

Maxwell's Equations: (In Integral Form)

$$\oint_{\text{surface}} \vec{E} \cdot \hat{n} da = \frac{Q_{\text{enclosed}}}{\epsilon_0} \quad \text{Gauss' Law}$$

$$\oint_{\text{surface}} \vec{B} \cdot \hat{n} da = 0 \quad \text{Gauss' Law for Magnetism}$$

$$\oint_{\text{curve}} \vec{E} \cdot d\vec{l} = -\frac{d\Phi_B}{dt} \quad \text{Faraday's Law}$$

$$\oint_{\text{curve}} \vec{B} \cdot d\vec{l} = \mu_0 I + \mu_0 \epsilon_0 \frac{d\Phi_E}{dt} \quad \text{Modified Ampere's Law}$$

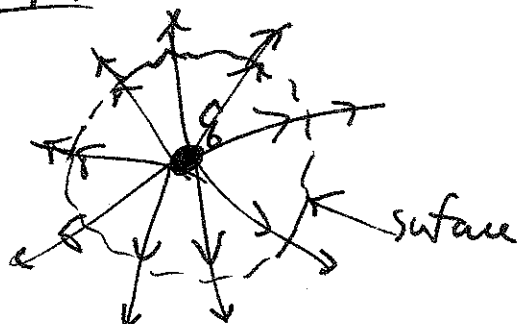
Broadly speaking, the left side of the equations tell us what the \vec{E} & \vec{B} field do, and the right side ~~says~~ tells us what causes them to do it.

Gauss Law: $\oint_{\text{surface}} \vec{E} \cdot \hat{n} da = \frac{Q_{\text{enclosed}}}{\epsilon_0}$

Flux of \vec{E} through a closed surface

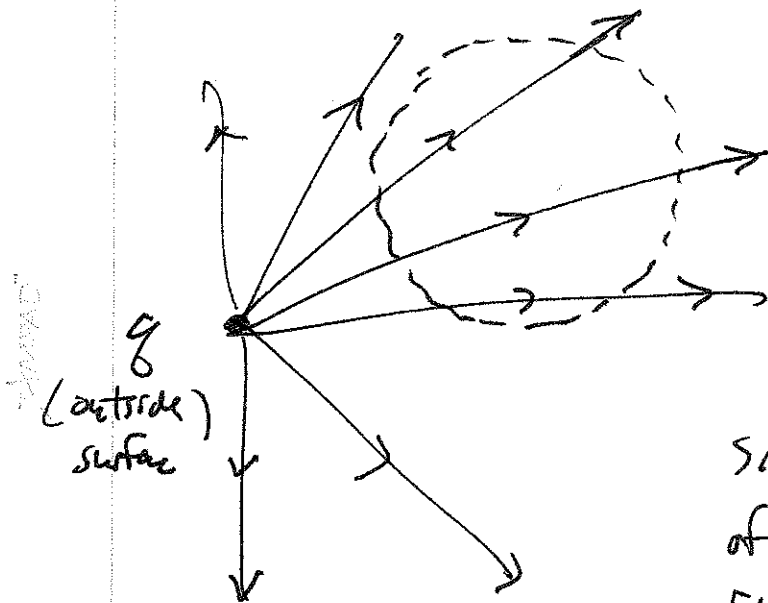
Is caused by the presence of charge inside that surface.

Examples:



← Non-zero flux caused by charge q.

Conversely, if there ^{are} no charges inside, then the flux must be zero:



Net Flux is zero: every electric field line penetrates through.

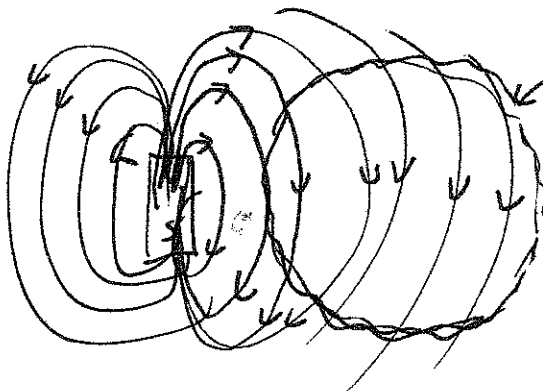
Since the flux is a measure of the number of electric field lines which ~~penetrate~~ originate inside, here the flux is zero.

Gauss' Law for Magnetism

$$\oint_{\text{surface}} \vec{B} \cdot \hat{n} da = 0$$

Flux through a closed surface

↑ is always zero



closed surface.

Flux of \vec{B} must be zero \Rightarrow there are no magnetic monopoles which can act as a source of \vec{B} field lines.

So Gauss' Law can be summarized.

"Electric Field lines can begin and end on charges. If charge is present, then there will be a non-zero flux through a surface

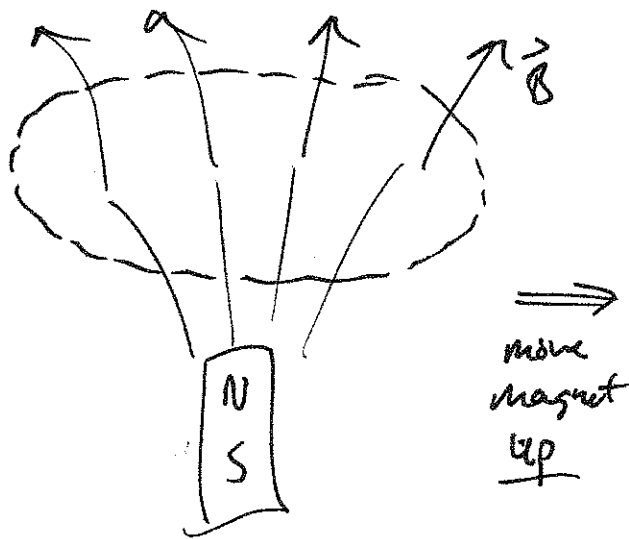
Magnetic field lines can never begin or end anywhere, because there are no magnetic monopoles. They can only go in circles?

Faraday's Law

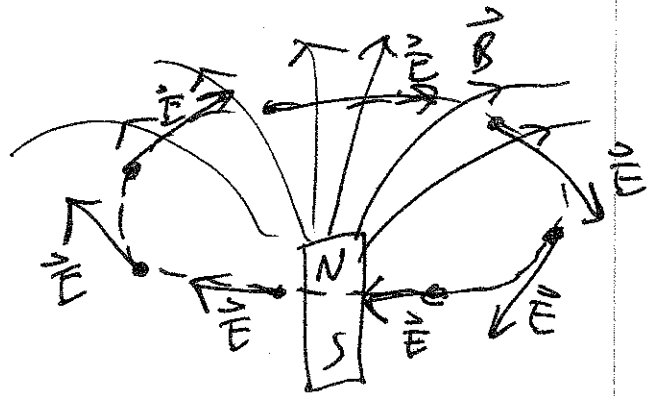
$$\oint \vec{E} \cdot d\vec{l} = -\frac{d\Phi_B}{dt}, \quad \Phi_B = \int \vec{B} \cdot \hat{n} da$$

open surface

"Electric Field lines will go in circles..."
"when the magnetic flux is changing in time."

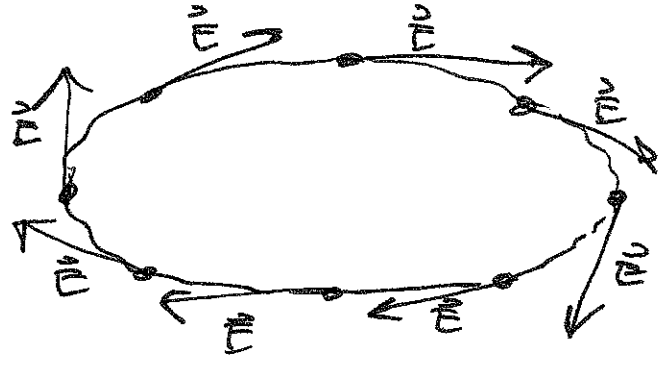


⇒
move magnet up



Φ_B is larger now.
This creates a circulating electric field

Just the \vec{E} field:



\vec{E} field is going in a circle due to the changing magnetic flux.

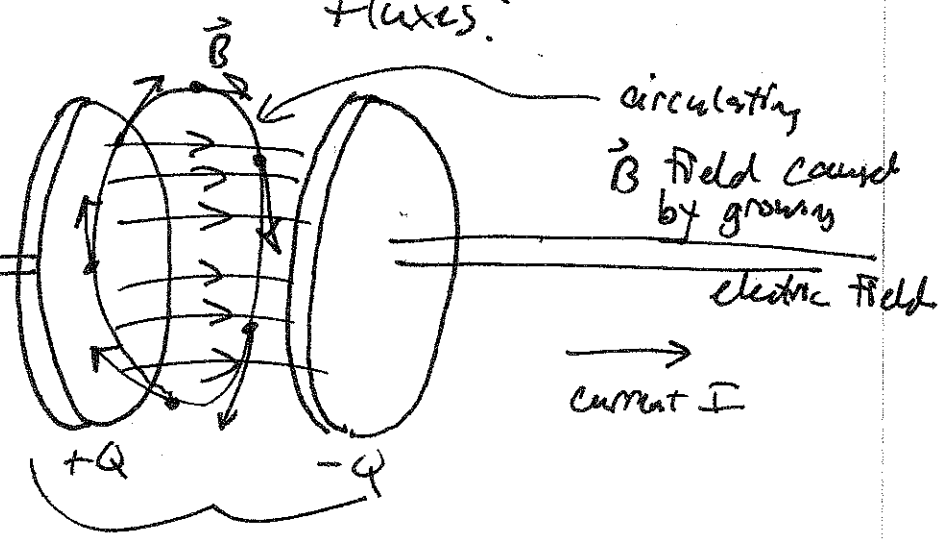
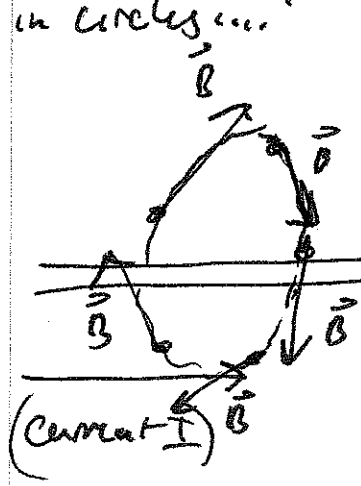
Modified Ampere's Law

$$\oint \vec{B} \cdot d\vec{l} = \mu_0 I + \mu_0 \epsilon_0 \frac{d\Phi_E}{dt}$$

Curve
"Magnetic fields go in circles..."

"caused by currents..."

"and/or caused by changing electric fluxes."



Charging capacitor plates, electric field is growing.

In Vacuum, Maxwell's Equations look like:

(set all charges and currents equal to zero)

$$\oint_{\text{surface}} \vec{E} \cdot \hat{n} da = 0$$

$$\oint_{\text{surface}} \vec{B} \cdot \hat{n} da = 0$$

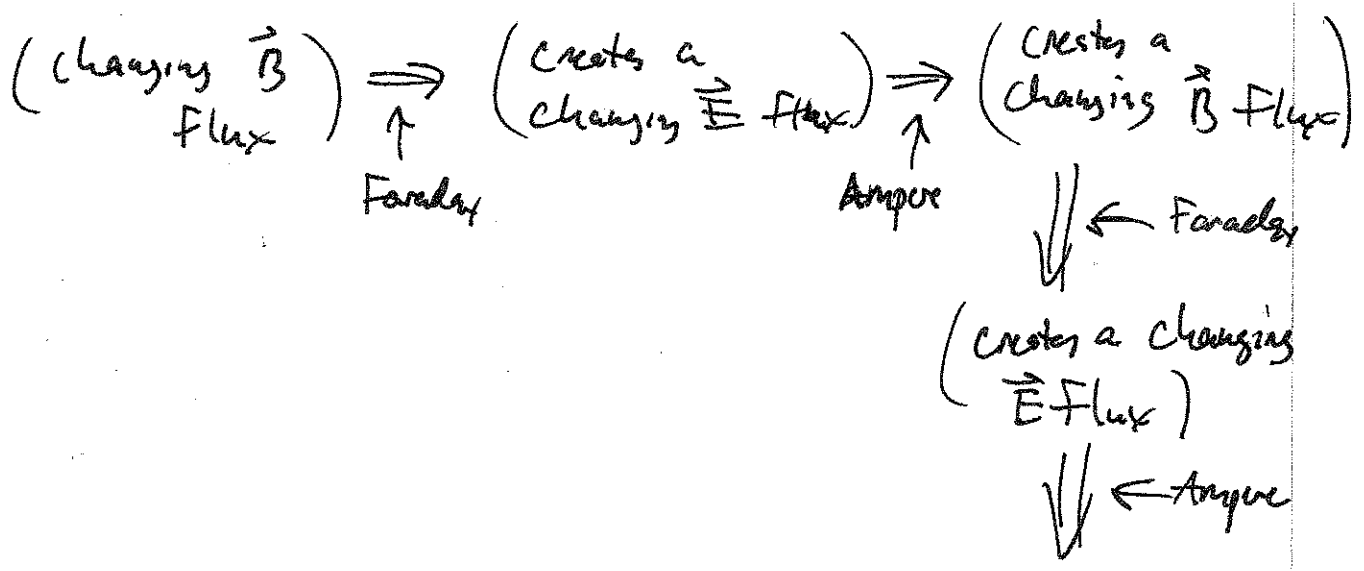
} "Neither \vec{E} nor \vec{B} can start or stop anywhere"

$$\oint \vec{E} \cdot d\vec{l} = - \frac{d\phi_B}{dt}$$

$$\oint \vec{B} \cdot d\vec{l} = \mu_0 \epsilon_0 \frac{d\phi_E}{dt}$$

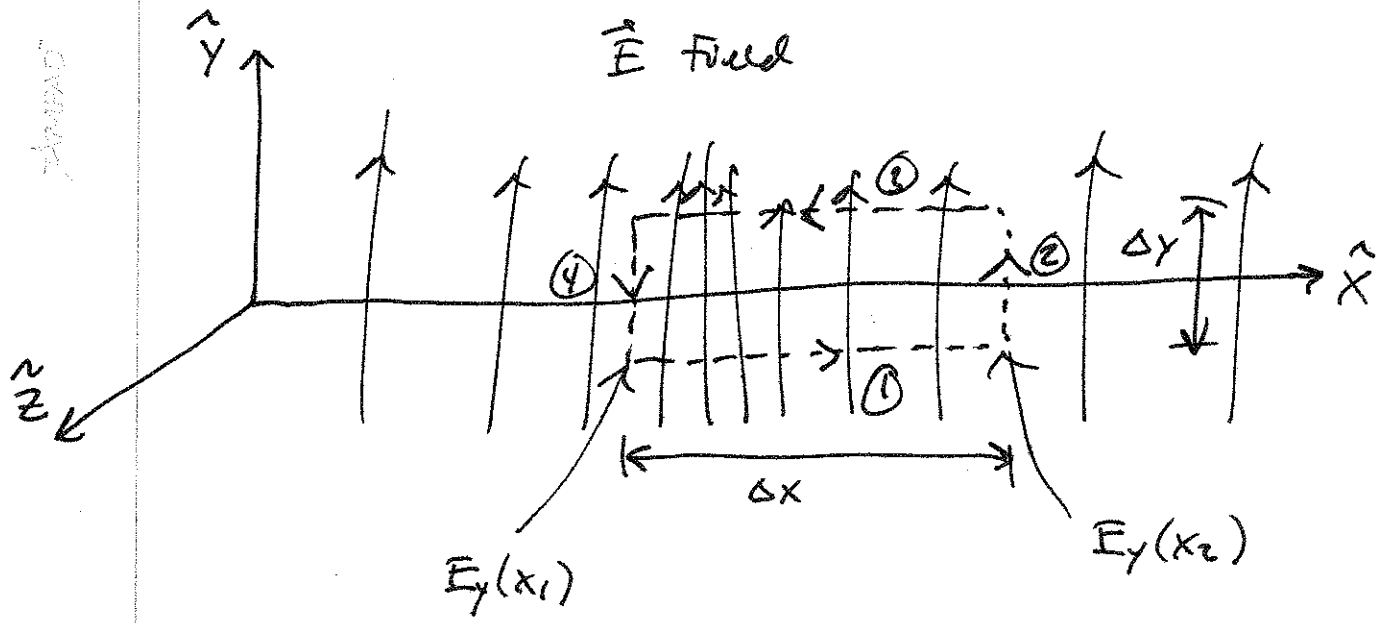
} "Both \vec{E} & \vec{B} will go ~~in~~ in circles, caused by a changing flux of the other field"

Faraday's Law & Ampere's Law work together to create propagating waves in the \vec{E} & \vec{B} fields. It works like this:



We can show that Faraday's Law & Ampere's Law imply that \vec{E} and \vec{B} both satisfy the Classical Wave Equation.

Argument: Apply Faraday's Law around a square region with an electric field in the \hat{y} direction.



The LHS hand side of Faraday's Law says

$$\oint_{\text{square}} \vec{E} \cdot d\vec{l} = \underbrace{E_y(x_2) \Delta y}_{\text{segment 2}} - \underbrace{E_y(x_1) \Delta y}_{\text{segment 4}} = [E_y(x_2) - E_y(x_1)] \Delta y$$

(segments 1 & 3 contribute zero because there $d\vec{l}$ is perpendicular to \vec{E}).

Now $E_y(x_2) - E_y(x_1)$ is the change in E_y over the small distance Δx .

Letting $\Delta x \rightarrow 0$,

We can write this change as

$$E_y(x_2) - E_y(x_1) = \Delta E_y \approx \frac{\partial E_y}{\partial x} \Delta x$$

Then the left hand side of Faraday's Law says

$$\oint_{\text{square}} \vec{E} \cdot d\vec{e} = \left(\frac{\partial E_y}{\partial x} \Delta x \right) \Delta y \quad (\text{LHS})$$

Left hand

side of Faraday's Law.

The Right hand side of Faraday's Law says that this circulation in \vec{E} must be caused by a changing flux of \vec{B} :

$$\frac{d\Phi_B}{dt} = \frac{d}{dt} \int \vec{B} \cdot \vec{n} da \approx \frac{d}{dt} (B_z \Delta x \Delta y) \quad (\text{RHS})$$

open surface
bounded by
square

↑
Approximate
 \vec{B} as roughly
constant over
the small area.

Putting the Left Hand Side together with the Right Hand Side:

$$\frac{\partial E_y}{\partial x} \Delta x \Delta y = \frac{dB_z}{dt} \Delta x \Delta y$$

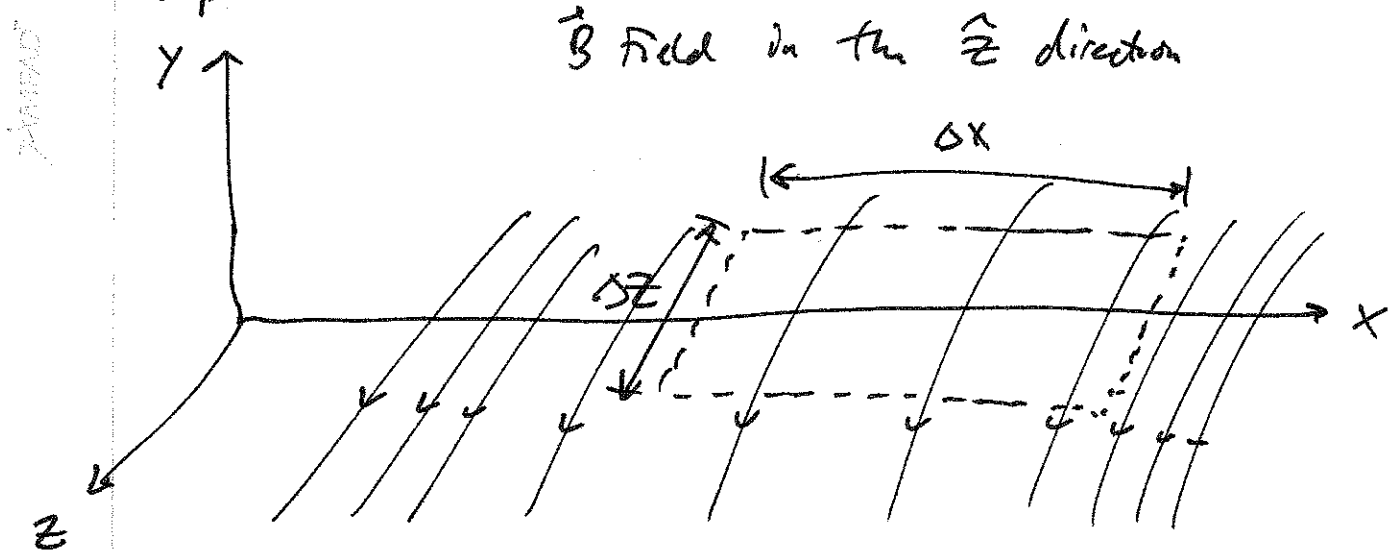
or

$$\frac{\partial E_y}{\partial x} = \frac{dB_z}{dt} = \frac{\partial B_z}{\partial t}$$

$$\boxed{\frac{\partial E_y}{\partial x} = -\frac{\partial B_z}{\partial t}} \quad (1)$$

This is Faraday's Law.

We can make a similar argument using modified Ampere's Law's



The mathematics is identical because the modified Ampere's Law is completely analogous to Faraday's Law (in the absence of charges & currents.)

The Result is

$$\boxed{\frac{\partial B_z}{\partial x} = -\mu_0 \epsilon_0 \frac{\partial E_y}{\partial t}} \quad (2)$$

This is the modified Ampere's Law

Now put (1) & (2) together. Take $\frac{\partial}{\partial x}$ of

Eq. (1):

$$\frac{\partial}{\partial x} \left(\frac{\partial E_y}{\partial x} \right) = - \frac{\partial}{\partial x} \left(\frac{\partial B_z}{\partial t} \right) = - \frac{\partial}{\partial t} \left(\frac{\partial B_z}{\partial x} \right)$$

Substitute
from Eq. (2).

$$= - \frac{\partial}{\partial t} \left(-\mu_0 \epsilon_0 \frac{\partial E_y}{\partial t} \right)$$

$$= \mu_0 \epsilon_0 \frac{\partial^2 E_y}{\partial t^2}$$

$$\therefore \boxed{\frac{\partial^2 E_y}{\partial x^2} = \mu_0 \epsilon_0 \frac{\partial^2 E_y}{\partial t^2}} \quad \text{Classical Wave Equation}$$

We can immediately see that:

- ① EM waves propagate with a phase velocity of $v_p = \frac{1}{\sqrt{\mu_0 \epsilon_0}} = 3 \times 10^8 \text{ m/s}$ in vacuum
speed of light
- ② EM waves can be identified with light.
- ③ EM waves will display no dispersion in vacuum.
 - \Rightarrow Pulses will propagate forever.
 - \Rightarrow Group velocity equal phase velocity
 - $\Rightarrow v_p$ is independent of wavelength

Maxwell's Equations in Integral form tell us about the global properties of \vec{E} & \vec{B} . This can be very useful, but in many cases it is also useful to know how \vec{E} & \vec{B} are behaving at a single point in space. For this we need to re-cast Maxwell's Equations in Differential Form.

The Operator $\vec{\nabla}$ (or ∇) (Gradient)

Let $F(x, y, z)$ be a scalar function of position. Then

$$\begin{aligned} \vec{\nabla} F &= \nabla F \equiv \frac{\partial F}{\partial x} \hat{x} + \frac{\partial F}{\partial y} \hat{y} + \frac{\partial F}{\partial z} \hat{z} \\ &= \left(\frac{\partial F}{\partial x}, \frac{\partial F}{\partial y}, \frac{\partial F}{\partial z} \right) \text{ a vector.} \end{aligned}$$

sometimes we put the arrow above ∇ , and

sometimes we don't

We can think of $\vec{\nabla}$ as being a vector:

$$\begin{aligned} \vec{\nabla} &\equiv \hat{x} \frac{\partial}{\partial x} + \hat{y} \frac{\partial}{\partial y} + \hat{z} \frac{\partial}{\partial z} \\ &= \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right) \end{aligned}$$

~~More~~

$\vec{\nabla}$ can act in 3 ways:

① Operate on a scalar ^{Function}, producing a vector:

$$\vec{\nabla} F = \text{vector} = \left(\frac{\partial F}{\partial x}, \frac{\partial F}{\partial y}, \frac{\partial F}{\partial z} \right) \text{ (The Gradient)}$$

② Operate on a vector function, via a dot-product:

$$\vec{\nabla} \cdot \vec{v} = \text{scalar} = \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z}$$

⊗ This is called "the Divergence"

③ Operate on a vector function, via a cross-product

$$\vec{\nabla} \times \vec{v} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ v_x & v_y & v_z \end{vmatrix}$$

$$= \hat{x} \left(\frac{\partial v_z}{\partial y} - \frac{\partial v_y}{\partial z} \right) + \hat{y} \left(\frac{\partial v_x}{\partial z} - \frac{\partial v_z}{\partial x} \right) + \hat{z} \left(\frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y} \right)$$

This is called "the curl"

or "the circulation" (older terminology).

Example Divergences

1) Let $\vec{v} = \vec{r} = x\hat{x} + y\hat{y} + z\hat{z}$.

$$\begin{aligned} \text{Then } \vec{\nabla} \cdot \vec{v} &= \frac{\partial}{\partial x}(x) + \frac{\partial}{\partial y}(y) + \frac{\partial}{\partial z}(z) \\ &= 1 + 1 + 1 \\ &= 3 \end{aligned}$$

Answers

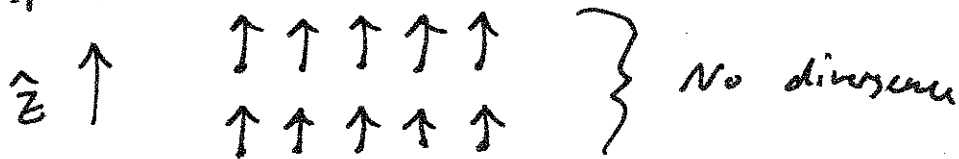
2) Let $\vec{v} = \hat{z}$

Then $\vec{\nabla} \cdot \vec{v} = \frac{\partial}{\partial x}(\emptyset) + \frac{\partial}{\partial y}(\emptyset) + \frac{\partial}{\partial z}(1) = \emptyset$

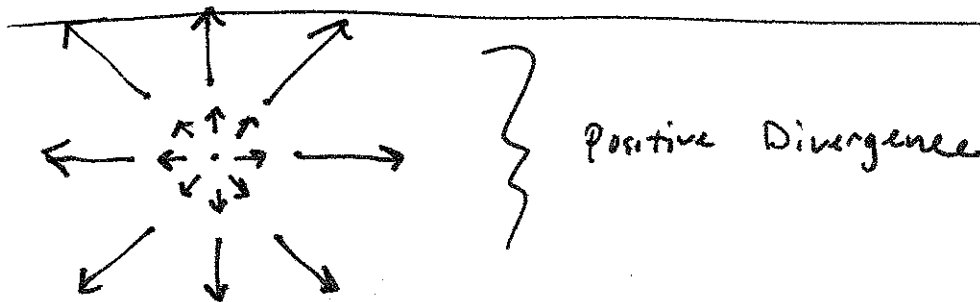
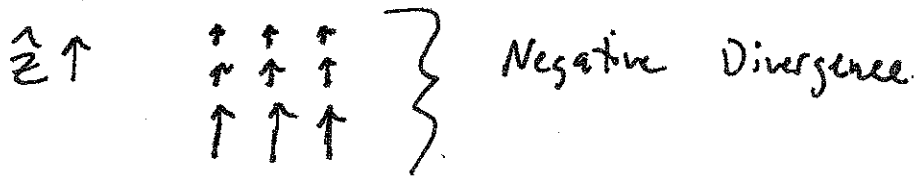
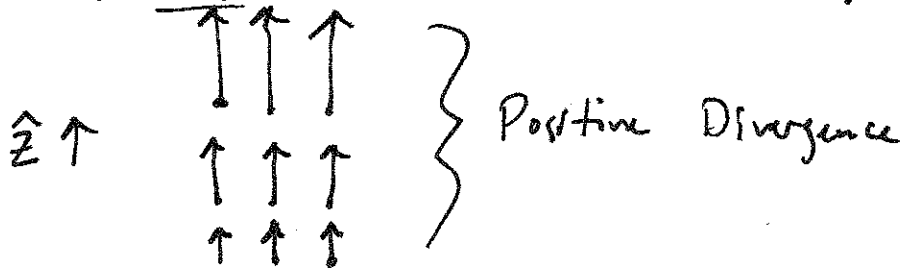
3) Let $\vec{v} = z\hat{z}$

Then $\vec{\nabla} \cdot \vec{v} = \frac{\partial}{\partial x}(\emptyset) + \frac{\partial}{\partial y}(\emptyset) + \frac{\partial}{\partial z}(z) = 1$

The Divergence is a measure of whether any particular point in space is acting like a "source" of the vector field. So a uniform field (like $\vec{v} = \hat{z}$) has no divergence anywhere in space:

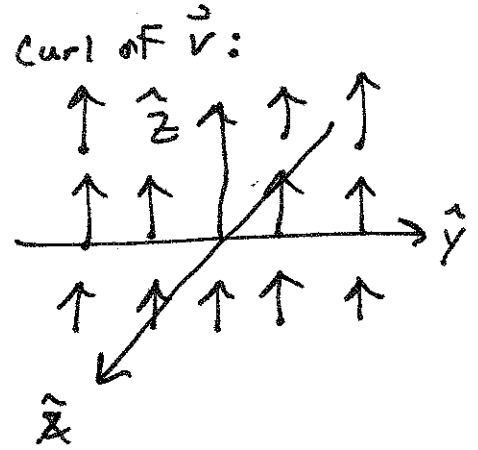
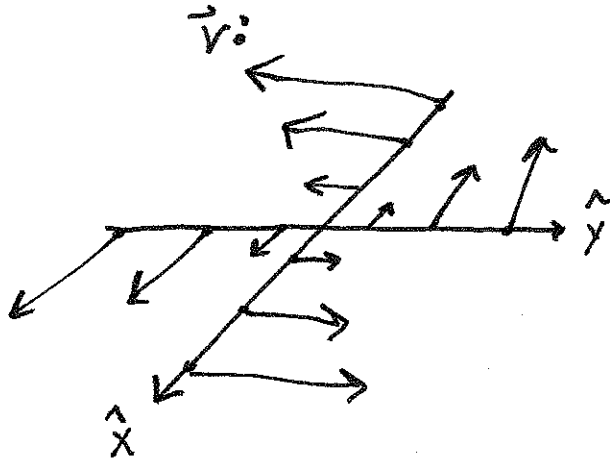


But a field which increases in intensity (magnitude) generally does have a non-zero divergence.



Example curls

① Let $\vec{v} = -y\hat{x} + x\hat{y}$:

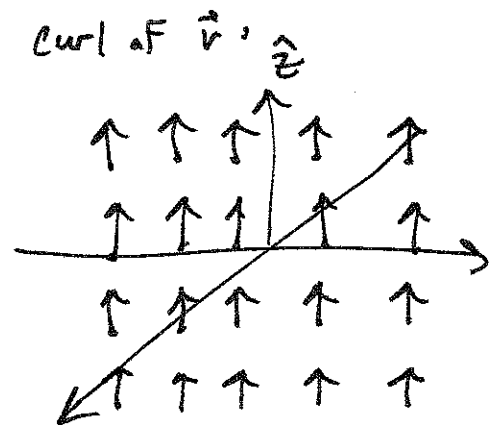
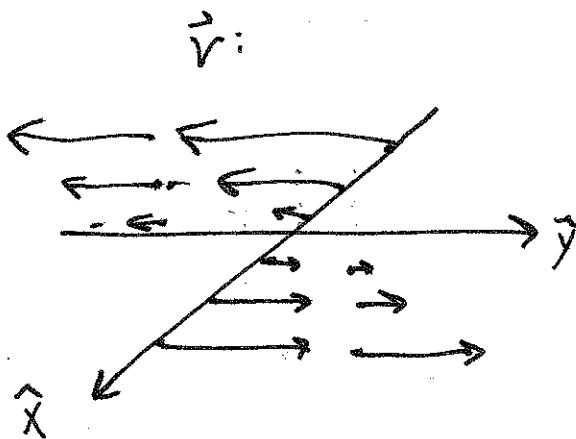


Then $\vec{\nabla} \times \vec{v} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -y & x & 0 \end{vmatrix} = \left(\frac{\partial}{\partial x}(x) - \frac{\partial}{\partial y}(-y) \right) \hat{z}$

$= 1+1 \hat{z}$

$= 2\hat{z}$

② Let $\vec{v} = xy$



Then $\vec{\nabla} \times \vec{v} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \phi & x & \phi \end{vmatrix}$

$$= (1) \hat{z}$$

$$= \hat{z}$$

STAMPED

The curl is a measure of the tendency of the vector field to rotate at each point in space. Imagine putting a tiny paddle wheel in the vector field. If it wants to rotate, then the curl is non-zero

2nd Derivatives

① The Divergence of a Gradient:

$$\vec{\nabla} \cdot \vec{\nabla} F = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right) \cdot \left(\frac{\partial F}{\partial x}, \frac{\partial F}{\partial y}, \frac{\partial F}{\partial z} \right)$$

$$= \frac{\partial^2 F}{\partial x^2} + \frac{\partial^2 F}{\partial y^2} + \frac{\partial^2 F}{\partial z^2}$$

We call this the Laplacian Operator:

$$\vec{\nabla} \cdot \vec{\nabla} F = \nabla^2 F = \text{Laplacian of } f.$$

② The curl of a Gradient:

$$\vec{\nabla} \times (\vec{\nabla} F) \leftarrow \text{This is always zero (prove from definition of } \vec{\nabla} \text{ and curl).}$$

③ Gradient of a Divergence:

$\vec{\nabla}(\vec{\nabla} \cdot \mathbf{F}) \leftarrow$ does not occur very often in physics

④ Divergence of a curl:

$\vec{\nabla} \cdot (\vec{\nabla} \times \vec{v}) \leftarrow$ this is always zero

⑤ Curl of a Curl:

$$\vec{\nabla} \times (\vec{\nabla} \times \vec{v}) = \vec{\nabla}(\vec{\nabla} \cdot \vec{v}) - \underbrace{\nabla^2 \vec{v}}_{\uparrow}$$

This is the Laplacian of a vector:

$$\nabla^2 \vec{v} \equiv \nabla^2 v_x \hat{x} + \nabla^2 v_y \hat{y} + \nabla^2 v_z \hat{z}$$

$$\begin{aligned} \nabla^2 \vec{v} &= \left(\frac{\partial^2 v_x}{\partial x^2} + \frac{\partial^2 v_x}{\partial y^2} + \frac{\partial^2 v_x}{\partial z^2} \right) \hat{x} \\ &+ \left(\frac{\partial^2 v_y}{\partial x^2} + \frac{\partial^2 v_y}{\partial y^2} + \frac{\partial^2 v_y}{\partial z^2} \right) \hat{y} \\ &+ \left(\frac{\partial^2 v_z}{\partial x^2} + \frac{\partial^2 v_z}{\partial y^2} + \frac{\partial^2 v_z}{\partial z^2} \right) \hat{z} \end{aligned}$$

The Fundamental Theorem of Calculus for Divergences

Also known as "Gauss' Theorem" and/or "the Divergence Theorem"

Recall the fundamental theorem of ordinary 1D calculus.

$$\int_a^b F(x) dx = F(b) - F(a) \quad \text{where } \frac{dF}{dx} = F'(x)$$

or $\int_a^b \frac{dF}{dx} dx = F(b) - F(a)$

In vector calculus we have several types of derivatives, so we have several versions of the fundamental theorem.

For divergences, the fundamental theorem says

$$\int_{\text{Volume}} (\vec{\nabla} \cdot \vec{v}) dV = \oint_{\text{Surface}} \vec{v} \cdot \hat{n} da$$

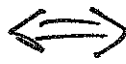
"Gauss' Theorem"
or
"Divergence Theorem"

Interpretation: Since $\vec{\nabla} \cdot \vec{v}$ measures how much the vector field \vec{v} spreads out at each point in space, if we integrate over a ~~the~~ volume it should be equal to the Flux of \vec{v} out of the surface of the volume.

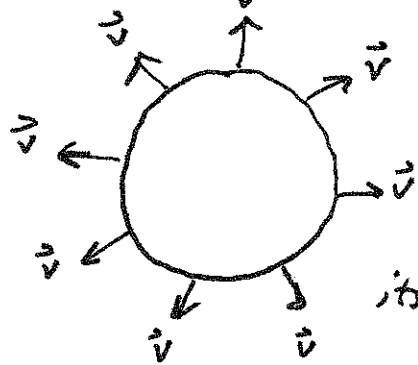
Volume Integral



A solid sphere



Surface integral



its spherical surface.

Fundamental Theorem of Calculus for Curves

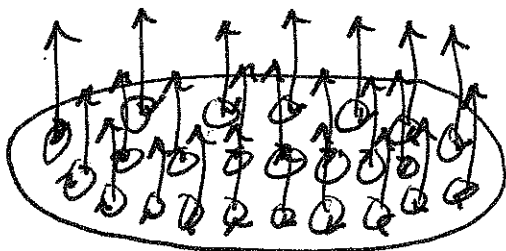
Also known as "Stokes Theorem"

$$\int_{\text{surface}} (\nabla \times \vec{v}) \cdot \hat{n} da = \int_{\text{curve}} \vec{v} \cdot d\vec{r}$$

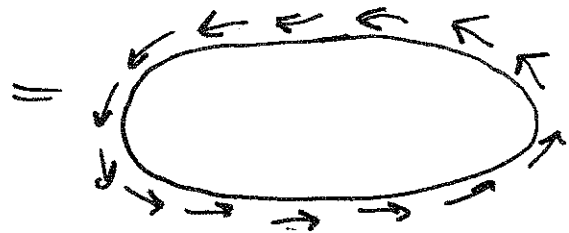
Flux of the curl through the surface

= Integral of \vec{v} around the edge

Interpretation: The curl measures how much the vector field \vec{v} tends to rotate at each point in space. If we add up lots of small rotation vectors, we should get the line integral around the edge:



Surface integral of the curl



line integral around the boundary

Maxwell's Equations in Differential Form

The integral form of Maxwell's Equations tell us how the \vec{E} field and \vec{B} field behave on average over a surface or around a closed curve.

This is sometimes useful, but it can also be clumsy. It is often more useful to know how the fields are behaving at each point in space.

This form is called the differential form.

~~Wikipedia~~

To convert from integral form to differential form, we use the Divergence Theorem and Stokes Theorem:

$$\int_{\text{Volume}} (\vec{\nabla} \cdot \vec{v}) dV = \oint_{\text{Surface}} \vec{v} \cdot \hat{n} da$$

"Divergence Theorem"

and

$$\int_{\text{Surface}} (\vec{\nabla} \times \vec{v}) \cdot \hat{n} da = \oint_{\text{Curve}} \vec{v} \cdot d\vec{l}$$

"Stokes Theorem"

Gauss' Law in Differential Form

$$\oint_{\text{Surface}} \vec{E} \cdot \hat{n} da = \frac{Q_{\text{enclosed}}}{\epsilon_0}$$

Surface

$\hookrightarrow = \int_{\text{Volume}} (\vec{\nabla} \cdot \vec{E}) dV$ according to the Divergence Theorem

$$\int_{\text{volume}} (\vec{\nabla} \cdot \vec{E}) dV = \frac{Q_{\text{enclosed}}}{\epsilon_0}$$

Now define $\rho(x, y, z) \equiv$ charge density at each point in space

$$= \frac{\text{Coulombs}}{\text{m}^3} \quad \text{in SI units}$$

$$\text{Then } Q_{\text{enclosed}} = \int_{\text{volume}} \rho dV$$

So that

$$\int_{\text{volume}} (\vec{\nabla} \cdot \vec{E}) dV = \frac{1}{\epsilon_0} \int_{\text{volume}} \rho dV = \int_{\text{volume}} \frac{\rho}{\epsilon_0} dV$$

For this to be true for any volume, it must be true that the integrands on the left hand side and right hand side are equal.

$$\boxed{\vec{\nabla} \cdot \vec{E} = \frac{\rho}{\epsilon_0}} \quad \text{"Gauss' Law in Differential Form"}$$

This says that at each point in space, the divergence of \vec{E} is proportional to the charge density at the same point in space.

(11)

Similarly, the Divergence Theorem can be used to convert Gauss' Law for magnetism:

$$\oint \vec{B} \cdot \hat{n} da = 0$$

$$\Downarrow$$
$$\boxed{\vec{\nabla} \cdot \vec{B} = 0}$$

"Gauss' Law for Magnetism in Differential Form"

Faraday's Law and Ampere's Law in Differential Form

Use Stokes' theorem to convert:

$$\underbrace{\oint_{\text{Curve}} \vec{E} \cdot d\vec{l}} = -\frac{d\phi_0}{dt} = -\frac{d}{dt} \int_{\text{Surface}} \vec{B} \cdot \hat{n} da$$

$$\hookrightarrow \int_{\text{Surface}} (\vec{\nabla} \times \vec{E}) \cdot \hat{n} da \quad \text{by Stokes Theorem}$$

$$\int_{\text{Surface}} (\vec{\nabla} \times \vec{E}) \cdot \hat{n} da = \int_{\text{Surface}} \left(-\frac{\partial \vec{B}}{\partial t} \right) \cdot \hat{n} da$$

Apparently the Integrand's are equal:

$$\boxed{\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}}$$

"Faraday's Law in Differential Form"

At every point in space, the curl of \vec{E} is proportional to the time rate change of \vec{B} at that same point.

Similarly for Ampere's Law we have

$$\oint_{\text{curve}} \vec{B} \cdot d\vec{l} = \mu_0 I + \mu_0 \epsilon_0 \frac{d\phi_E}{dt}$$



$$\vec{\nabla} \times \vec{B} = \mu_0 \vec{J} + \mu_0 \epsilon_0 \frac{\partial \vec{E}}{\partial t}$$

"Ampere's Law" in Differential Form.

where \vec{J} = current per unit area perpendicular to the flow.

In vacuum, with no charges and no currents, the four Maxwell Equations are even simpler:

$$\begin{aligned} \vec{\nabla} \cdot \vec{E} &= 0 \\ \vec{\nabla} \cdot \vec{B} &= 0 \\ \vec{\nabla} \times \vec{E} &= -\frac{\partial \vec{B}}{\partial t} \\ \vec{\nabla} \times \vec{B} &= \mu_0 \epsilon_0 \frac{\partial \vec{E}}{\partial t} \end{aligned}$$

In vacuum, no charges or currents

$$\begin{aligned} \vec{\nabla} \cdot \vec{E} &= \rho/\epsilon_0 \\ \vec{\nabla} \cdot \vec{B} &= 0 \\ \vec{\nabla} \times \vec{E} &= -\frac{\partial \vec{B}}{\partial t} \\ \vec{\nabla} \times \vec{B} &= \mu_0 \vec{J} + \mu_0 \epsilon_0 \frac{\partial \vec{E}}{\partial t} \end{aligned}$$

with charges and currents

Wave Equation from Differential Form of Maxwell's Eq.

Start with Faraday's Law:

$$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

Take the curl of both sides:

$$\nabla \times (\nabla \times \vec{E}) = \nabla \times \left(-\frac{\partial \vec{B}}{\partial t} \right) = -\frac{\partial}{\partial t} (\nabla \times \vec{B})$$

↳ vector calculus identity:

$$\nabla \times \nabla \times \vec{v} = \nabla (\nabla \cdot \vec{v}) - \nabla^2 \vec{v}$$

$$\text{so } \nabla \times \nabla \times \vec{E} = \nabla (\nabla \cdot \vec{E}) - \nabla^2 \vec{E}$$

↳ zero in vacuum (no charges)

$$\nabla \times \nabla \times \vec{E} = -\nabla^2 \vec{E}$$

So the curl of Faraday's law says

~~$$\nabla \times \nabla \times \vec{E} = -\nabla^2 \vec{E}$$~~

So we have

$$-\nabla^2 \vec{E} = -\frac{\partial}{\partial t} (\nabla \times \vec{B})$$

↳ $\mu_0 \epsilon_0 \frac{\partial \vec{E}}{\partial t}$ by Ampere's Law

$$-\nabla^2 \vec{E} = -\mu_0 \epsilon_0 \frac{\partial^2 \vec{E}}{\partial t^2}$$

$$\text{or } \boxed{\nabla^2 \vec{E} = \mu_0 \epsilon_0 \frac{\partial^2 \vec{E}}{\partial t^2}}$$

This is the wave Equation for \vec{E} .

Component-by-component it reads as

$$x\text{-Component: } \frac{\partial^2 E_x}{\partial x^2} + \frac{\partial^2 E_x}{\partial y^2} + \frac{\partial^2 E_x}{\partial z^2} = \mu_0 \epsilon_0 \frac{\partial^2 E_x}{\partial t^2}$$

$$y\text{-Component: } \frac{\partial^2 E_y}{\partial x^2} + \frac{\partial^2 E_y}{\partial y^2} + \frac{\partial^2 E_y}{\partial z^2} = \mu_0 \epsilon_0 \frac{\partial^2 E_y}{\partial t^2}$$

$$z\text{-Component: } \frac{\partial^2 E_z}{\partial x^2} + \frac{\partial^2 E_z}{\partial y^2} + \frac{\partial^2 E_z}{\partial z^2} = \mu_0 \epsilon_0 \frac{\partial^2 E_z}{\partial t^2}$$

Similarly, by taking the curl of Ampere's Law, we get

$$\boxed{\nabla^2 \vec{B} = \mu_0 \epsilon_0 \frac{\partial^2 \vec{B}}{\partial t^2}} \quad \text{Wave Equation for } \vec{B} \text{ Field.}$$

Monochromatic Plane Wave Solution

A plane wave solution traveling in the z-direction can be written as

$$\vec{E}(z,t) = \vec{E}_0 e^{i(kz - \omega t)}, \quad \vec{B}(z,t) = \vec{B}_0 e^{i(kz - \omega t)}$$

Clearly these harmonic functions will satisfy the wave equation. ~~But~~ But Maxwell's Eqs place some additional constraints on:

- i) The direction that the amplitude vectors \vec{E}_0 and \vec{B}_0 are allowed to point
- ii) The relationship between \vec{E} & \vec{B} .

Keep in mind that, in principle, \vec{E}_0 and \vec{B}_0 could be complex, so that there is an additional phase hidden inside them

Additional constraints on \vec{E} & \vec{B}

i) EM waves are transverse: \vec{E}_0 and \vec{B}_0 should be exactly perpendicular to the direction of travel.

This follows from $\vec{\nabla} \cdot \vec{E} = 0$ and $\vec{\nabla} \cdot \vec{B} = 0$.

For example, for our plane wave solution we have

$$\vec{\nabla} \cdot (\vec{E}_0 e^{i(kz - \omega t)}) = 0$$

$$\frac{\partial}{\partial x} (\underbrace{E_{0x} e^{i(kz - \omega t)}}_{\text{no x-dependence}}) + \frac{\partial}{\partial y} (\underbrace{E_{0y} e^{i(kz - \omega t)}}_{\text{no y-dependence}}) + \frac{\partial}{\partial z} (E_{0z} e^{i(kz - \omega t)})$$

So this reduces to $ik E_{0z} e^{i(kz - \omega t)} = 0$ = 0

$\therefore E_{0z} = 0$ } for a plane wave travelling in the z-direction.

Similarly $B_{0z} = 0$

(follows from $\vec{\nabla} \cdot \vec{B} = 0$)

~~For~~ For wave solutions, \vec{E} and \vec{B} should have no components in the direction of travel.

2) \vec{E} and \vec{B} are in phase with each other and mutually perpendicular. This follows from Faraday's Law:

$$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

The x-component of this equation is

$$\frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial z} = -\frac{\partial B_x}{\partial t}$$

For our plane wave solution this requirement is

$$\frac{\partial}{\partial y} \left(E_{0z} e^{i(kz - \omega t)} \right) - \frac{\partial}{\partial z} \left(E_{0y} e^{i(kz - \omega t)} \right) = -\frac{\partial}{\partial t} \left(B_{0x} e^{i(kz - \omega t)} \right)$$

no y-dependence

$$-ikE_{0y} = i\omega B_{0x}$$

$$\boxed{-kE_{0y} = \omega B_{0x}}$$

Similarly, the y-component of Faraday's Law requires that

$$\boxed{kE_{0x} = \omega B_{0y}}$$

And we already know that $\boxed{B_{0z} = E_{0z} = 0}$

We can write all three of these equations on one line using a cross product:

$$\vec{B}_0 = \frac{k}{\omega} (\hat{z} \times \vec{E}_0)$$

Or, since $\frac{k}{\omega} = \frac{1}{v_{\text{phase}}} = \frac{1}{c}$,

$$\vec{B}_0 = \frac{1}{c}(\hat{z} \times \vec{E}_0)$$

or $\boxed{\hat{z} \times \vec{E}_0 = c\vec{B}_0}$

This equation says that

- \vec{B}_0 is perpendicular to \hat{z} (direction of travel)
- \vec{E}_0 is perpendicular to \vec{B}_0
- \vec{E}_0 is perpendicular to \hat{z} (direction of travel)

Also \vec{E}_0 and \vec{B}_0 are related in their magnitudes by

$$\boxed{E_0 = cB_0}$$

And they are in phase \Rightarrow when \vec{E} is maximal, \vec{B} is also maximal at that same location in space

AMBIENT