Why you should hate Sine and Cosine.

Sine and Cosine are not the best way to describe an oscillator.

Reason #1 They don't obey the normal rules of algebra.

Ex: $\frac{\cos\theta}{s} \stackrel{?}{=} \mathrm{co}\,\theta$? (of course not)

↑ s does not divide out.

Since they don't follow the rules of algebra, we must use long tables of trigonometric identities (!!) to manipulate them.

Reason #2 Sine and Cosine obscure the amplitude, phase, and frequency of an oscillation.

Ex: $\left.\begin{aligned} f(\theta) &= \cos\theta \\ g(\theta) &= \sin\theta \end{aligned}\right\}$ Question: what is the phase difference between $f(\theta)$ and $g(\theta)$?

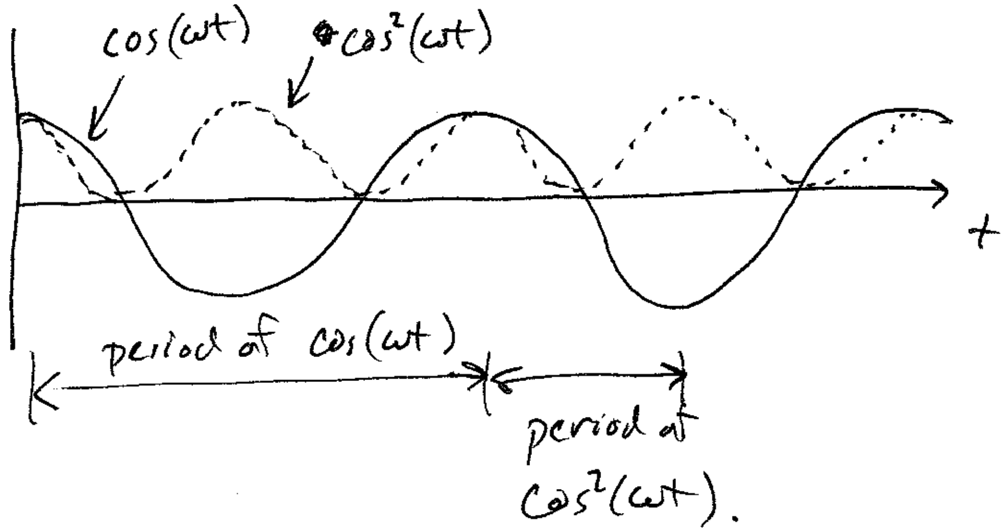
Answer: It appears to be zero, but $\cos\theta$ and $\sin\theta$ have a built-in phase difference of $\pi/2$. But you can't see that explicitly in their arguments

Ex: $f(\theta) = \cos\theta$, $g(\theta) = \cos^2\theta$. } Do these functions have the same amplitude?

Answer: They appear to have the same amplitude, but the peak-to-peak variation of $\cos\theta$ is 2 (−1 to 1) and the peak-to-peak variation of $\cos^2\theta$ is (zero to 1). So $\cos\theta$ has twice the amplitude as $\cos^2\theta$.

Ex: $f(\theta) = \cos(\omega t)$, $g(\theta) = \cos^2(\omega t)$ } Do these functions have the same frequency?

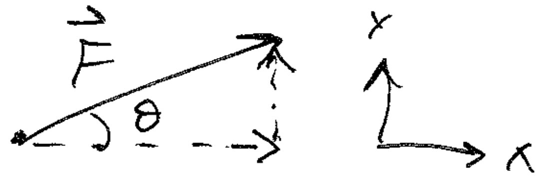
Answer: $\cos^2(\omega t)$ has twice the frequency of $\cos(\omega t)$:



We must use a ~~trig~~ trig identity to see this:

$$\cos^2(\omega t) = \frac{1}{2}\left(1 + \cos(\underbrace{2\omega}_{\text{twice the frequency}} t)\right)$$

The problem with Sine & Cosine is that they are designed to work well in geometry:



What is F_x?

Answer: simply $|\vec{F}|\cos\theta$

What is F_y?

Answer: simply $|\vec{F}|\sin\theta$.

However, they are awkward to use in algebra.

In most dynamical problems in physics, the nature of the problem is not fundamentally geometrical. So Sine & Cosine are not the best choice to describe the dynamics.

SHO Equation of Motion:

$$\ddot{x} + \omega_0^2 x = \emptyset$$

The solution must be a function which is proportional to its 2nd derivative. Sine & Cosine satisfy this requirement, but so do exponential functions.

Try $\quad x(t) = e^{\omega_0 t} \quad \leftarrow$ guessed solution.

Then $\quad \ddot{x}(t) = \omega_0^2 e^{\omega_0 t}$

Substitute:
$$\ddot{x} + \omega_0^2 x \stackrel{?}{=} \emptyset$$
$$\downarrow$$
$$\omega_0^2 x + \omega_0^2 x \stackrel{?}{=} \emptyset$$
$$2\omega_0^2 x \stackrel{?}{=} \emptyset$$

This doesn't work unless $x(t) = \emptyset$ (trivial solution)
The problem is that the two terms need to <u>cancel</u>, but in this case they added together.

So try instead:

$$x(t) = e^{+i\omega_0 t} \leftarrow \text{guessed solution}$$

Then $\ddot{x}(t) = -\omega_0^2 e^{i\omega_0 t}$ because $(i)^2 = -1$.

Substitute:
$$\ddot{x} + \omega_0^2 x \stackrel{?}{=} \emptyset$$
$$\downarrow \quad\quad \downarrow$$
$$-\omega_0^2 x + \omega_0^2 x \stackrel{?}{=} \emptyset$$
$$\emptyset = \emptyset \leftarrow \text{yes} \checkmark$$

So this solution satisfies the equation of motion.

But what does it mean to exponentiate a complex number? And how can this represent the position of an oscillator, which is always a real number?

Complex Analysis

Complex numbers can be thought of as a pair of numbers:

$$z = (x, y)$$

Addition is defined as:

$$z_1 + z_2 = (x_1 + x_2, \; y_1 + y_2)$$

and multiplication is defined as

$$z_1 z_2 = (x_1 x_2 - y_1 y_2, \; y_1 x_2 + x_1 y_2)$$

The multiplication rule is complicated, but it is easy to remember if you define $i \equiv \sqrt{-1}$ and let

$$z_1 = x_1 + i y_1$$

and $$z_2 = x_2 + i y_2$$

Then, just following the normal rules of algebra:

$$z_1 z_2 = (x_1 + i y_1)(x_2 + i y_2)$$

$$= x_1 x_2 - y_1 y_2 + i y_1 x_2 + i x_1 y_2$$

$$= (x_1 x_2 - y_1 y_2) + i (y_1 x_2 + y_2 x_1)$$

So we usually do not use the (x, y) notation, and instead we use the $x + iy$ notation.

Complex conjugation is defined by:

$$z^* \equiv (x+iy)^* \equiv x - iy$$

reverse the sign of the imaginary component.

When you multiply a "c-number" by its own complex conjugate, the result is

"complex number"

$$zz^* = (x+iy)(x-iy)$$

$$zz^* = (x^2+y^2) + i\underbrace{(xy-yx)}_{\text{zero}}$$

$$\boxed{zz^* = x^2+y^2}$$

Division: If a ~~com~~ c-number is in the denominator, the trick is to multiply the numerator and denominator by its complex conjugate:

$$\frac{1}{x+iy} = \frac{(x-iy)}{(x+iy)(x-iy)} = \frac{x-iy}{x^2+y^2}$$

$$= \left(\frac{x}{x^2+y^2}\right) - i\left(\frac{y}{x^2+y^2}\right)$$

real part $\qquad$ imaginary part.

Question: Can we define a cosine function for a complex number:

$$\cos(z) = ??? \leftarrow \text{What does this mean?}$$

Answer: Yes, use the Taylor Series definition of Cosine:

$$\cos(x) = 1 - \frac{1}{2}x^2 + \frac{1}{4!}x^4 + \ldots$$

So let's define:

$$\boxed{\cos(z) \equiv 1 - \frac{1}{2}z^2 + \frac{1}{4!}z^4 + \ldots}$$

Note that $z^2 = zz$, not zz^*.

Example: What is $\cos(i)$?

Answer: $\cos(i) = 1 - \frac{1}{2}(i)^2 + \frac{1}{4!}(i)^4 + \ldots$

$$= \underbrace{1 + \frac{1}{2} + \frac{1}{4!} + \ldots}_{\cos(i) \text{ is a completely } \underline{\text{real}} \text{ number!}}$$

How about exponentials? Can we exponentiate a complex number?

Answer: Yes, just extend the meaning of the exponential function using its Taylor Series:

real numbers: $e^x = 1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \ldots$

So we <u>define</u>:

$$e^z \equiv 1 + z + \frac{1}{2!}z^2 + \frac{1}{3!}z^3 + \ldots$$

Here's the best part: What happens if we exponentiate a <u>purely</u> <u>imaginary</u> number?

Let $z = iy \Leftarrow$ no real part, purely imaginary

Then $e^z = e^{iy} = 1 + (iy) + \frac{1}{2!}(iy)^2 + \frac{1}{3!}(iy)^3 + \ldots$

$$= \underbrace{\left(1 - \frac{1}{2}y^2 + \frac{1}{4!}y^4 + \ldots\right)}_{\cos(y)} + i\underbrace{\left(y - \frac{1}{3!}y^3 + \frac{1}{5!}y^5 + \ldots\right)}_{\sin(y)}$$

$$\therefore \boxed{e^{iy} = \cos(y) + i\sin(y)}$$

Often this is written using θ instead of y:

$$\boxed{e^{i\theta} = \cos(\theta) + i\sin(\theta)}$$ "Euler's Formula" (1748)

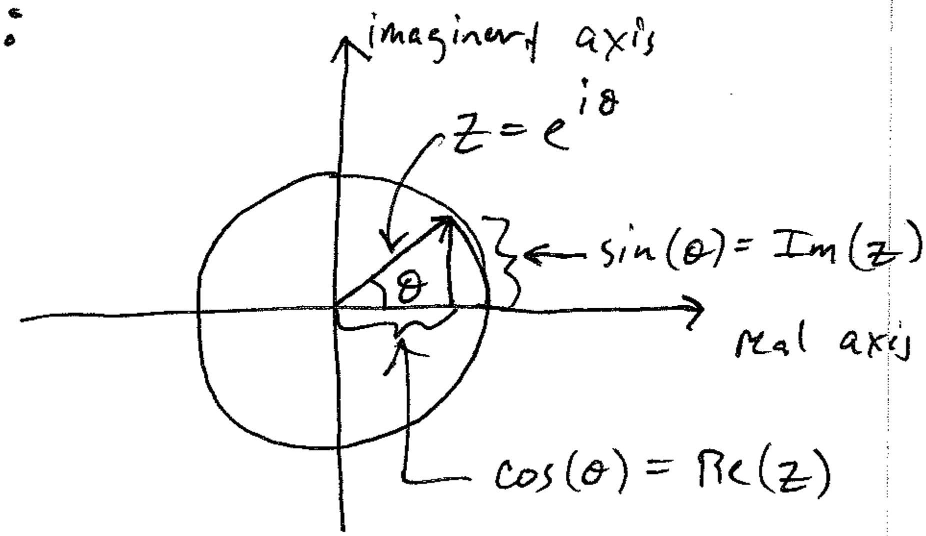
Here we have seen the Sine and Cosine function appear, as if by magic, in a context which is completely algebraic, not geometrical. This shows that there is a deep connection between algebra and geometry.

Euler's Formula says: when you exponentiate an imaginary number, you get another complex number whose real and imaginary components oscillate forever as the argument of the exponential increases.

Complex numbers as vectors

50 years after Euler, Wessel realized that Euler's Formula allows us to think of complex numbers as being vectors in the complex plane:

Wessel's Picture:



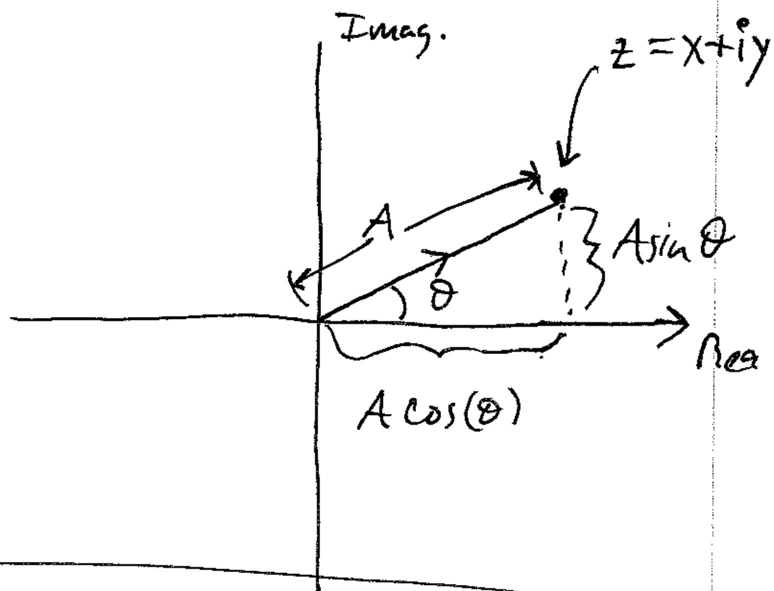
The magnitude of $e^{i\theta}$ is, according to the pythagorean theorem,

$$|z| = \sqrt{\cos^2\theta + \sin^2\theta} = 1$$

$\therefore$ $\boxed{e^{i\theta} \text{ is a unit vecter in the complex plane. It has length 1.}}$

Any complex number can be represented this way simply by scaling the unit vector:

Let $z = x + iy$.



Then choose

$A = \sqrt{x^2+y^2}$

and $\theta = \tan^{-1}(y/x)$

Then $\boxed{z = Ae^{i\theta}}$.

$Ae^{i\theta}$ is a vector in the complex plane. It has magnitude A and makes an angle of θ with the real axis.

So we can think of complex numbers as vectors, and we can write them using two notations:

① Cartesian notation: $z = x + iy$ ← a vector

② Polar notation: $z = Ae^{i\theta}$ ← the same vector.

We can go back and forth between notations:

$$\boxed{\begin{aligned} A &= \sqrt{x^2+y^2} \\ \theta &= \tan^{-1}(y/x) \end{aligned}}$$

Cartesian to Polar

$$\boxed{\begin{aligned} x &= A\cos(\theta) \\ y &= A\sin(\theta) \end{aligned}}$$

Polar to Cartesian.

Polar form is the ~~mo~~ most useful, because the amplitude and phase are shown explicitly:

$$z = \underbrace{A}_{\text{Amplitude}} e^{\overbrace{i\theta}^{\text{phase}}}$$

Also, every symbol used here obeys the normal rules of algebra.

Example: Can we divide by e? Answer: Yes.

$$\frac{z}{e} = \frac{Ae^{i\theta}}{e} = \left(Ae^{i\theta}\right)\left(e^{-1}\right) = Ae^{i\theta-1}$$

$$\text{or } \underbrace{\left(\frac{A}{e}\right)}_{\text{amplitude}} e^{\overbrace{i\theta}^{\text{phase}}}$$

Polar form has these characteristics:

① The exponential argument is purely ~~rea~~ imaginary:

$$e^{i\theta}$$

② The amplitude is purely real:

$$A \leftarrow \text{real}$$

To manipulate a complex number into polar form, just use the normal rules of algebra. (No need to use trig identities.):

Example: $z = e^{i\omega t - \alpha}$ ← Is this polar form?

Answer: No, because the exponent argument has a real component. But we can put it into polar form:

$$z = e^{i\omega t - \alpha} = \left(e^{i\omega t}\right)\left(e^{-\alpha}\right) = \underbrace{\left(e^{-\alpha}\right)}_{\text{purely real}} e^{\overbrace{i(\omega t)}^{\text{purely imaginary}}}$$

So this is polar form.

What's the amplitude? Answer: $e^{-\alpha}$

What's the phase? Answer: ωt.

Example: $z = i e^{i\omega t}$ ← Is this polar form?

Answer: No, because the (i) in front is not purely real.

Manipulate it into polar form:

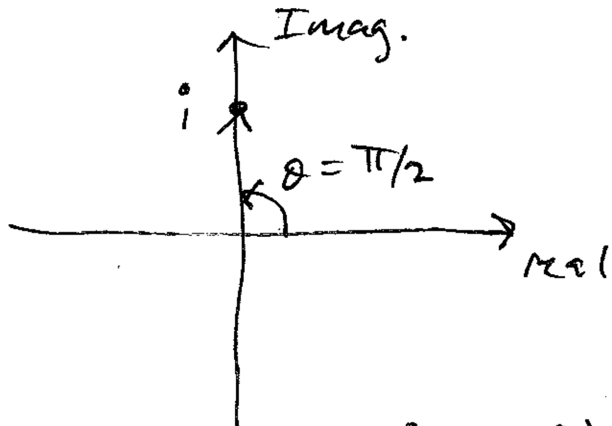
$$z = \underbrace{i}\, e^{i\omega t}$$

First convert (i) to polar form:

$$i = e^{i\pi/2} \quad \text{because} \quad e^{i\pi/2} = \underbrace{\cos(\pi/2)}_{\emptyset} + i\,\underbrace{\sin(\pi/2)}_{1}$$

$$= i$$

Or picture the (i) vector in the complex plane:



$$\text{So} \quad z = (i)\left(e^{i\omega t}\right) = \left(e^{i\pi/2}\right)\left(e^{i\omega t}\right) = e^{i(\omega t + \pi/2)} \checkmark$$

Polar form.

What's the amplitude? Answer: 1
What's the phase? Answer: $\omega t + \pi/2$.

This example shows that

$$\boxed{\text{Multiplication by } (i) \text{ represents a phase shift of } \pi/2.}$$

Example: $z = -e^{i\omega t}$ ← Is this polar form?

Answer: Yes, more or less. But there is one subtlety:

What's the phase?

$$z = -e^{i\omega t} = \underbrace{(-1)}_{-1 = e^{i\pi}}\left(e^{i\omega t}\right)$$

$$z = \left(e^{i\pi}\right)\left(e^{i\omega t}\right)$$

$$z = e^{i\overbrace{(\omega t + \pi)}^{}}$$

$$\underbrace{}_{\text{phase.}}$$

So the phase is $\underline{\omega t + \pi}$. The amplitude is 1.

$$\boxed{\text{Multiplication by } -1 \text{ represents a phase shift of } \pi.}$$

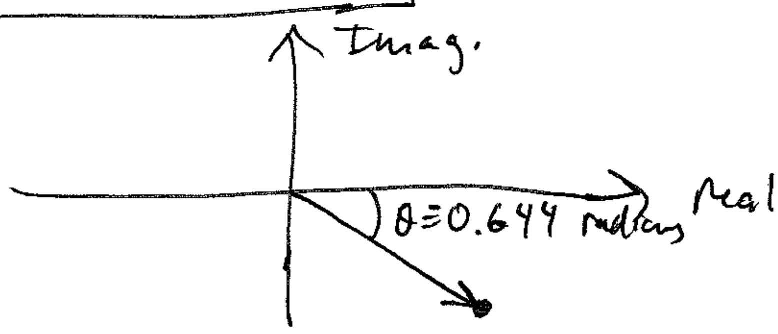
Example: $z = 4 - 3i$ ← Manipulate it into Polar form

$$A = \sqrt{4^2 + (-3)^2} = 5$$

$$\theta = \tan^{-1}(-3/4) = -0.644 \text{ radians}$$

$$\therefore \boxed{z = 5e^{-i(0.644)}}$$

Picture it:



Application to Physics

In classical physics, all quantities are real numbers. So how can we use complex numbers?

Answer: We will solve the equations using complex numbers, and then at the very end, we will simply agree to take the real components.

In fact, we usually won't even bother to take the real part explicitly. We will just write down the complex solution, while keeping in mind that ~~the~~ we only care about the real part.

Example: Simple Harmonic Oscillator

$$\ddot{x} + \omega_0^2 x = 0.$$

Guessed Solution: $x(t) = Ae^{i(\omega_0 t + \delta)}$

Check it:

$$\dot{x}(t) = i\omega_0 A e^{i(\omega_0 t + \delta)}$$

$$\ddot{x}(t) = -\omega_0^2 A e^{i(\omega_0 t + \delta)}$$

Substitute:

$$\ddot{x} + \omega_0^2 x = 0$$

$$\left(-\omega_0^2 A e^{i(\omega_0 t + \delta)}\right) + \omega_0^2 \left(A e^{i(\omega_0 t + \delta)}\right) = 0$$

$$0 = 0 \checkmark$$ Yes, it works.

Final Solution:

$$\boxed{x(t) = Ae^{i(\omega_0 t + \delta)},\quad \text{A \& δ to be determined by the initial conditions.}}$$

Of course, we know that the actual position of the oscillator as a function of time is given by the <u>real part</u> of this solution:

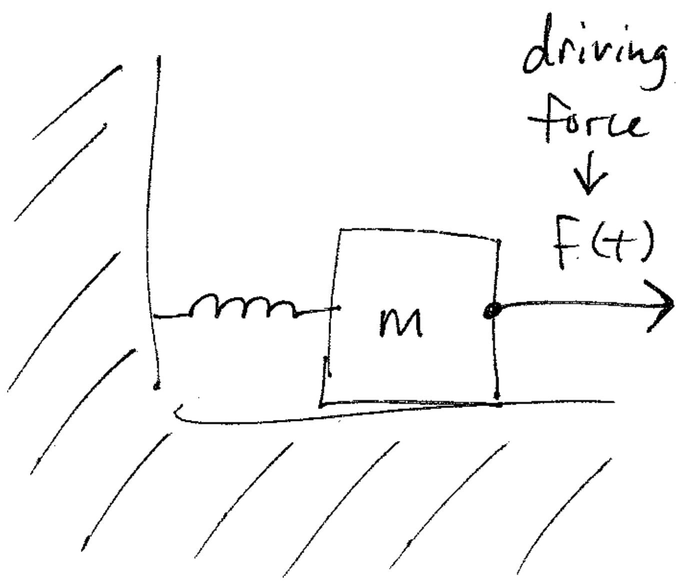
$$\mathrm{Re}(x(t)) = \mathrm{Re}\left(Ae^{i(\omega_0 t + \delta)}\right)$$
$$= A\cos(\omega_0 t + \delta)$$

But we won't even bother to write down this final step.

<u>Forced Oscillator</u>

Imagine a mass on a spring which is subject to a <u>second force</u> which varies in time. The equation of motion is

$$F = ma$$
$$-kx + F(t) = m\ddot{x}$$



$$\boxed{\ddot{x} + \omega_0^2 x = \frac{F(t)}{m}}$$ where $\omega_0 \equiv \sqrt{\frac{k}{m}}$

To be concrete, let's assume that the force has the form:

$$F(t) = F_0 \cos(\omega_f t),$$

"forcing frequency"

where ω_f is the frequency of the motor which creates the force. Note that ω could be any real number. We can choose ω_f freely by choosing different motor rotation speeds, for example.

ω_0 = natural frequency of the oscillator, with no external force present.

ω_f = frequency of the external force, which we can choose.

Let's solve the problem with complex exponentials.

Let's imagine that we are solving a fictious problem where x and F are complex. Let

$$x \equiv x_r + i x_i$$

real part of x — imaginary part of x

and $F = F_r + iF_i$

F_r: real part; F_i: imaginary part

Then our fictious equation of motion is:

$$\frac{d^2(x_r + i x_i)}{dt^2} + \omega_0^2 (x_r + i x_i) = \frac{F_r + iF_i}{m}$$

Gather together real & imaginary parts:

$$\left[\frac{d^2 x_r}{dt^2} + \omega_0^2 x_r - \frac{F_r}{m}\right] + i\left[\frac{d^2 x_i}{dt^2} + \omega_0^2 x_i - \frac{F_i}{m}\right] = \phi$$

For this to be true at all times, both the real and imaginary parts should be zero at all times:

$$\boxed{\begin{aligned} &\frac{d^2 x_r}{dt^2} + \omega_0^2 x_r = \frac{F_r}{m} \\ \text{and } &\frac{d^2 x_i}{dt^2} + \omega_0^2 x_i = \frac{F_i}{m} \end{aligned}}$$

This is the true physics equation of motion.

This is exactly the same equation of motion, just written for x_i rather than x_r.

$\boxed{\text{By solving the fictious (complex) equation of motion, we are effectively solving the true physics equation. It is the } \underline{\text{real}} \text{ component. (Or the imaginary component as well.)}}$

So let's assume a driving force which looks like:

$$F(t) = F_0 e^{i\omega_f t}$$, which really means $$F(t) = F_0 \cos(\omega_f t).$$

And let's pretend that x is a complex variable.

Now we solve:

$$\boxed{\ddot{x} + \omega_0^2 x = \frac{F_0}{m} e^{i\omega_f t}}$$

(recall that $\omega_f \neq \omega_0$ in general!)

Guessed solution: Let's try our old solution from the simple harmonic oscillator:

$$x(t) = A e^{i(\omega_0 t + \delta)}$$, which really means $$x(t) = A\cos(\omega_0 t + \delta)$$

Here we are guessing that the oscillator still goes at its old, natural frequency, ω_0.

Is this guess correct?

Try it:

$$\ddot{x} = -\omega_0^2 x$$

Therefore, when we substitute,

$$-\omega_0^2 x + \omega_0^2 x \stackrel{?}{=} \frac{F_0}{m} e^{i\omega_f t}$$

$$\emptyset = \frac{F_0}{m} e^{i\omega_f t}$$

$\boxed{F_0 = \emptyset}$ ← This solution only works if the driving force is <u>zero</u>.

We need a better guess. This time, let's guess that it oscillates at the driving frequency (ω), not ω_0:

Guessed Solution: $x(t) = Ae^{i(\omega_f t + \delta)}$ ← Forcing frequency or driving frequency

Try it: $\ddot{x} = -\omega_f^2 x = -\omega_f^2 \left(Ae^{i(\omega_f t + \delta)}\right)$

Therefore when substituting,

$$-\omega_f^2\left(Ae^{i(\omega_f t+\delta)}\right) + \omega_0^2\left(Ae^{i(\omega_f t+\delta)}\right) = \frac{F_0}{m} e^{i\omega_f t}$$

All the factors of $e^{i\omega_f t}$ cancel:

$$Ae^{i\delta}\left(\omega_0^2 - \omega_f^2\right) = \frac{F_0}{m}$$

$$\boxed{A = \frac{F_0 e^{-i\delta}}{m\left(\omega_0^2 - \omega_f^2\right)}}$$

A must be real, because it is the amplitude of a complex number in polar form. Therefore the Right Hand Side must be real:

$$e^{-i\delta} = \text{real number} \quad (\text{because } F_0, m, \omega_0^2, \omega^2 \text{ are real.})$$

$$\underbrace{\cos(\delta) - i\sin(\delta)} = \text{real number}$$

$$\sin(\delta) = \emptyset$$

$$\Rightarrow \boxed{\delta = \emptyset}$$

The phase shift is <u>zero</u>. What does this mean?

<u>Answer</u>: By assumption, the driving force has a phase of zero: (at $t=\emptyset$)

$$F(t) = F_0 e^{i\omega_f t} \leftarrow \text{our assumed driving force.}$$

This just means that we have chosen $t=\emptyset$ to be a maximum point in the driving function.

The position of the oscillator, however, we have allowed to have a phase: (at $t=\emptyset$)

$$x(t) = Ae^{i(\omega_f t + \delta)}$$

↑ we allow the oscillator to have a non-zero phase at $t=\emptyset$.

But after substituting into the Eq. of motion, we find that in order for the guessed solution to work, we must have $\delta = \phi$ after all.

$\Rightarrow$ The oscillator moves in phase with the driving force. When the driving force is maximal, the oscillator position is also maximal.

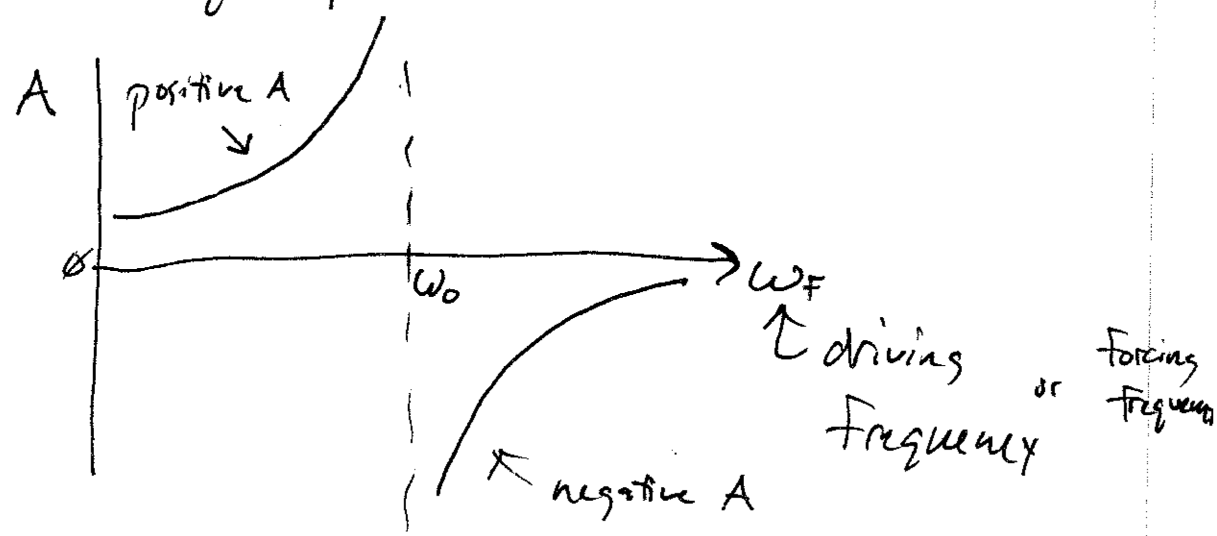
(however, see the caveat below.)

So our result is

$$\boxed{A = \frac{F_0}{m(\omega_0^2 - \omega^2)}}$$

Our guessed solution works, but the amplitude is no longer a free parameter determined by initial conditions. Instead, the amplitude is fixed by the magnitude of the driving force, the mass, and the driving frequency and natural frequency.

Amplitude vs Driving Frequency:
The size of the amplitude depends on how close the driving frequency is to the natural frequency:



Comments:

① If the driving frequency is small compared to ω_0, then A is <u>small</u> and <u>positive</u>. (small oscillation, 100% in phase with driving force.)

② If driving frequency is large compared to ω_0, then A is <u>small</u> and <u>negative</u>. Negative A means that it is 180° out-of-phase with the driving force.

③ If driving frequency is similiar to the natural frequency ω_0, then the amplitude of oscillation will be <u>large</u>. This is an example of <u>resonance</u>.