i\omega_0)\left(q_0 e^{i(\omega_0 t+\delta)}\right)$$
$$= i\omega_0 q(t)$$

current has a phase shift of 90° compared to charge

(c)

<u>Energy</u>: U_E = electric energy

$$= \frac{1}{2C} q^2 = \frac{1}{2C}\left[q_0 \cos(\omega_0 t + \delta)\right]^2$$

$$= \frac{q_0^2}{2C} \cos^2(\omega_0 t + \delta)$$

U_B = magnetic energy

$$= \frac{1}{2} L I^2$$

$$= \frac{1}{2} L \Big[\underbrace{Re\left(i\omega_0 q_0 e^{i(\omega_0 t + \delta)}\right)}_{-\omega_0 q_0 \sin(\omega_0 t + \delta)}\Big]^2$$

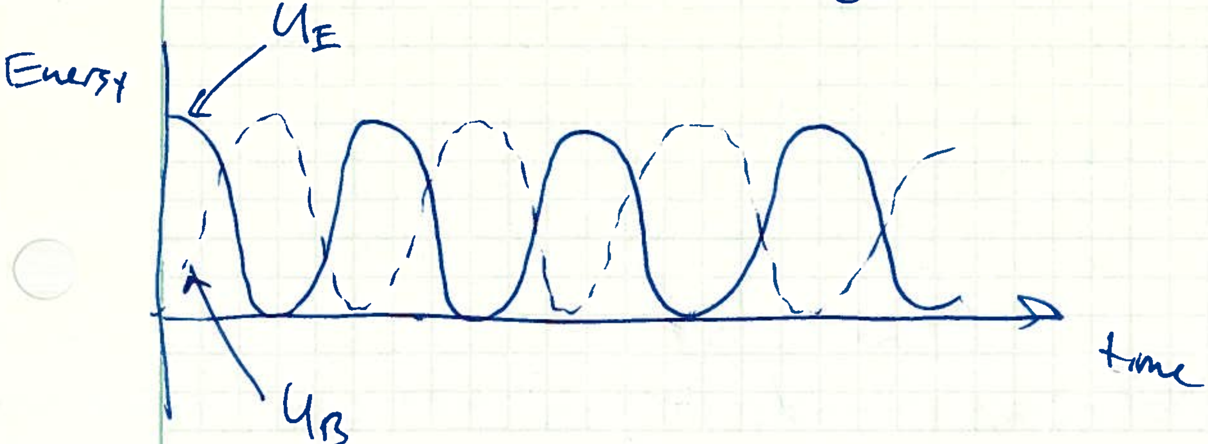
$$= \frac{1}{2} q_0^2 \underbrace{\omega_0^2 L} \sin^2(\omega_0 t + \delta)$$

$$\omega_0^2 = \frac{1}{LC} \quad \text{so} \quad \omega_0^2 L = \frac{1}{C}$$

$$= \frac{q_0^2}{2C} \sin^2(\omega_0 t + \delta)$$

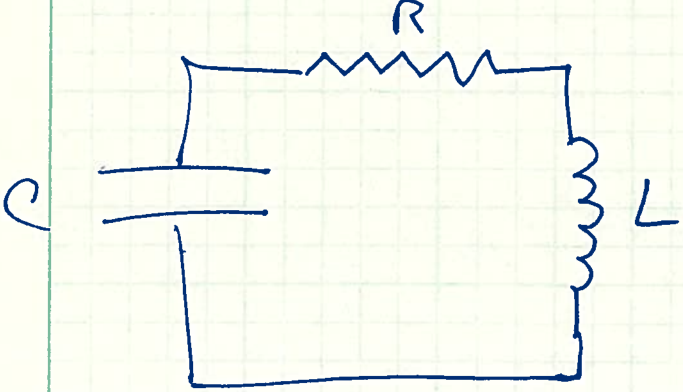
Total energy $= U_E + U_B = \frac{q_0^2}{2C}\Big[\underbrace{\cos^2(\omega_0 t + \delta) + \sin^2(\omega_0 t + \delta)}_{1}\Big]$

$$= \frac{q_0^2}{2C} = \text{constant}$$



LC oscillilator with damping — RLC circuit

Add a resistor to the circuit: Electrical energy will be converted to heat in the resistor



Voltage Rule:

$$V_C + V_R + V_L = \emptyset$$

$V_C \to \frac{q}{C}$, $V_R \to IR = \dot{q}R$, $V_L \to L\frac{dI}{dt} = L\frac{d^2q}{dt^2} = L\ddot{q}$

$$\therefore \frac{q}{C} + R\dot{q} + L\ddot{q} = \emptyset$$

$$\boxed{\ddot{q} + \frac{R}{L}\dot{q} + \frac{1}{LC}q = \emptyset}$$

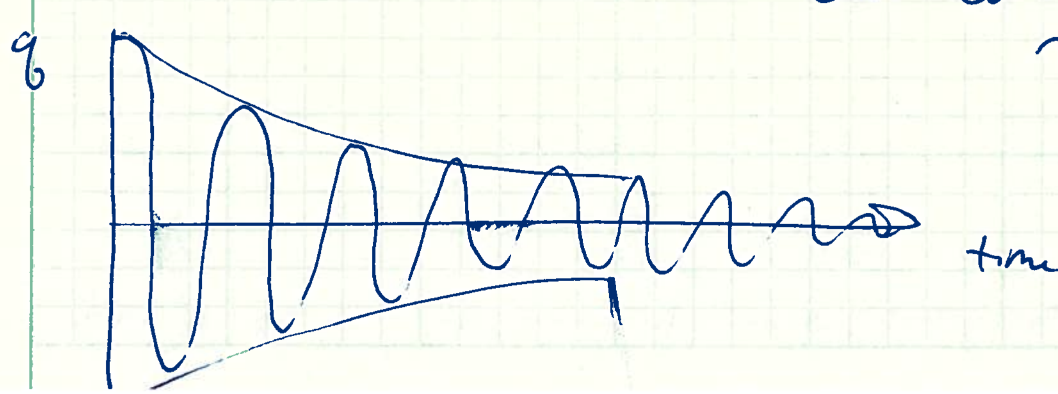
Simple Harmonic Oscillator with damping

Solution (light damping):

$$\boxed{q(t) = q_0 e^{-\gamma t/2} e^{i(\omega t+\delta)}}$$

where $\gamma = \frac{R}{L}$

and $\omega = \sqrt{\omega_0^2 - \frac{\gamma^2}{4}}$



Driven RLC circuit — Series

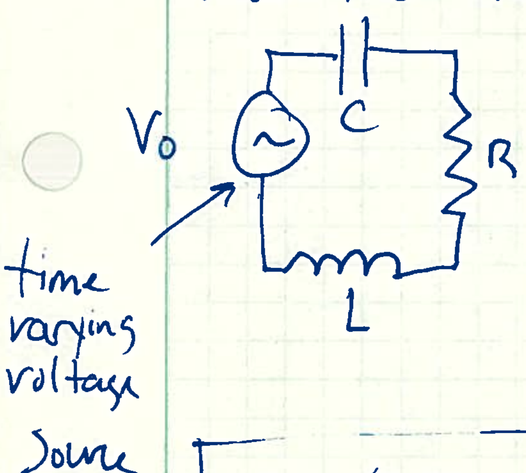
Suppose we have an oscillating circuit with a voltage source that varies in time as a cosine:

$$V_{source}(t) = V_0 \cos(\omega t) = V_0 e^{i\omega t}$$

forcing frequency

We choose δ (phase shift) $= \emptyset$ be choosing $t = \emptyset$ correctly.

Then the RLC series circuit looks like:



Voltage Loop Rule:

$$V_S = \cancel{V_R}\; V_c + V_R + V_L$$

$$(V_0 e^{i\omega t}) \quad \left(\frac{1}{C} q\right) \quad (R\dot{q}) \quad (L\ddot{q})$$

$$\boxed{\ddot{q} + \left(\frac{R}{L}\right)\dot{q} + \left(\frac{1}{LC}\right) q = \left(\frac{V_0}{L}\right) e^{i\omega t}}$$

Driven Harmonic Oscillator.

Steady State Solution: $q(t) = q_0 e^{i(\omega t + \delta)}$ or $A e^{i(\omega t + \delta)}$

where ~~A = q_0~~

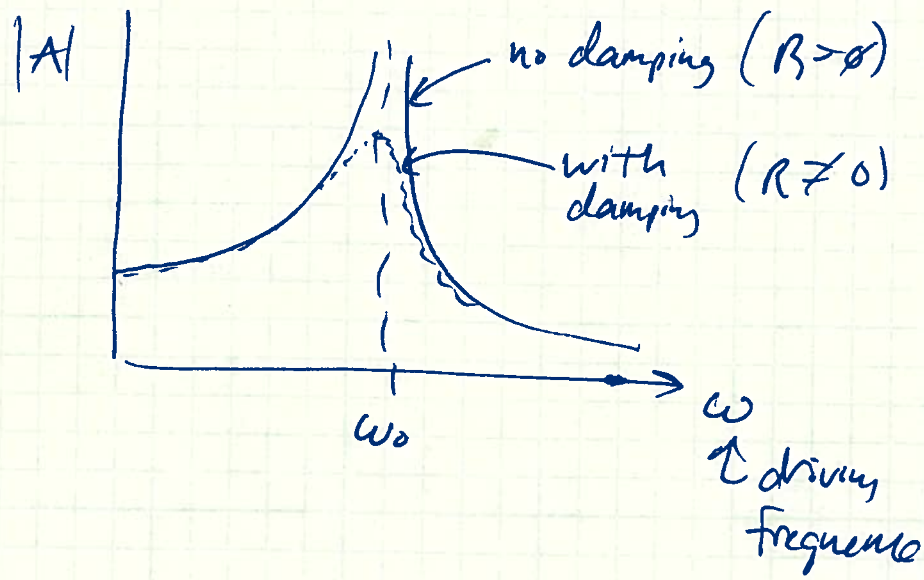
$$q_0(\omega) = \frac{(V_0/L)}{\sqrt{(\omega_0^2 - \omega^2)^2 + (\omega \gamma)^2}}$$

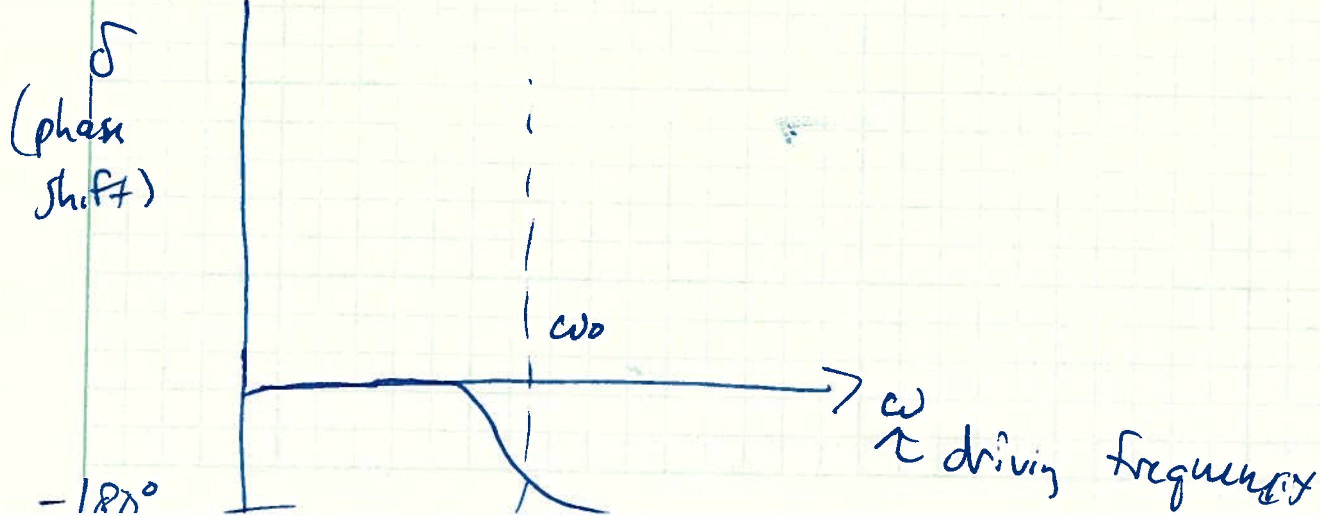
ω = driving frequency, $\omega_0 = \frac{1}{\sqrt{LC}}$, $\gamma = (R/L)$

and $\delta(\omega) = -\tan^{-1}\left[\frac{\omega\tau}{(\omega_0^2 - \omega^2)}\right]$

Just like the forced mechanical oscillator we have a <u>resonance</u> when $\omega \approx \omega_0$.

When $\omega \approx \omega_0$, the following things become very large:
1) the peak charge on the capacitor
2) the peak current in the circuit
3) the total energy stored in the C & L.





Phasor Analysis of AC circuits

Phasors give us a geometric method for thinking about harmonic oscillators. Particularly useful for AC circuits.

Basic Principles

1. Resistors: Voltage Rule: $V = IR$

If $V_R = V_0 e^{i\omega t}$, then $I = \underbrace{\frac{V_0}{R}}_{\text{no phase shift}} e^{i\omega t}$

$\boxed{V_R = IR}$ Ohm's Law for resistors

2. Capacitors: $Q = CV$

$$I = \frac{dQ}{dt} = C\frac{dV}{dt}$$

If $V = V_0 e^{i\omega t}$, then $I = i\omega C \underbrace{V_0 e^{i\omega t}}_{V}$

or

$$V = I\underbrace{\left(\frac{1}{i\omega C}\right)}$$

Define $\boxed{X_C \equiv \frac{1}{i\omega C}}$ "Capacitive Reactance"

Then $\boxed{V = IX_C}$ "Ohm's Law for Capacitors"

③ Inductors: $V = L\dfrac{dI}{dt}$

If $V = V_0 e^{i\omega t}$

Then $I = \dfrac{1}{i\omega L}\underbrace{\left(V_0 e^{i\omega t}\right)}_{V}$

OR

$V = I\underbrace{(i\omega L)}$

Define $i\omega L \equiv X_L$ = "inductive reactance"

Then $\boxed{V = IX_L}$ Ohm's Law for Inductors

Summarizing

Resistors: $V = IR$, $\Rightarrow$ V & I are 100% in phase (no phase difference)

Capacitors: $V = IX_c$, $X_c = \dfrac{1}{i\omega C} = \dfrac{-i}{\omega C}$ ← phase difference

$\Rightarrow$ Voltage lags the current by 90° ($\pi/4$)

Inductors: $V = IX_L$, $X_L = i\omega L$ ← phase difference

$\Rightarrow$ Voltage leads the current by 90°.

In AC circuits, Capacitors and Inductors behave like resistors in that the peak voltage is proportional to the peak current. (Like Ohm's Law for resistors.) On the other hand,

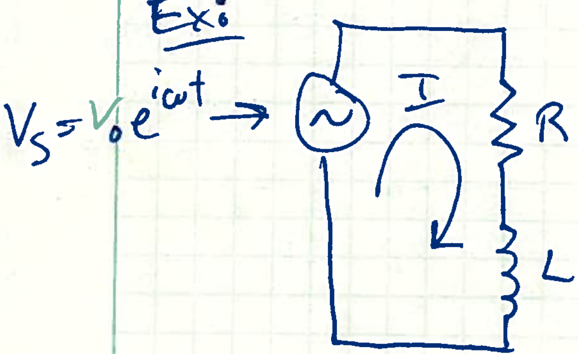
1. There is a phase shift of $+90°$ or $-90°$ between voltage & current.

2. The proportionality "constant" ~~for~~ depends on the frequency: $X_C = \frac{-i}{\omega C}$, $X_L = i\omega L$.

frequency dependent

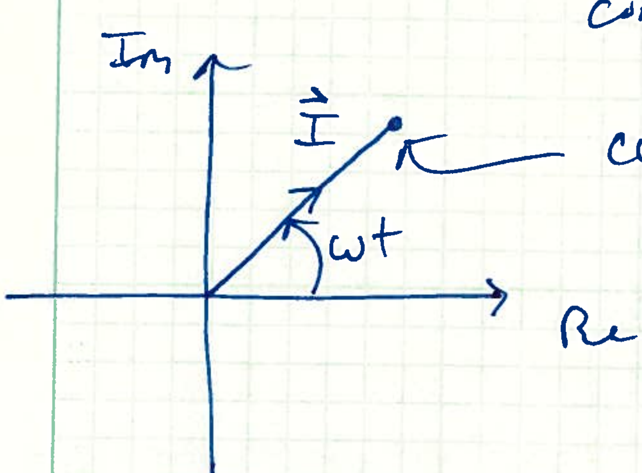
Application to Phasor Analysis

Ex:



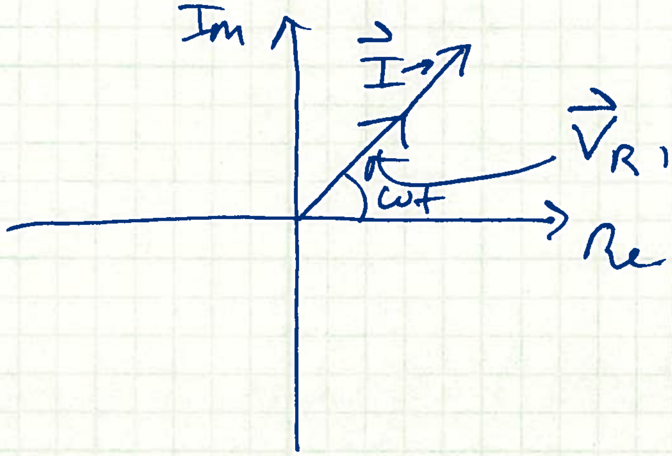
Driven RL circuit.
This circuit has one and only one current.

Let's draw it in the complex plane:



current phasor, rotating at frequency ω. Makes an angle ωt with the real axis.

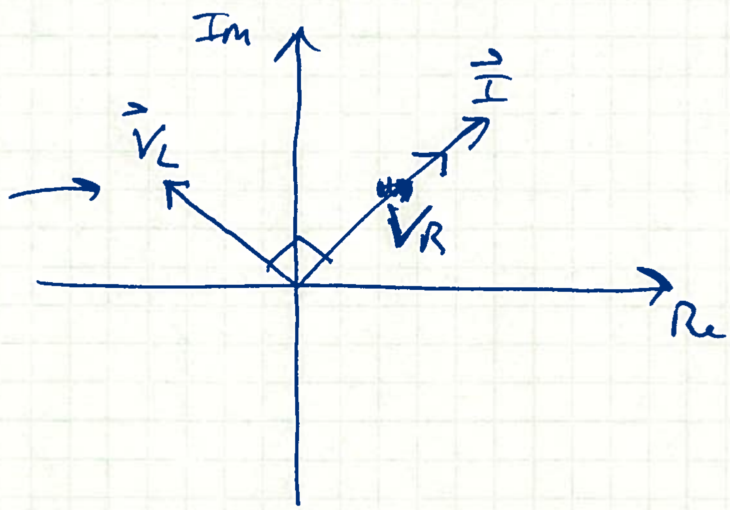
Add the voltage phasor for the resistor



voltage phasor for resistor

In-phase with the current

Now add the voltage phasor for the inductor:



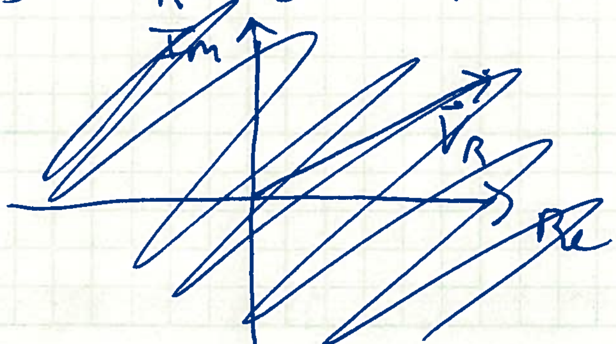
Voltage of Inductor leads the current by 90°.

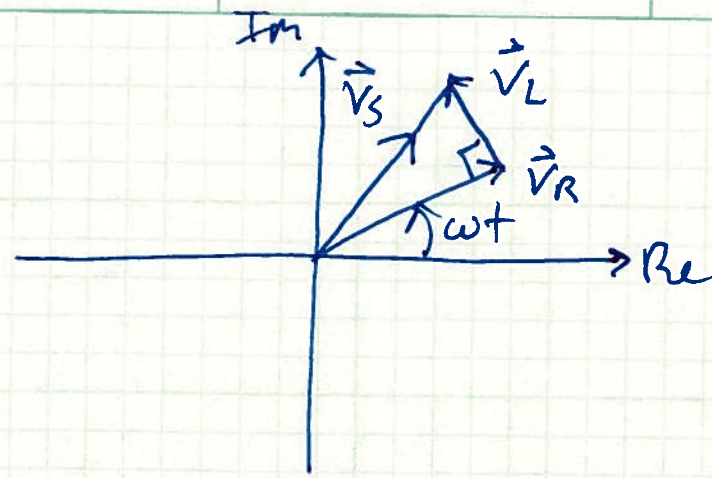
Now the voltage loop rule says:

$$V_S = V_R + V_L$$ which we can interpret geometrically.

$$\vec{V}_S = \vec{V}_R + \vec{V}_L$$ ⟸ phasors

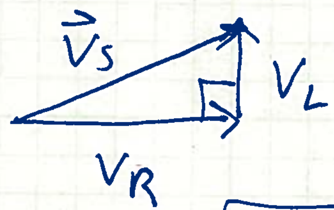
or





This is a phasor diagram for the voltages in the circuit

This is useful because we can use ~~the~~ geometry to figure out the relationships:



$$\therefore \quad |\vec{V}_S| = V_0 = \sqrt{|V_R|^2 + |V_L|^2}$$

$$|V_R| = I_0 R$$

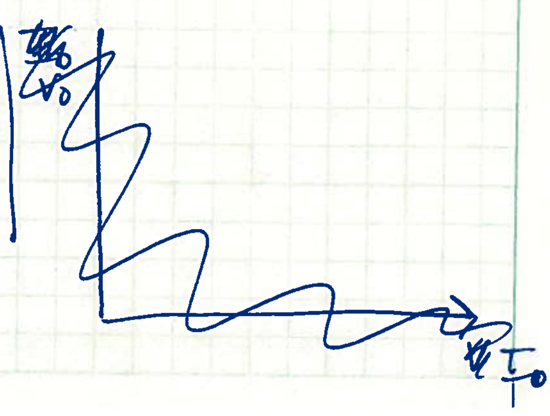
$$|V_L| = I_0 \, |(i\omega L)| = I_0(\omega L)$$

$$V_0 = \sqrt{(I_0 R)^2 + (\omega L I_0)^2}$$

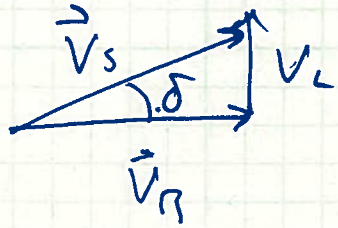
$$V_0 = I_0 \underbrace{\left[\sqrt{R^2 + (\omega L)^2}\right]}_{\text{Define } Z = \text{"impedance"}} = I_0 Z$$

Peak current $\boxed{I_0 = \dfrac{V_0}{\sqrt{R^2 + (\omega L)^2}}}$

Peak current drops as driving frequency increases



How about phase differences?



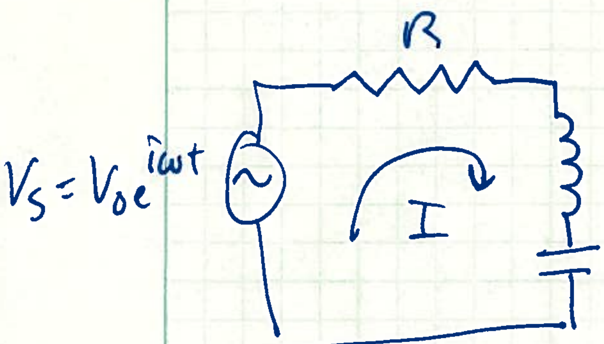
phase difference between V_R & V_S: (or I & V_S)

$$\delta = \tan^{-1}\left(\frac{|V_L|}{|V_R|}\right) = \tan^{-1}\left[\frac{\omega L I_0}{R I_0}\right]$$

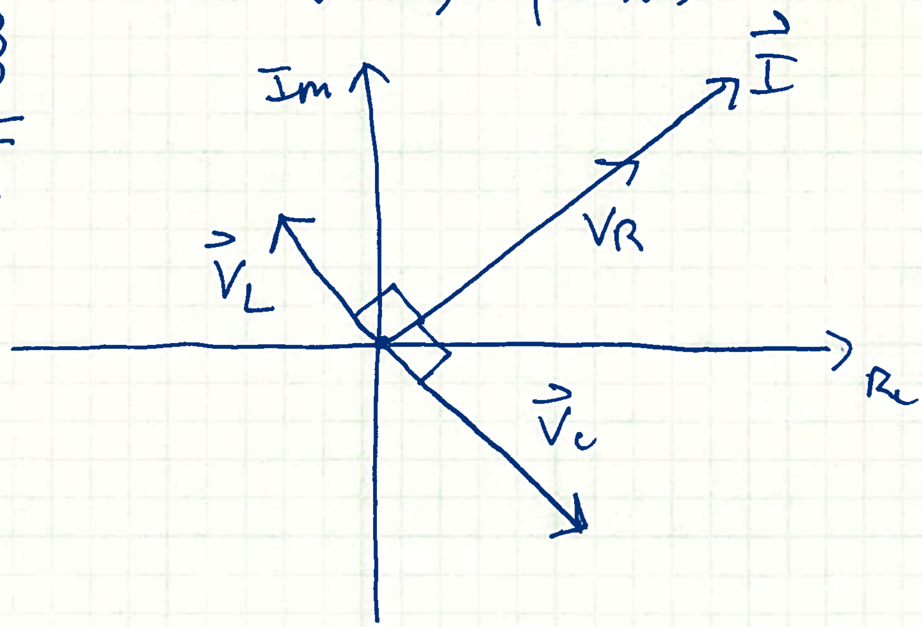
$$\boxed{\delta = \tan^{-1}\left(\frac{\omega L}{R}\right)}$$

Example

Driven RLC circuit:

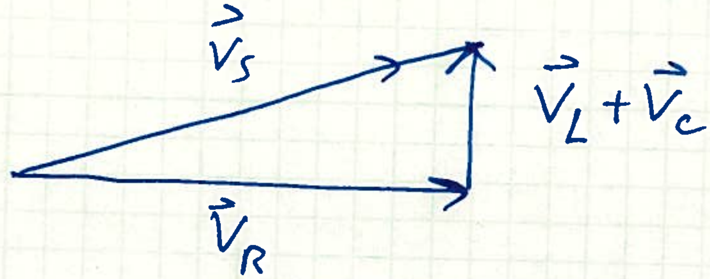


One current phasor, four voltage phasors:



Voltage Loop Rule: $\vec{V}_S = \vec{V}_R + \vec{V}_L + \vec{V}_C$

Note that $\vec{V}_L$ & $\vec{V}_C$ are in opposite directions.



$$\therefore \quad |\vec{V}_S| = V_0 = \sqrt{|V_R|^2 + \left(|\vec{V}_L + \vec{V}_C|\right)^2}$$

$$= \sqrt{(I_0 R)^2 + \left(I_0 \omega L - \frac{I_0}{\omega C}\right)^2}$$

$$= I_0 \underbrace{\sqrt{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2}}_{Z \,=\, \text{impedance of this circuit}}$$

$$\therefore \quad I_0 = \text{peak current} = \frac{V_0}{\sqrt{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2}}$$

$$= \frac{(V_0/L)}{\sqrt{\left(\frac{R^2}{L^2}\right) + \left(\omega - \frac{1}{\omega L C}\right)^2}}$$

$$\therefore \quad \frac{R}{L} = \gamma \quad , \quad \frac{1}{LC} = \omega_0^2$$

$$\boxed{I_0 = \frac{\omega (V_0/L)}{\sqrt{(\omega\gamma)^2 + (\omega^2 - \omega_0^2)^2}}}$$ Resonance when $\omega \approx \omega_0$.

Previously we found by solving the differential equation that

$$q_0 = \frac{(V_0/L)}{\sqrt{(\omega\gamma)^2 + (\omega^2 - \omega_0^2)^2}} \quad \text{or} \quad q(t) = q_0 e^{i\omega t}$$

which means that

$$I(t) = \dot{q}(t) = i\underbrace{\omega q_0}_{I_0 = \omega q_0} e^{i\omega t}$$

or $$I_0 = \frac{\omega (V_0/L)}{\sqrt{(\omega\gamma)^2 + (\omega^2 - \omega_0^2)^2}}$$

So we got the same result without calculating with the differential equation.