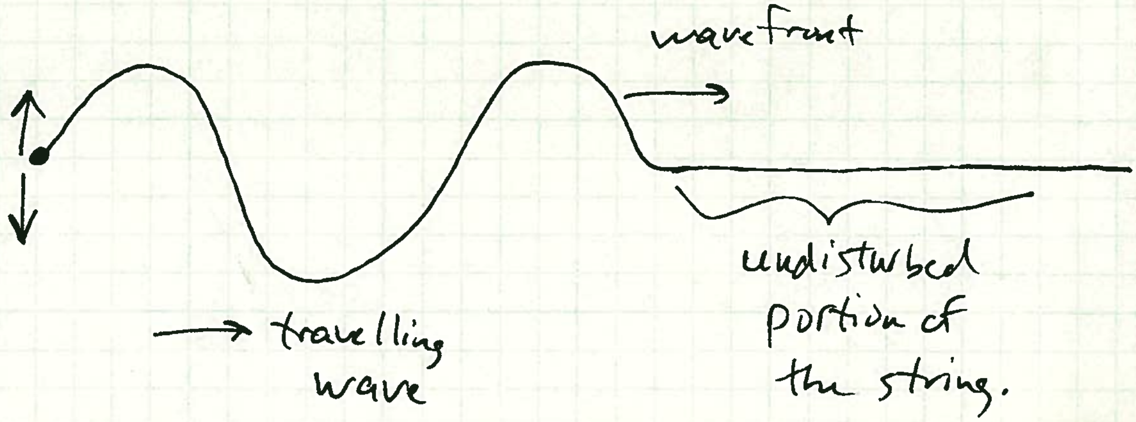
Phys 223

Transport of Energy by a mechanical wave

If we grab the end of a string and oscillate it up and down, we set up a traveling wave which propagates down the string:



The part of the string which is oscillating clearly possesses kinetic energy. It also possesses potential energy due to the stretching of the string.

As time goes forward, new sections of the string are brought into motion as the wave propagates, so the total energy contained in the string is increasing, and the energy travels with the wave down the string. This energy comes from the work done by the driving force which makes the string oscillate.

We can calculate the rate of energy transport by calculating the work done by the driving force per unit time. Remember, in SI units

$$\text{Rate of Energy transmission} = \frac{\text{Joules}}{\text{sec}} = \text{Power} = \text{Watts}$$

So consider a sinusoidal traveling wave:

$$y(x,t) = A\sin(kx - \omega t)$$

The driving force is applied at $x = \emptyset$, and the transverse motion of the string at that location is

$$y(x=\emptyset,t) = A\sin(-\omega t) = -A\sin(\omega t)$$

The differential work done in a small displacement (dy)

$$dW = F_t\, dy = F_t\, d(-A\sin(\omega t)) = -F_t A\omega\cos(\omega t)\,dt$$

↑ transverse component of driving force

The transverse force is

$$F_t = -T\frac{\partial y}{\partial x}\bigg|_{x=0} = -T\left[\frac{\partial}{\partial x}\left(A\sin(kx-\omega t)\right)\right]\bigg|_{x=0}$$

$$= -TAk\cos(kx-\omega t)\Big|_{x=0}$$

$$= -TAk\cos(-\omega t)$$

$$F_t = -TAk\cos(\omega t)$$

~~The total~~ $\therefore$ In a short time period (dt) the work done is

$$dW = -F_T A\omega\cos(\omega t)\,dt = \cancel{\;}\left(-TAk\cos(\omega t)\right)\left(A\omega\cos(\omega t)\,dt\right)$$

$$dW = TA^2k\omega\cos^2(\omega t)\,dt$$

The total work done in <u>one</u> <u>complete</u> cycle is

$$\underset{\uparrow \text{one cycle}}{W_1} = \int_0^{2\pi/\omega} TA^2k\omega\cos^2(\omega t)\,dt$$

$$= TA^2k\omega \underbrace{\int_0^{2\pi/\omega}\cos^2(\omega t)\,dt}_{\frac{\pi}{\omega} = \frac{\pi}{2\pi f} = \frac{1}{2f}}$$

$$W_1 = \frac{\frac{1}{2}TA^2k\omega}{f}$$ ← This is the work done per cycle.

The work done per second is the work done per cycle times the number of cycles per second, which is (f).

$\therefore$ Work per second = Power = $W_1 f = \frac{1}{2}TA^2k\omega$

We can re-write this: $T = \rho v^2 = \rho v\left(\frac{\omega}{k}\right)$

mass density → ρ; v ← phase velocity

$\therefore$ Power = Energy transmission $= \frac{1}{2}\left(\rho v\frac{\omega}{k}\right)A^2k\omega$

$$\boxed{P = \frac{1}{2}\rho v\omega^2A^2}$$

Another way to write it: $Z = \rho v$, so

$$\boxed{P = \tfrac{1}{2} Z \omega^2 A^2}$$

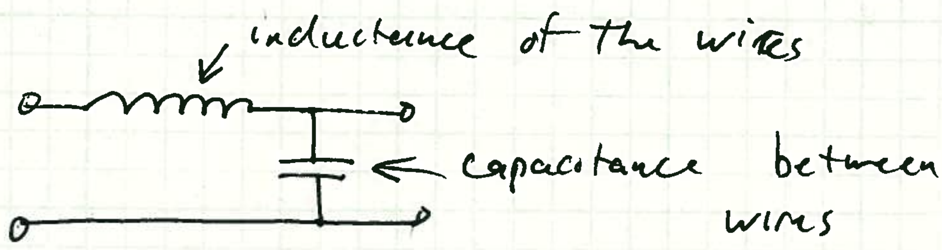
Transmission Lines.

(1) Any two conductors that are seperated by a finite distance have some capacitance.

(2) Any length of wire, even a straight wire, has some inductance.

So consider two parallel straight wires:

We can model a small section of these wires as



Let L_0 = inductance per unit length

and C_0 = capacitance per unit length.

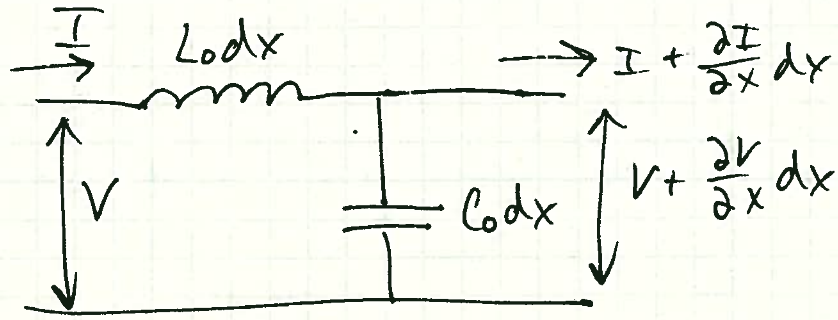
Then a short section (dx) has $L = L_o dx$
and $C = C_o dx$.

Let the input current and voltage by I & V.

Then the output current and voltage will be

$$I_{output} = I + \underbrace{\frac{\partial I}{\partial x} dx}_{\text{change in current}}$$

$$V_{output} = V + \underbrace{\frac{\partial V}{\partial x} dx}_{\text{change in voltage}}$$



The change in voltage must be due to the inductor:

$$\frac{\partial V}{\partial x} dx = \underbrace{-(L_o dx)\frac{\partial I}{\partial t}}_{\substack{\text{Voltage drop} \\ \text{across an inductor,} \\ \text{where } L = L_o dx}}$$

or $\quad \dfrac{\partial V}{\partial x} = -L_o \dfrac{\partial I}{\partial t} \quad$ ①

The change in current must mean that the capacitor has charged up a little bit:

$$dI = \frac{dQ}{dt} = \frac{\partial}{\partial t}(cv) = \frac{\partial}{\partial t}(C_0 dx\, v)$$

$$-\frac{\partial I}{\partial x} dx = C_0 dx \frac{\partial v}{\partial t}$$

or $$-\frac{\partial I}{\partial x} = C_0 \frac{\partial v}{\partial t} \quad (2)$$

Now take $\frac{\partial}{\partial x}$ of (1) and $\frac{\partial}{\partial t}$ of (2):

$\frac{\partial}{\partial x}$ (1): $$\frac{\partial^2 v}{\partial x^2} = -L_0 \frac{\partial I}{\partial x \partial t}$$

$\frac{\partial}{\partial t}$ (2): $$-\frac{\partial^2 I}{\partial t \partial x} = C_0 \frac{\partial^2 v}{\partial t^2}$$

Since $\frac{\partial^2 I}{\partial t \partial x} = \frac{\partial^2 I}{\partial x \partial t}$, we have

$$\boxed{\frac{\partial^2 v}{\partial x^2} = L_0 C_0 \frac{\partial^2 v}{\partial t^2}}$$

These are the wave equations again.

Also, we have

$$\boxed{\frac{\partial^2 I}{\partial x^2} = L_0 C_0 \frac{\partial^2 I}{\partial t^2}}$$

We can immediately infer that voltage & current waves can propagate down the pair of conductors.

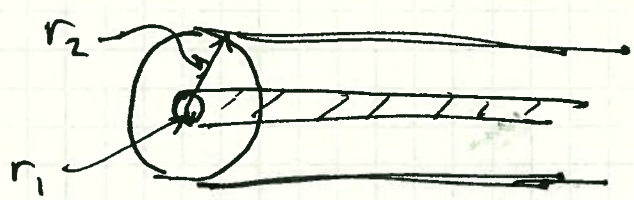
What's the phase velocity of propagation? We simply read it from the wave equation:

$$\frac{\partial^2 V}{\partial x^2} = L_0 C_0 \frac{\partial^2 V}{\partial t^2}$$

$$\therefore \quad \frac{1}{v^2} = L_0 C_0$$

$$\boxed{v = \frac{1}{\sqrt{L_0 C_0}}}$$

Coaxial Cable



Has inductance per unit length

$$L_0 = \frac{\mu}{2\pi} \ln\left(\frac{r_2}{r_1}\right)$$ where μ = magnetic permeability of the material between the conductors.

Also, it has capacitance per unit length

$$C_0 = \frac{2\pi\epsilon}{\ln(r_2/r_1)} \quad \text{where } \epsilon = \text{permittivity of the dielectric}$$

What's the phase velocity of an electrical wave travelling in this cable?

$$v = \frac{1}{\sqrt{C_0 L_0}} = \frac{1}{\sqrt{\left(\frac{2\pi\epsilon}{\ln(r_2/r_1)}\right)\frac{\mu}{2\pi}\ln(r_2/r_1)}}$$

$$= \frac{1}{\sqrt{\epsilon\mu}}$$

For a typical coaxial cable, with <u>polyethylene</u> as the dielectric,

$$\epsilon \approx 2.25\,\epsilon_0$$

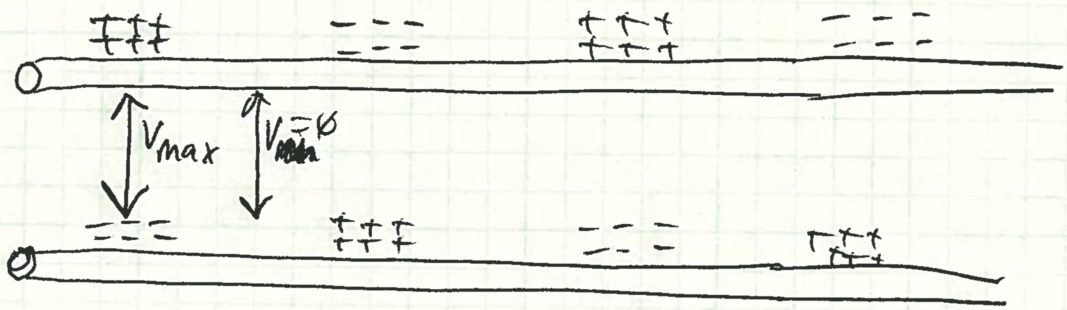
$$\mu \approx \mu_0$$

$$\text{so} \quad V \approx \frac{1}{\sqrt{2.25}}\underbrace{\frac{1}{\sqrt{\epsilon_0\mu_0}}}_{\text{speed of light } = c = 3.0\times10^8\ \text{m/s}}$$

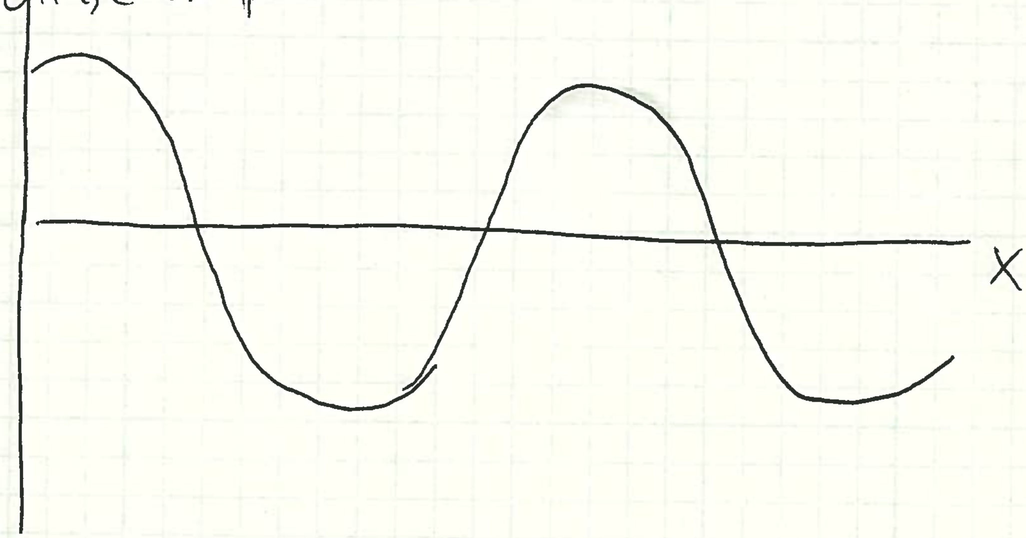
$$v \approx \frac{2}{3}c \approx 2.0\times10^8\ \text{m/s}$$

Characteristic Impedance of a Transmission Line

Transmission Lines support voltage & current waves:



Voltage vs. position



Voltage Satisfies the wave Equation:

$$\frac{\partial^2 V}{\partial x^2} = L_o C_o \frac{\partial^2 V}{\partial t^2}$$, where L_o = inductance per unit length

C_o = Capacitance per unit length

Wave phase velocity is $\frac{1}{\sqrt{L_o C_o}}$

For a coaxial cable, $\frac{1}{\sqrt{L_o C_o}} = \frac{1}{\sqrt{\mu \epsilon}} = v_p$

So the voltage wave in a coaxial cable ~~wave~~ travels at a speed equal to an electromagnetic wave in that dielectric.

The current also satisfies a wave equation:

$$\frac{\partial^2 I}{\partial x^2} = L_0 C_0 \frac{\partial^2 I}{\partial t^2}$$

The Voltage ~~a~~ and Current are related by two equations:

$$\frac{\partial V}{\partial x} = -L_0 \frac{\partial I}{\partial t} \quad \text{and} \quad \frac{\partial I}{\partial x} = -C_0 \frac{\partial V}{\partial t}.$$

We can use these to determine the ratio of <u>peak voltage</u> to <u>peak current</u>, which is called the <u>characteristic impedance</u> of the transmission line.

~~Then~~ A solution to the voltage wave Equation is

$$V_+(x,t) = V_{0+} \sin(kx - \omega t)$$

plus sign means this wave travels in the positive x direction

The current associated with this voltage is

$$I_+(x,t) = I_{o+} \sin(kx - \omega t) \,.$$

We can confirm that this is correct by substituting into the differential equations:

$$\frac{\partial V}{\partial x} = V_{o+} k \cos(kx - \omega t)$$

$$\frac{\partial I}{\partial t} = I_{o+} (-\omega) \cos(kx - \omega t)$$

We require that $\frac{\partial V}{\partial x} = -L_o \frac{\partial I}{\partial t}$

$$\text{i.e.} \quad V_{o+} k \cos(kx - \omega t) = -L_o I_{o+} (-\omega) \cos(kx - \omega t)$$

$$\frac{V_{o+}}{I_{o+}} = L_o \underbrace{\frac{\omega}{k}}_{\substack{\text{phase} \\ \text{velocity.}}} = L_o V_{phase}$$

Therefore there is a fixed relationship between the magnitude of the peak voltage and the peak current (these magnitudes cannot be chosen independently.) We call that ratio the <u>characteristic</u> <u>impedance</u> of the transmission line.

Z_0 = "characteristic Impedance" $\equiv \dfrac{V_{0+}}{I_{0+}}$

$\therefore \quad Z_0 = L_0 v = L_0 \underbrace{\left(\dfrac{1}{\sqrt{L_0 C_0}}\right)} = \sqrt{\dfrac{L_0}{C_0}}$

$\boxed{Z_0 = \sqrt{\dfrac{L_0}{C_0}}}$ = units of Ω (Ohms).

For a coaxial cable, $L_0 = \dfrac{\mu}{2\pi} \ln\left(\dfrac{r_2}{r_1}\right)$

$C_0 = \dfrac{2\pi\varepsilon}{\ln(r_2/r_1)}$

so $Z_0 = \sqrt{\dfrac{\dfrac{\mu}{2\pi}\ln\left(\dfrac{r_2}{r_1}\right)}{\dfrac{2\pi\varepsilon}{\ln(r_2/r_1)}}} = \underbrace{\dfrac{1}{2\pi}\sqrt{\dfrac{\mu}{\varepsilon}} \ln\dfrac{r_2}{r_1}}$

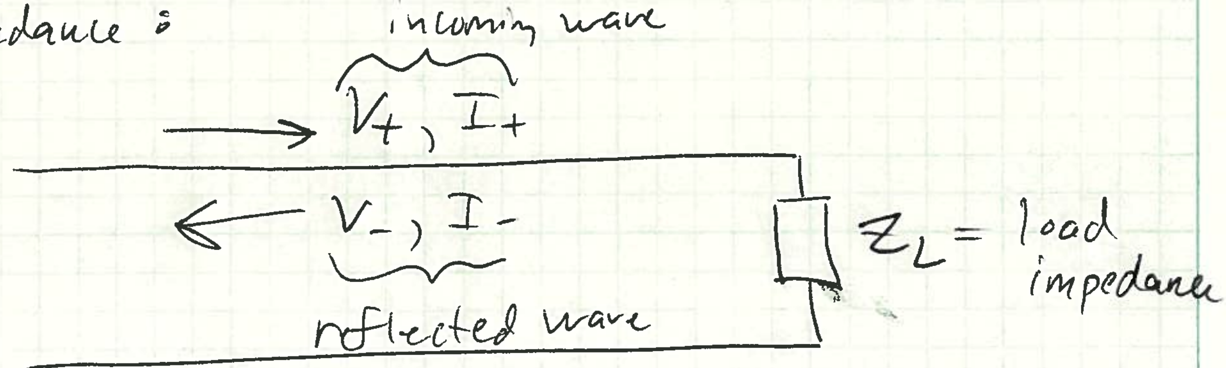
coaxial cable characteristic impedance

Typical Coaxial cables have μ, ε, r_2 & r_1 such that $Z_0 = 50\,\Omega$. But sometimes you find cables that have $Z_0 = 75\,\Omega$, $100\,\Omega$, etc.

Note that $Z_0 = \sqrt{\dfrac{L_0}{C_0}}$ = purely real $\Rightarrow$ Voltage and Current are 100% in phase.

Load Impedance & Reflections

At the end of a transmission line we can connect an electrical device with some impedance:



We can expect to have reflections from this load impedance, just like for mechanical waves. Since the ~~4a~~ mathematics is the same as before (just the wave equation with boundary conditions), the results will be the same as with mechanical waves:

$$\frac{\text{~~Ratio of~~ Reflected Amplitude}}{\text{Incoming Amplitude}} = \frac{V_-}{V_+} = \frac{Z_L - Z_0}{Z_L + Z_0}$$

$$\frac{\text{Transmitted Amplitude}}{\text{Incoming Amplitude}} = \frac{V_L}{V_+} = \frac{2Z_L}{Z_L + Z_0}.$$

Notice that there is one important distinction between the reflected amplitude of a voltage wave on a transmission line and the same thing in a mechanical wave: If ~~the~~ $Z_L \to \infty$, we get 100% reflection, but without a phase shift.

$$\frac{V_-}{V_+} = \frac{Z_L - Z_0}{Z_L + Z_0} \to 1 \text{ as } Z_L \to \infty.$$

(open circuit)

In the case of a mechanical wave, $Z_2 \to \infty$ means that the last particle is not allowed to move (boundary condition). For a voltage wave, however, the end of the transmission line is allowed to have a non-zero voltage even when $Z_L \to \infty$. So the reflected wave is not forced to cancel the incoming wave

For the current wave, on the other hand, the expressions are

$$\frac{I_-}{I_+} = \frac{Z_0 - Z_L}{Z_0 + Z_L}$$

and $$\frac{I_L}{I_+} = \frac{2 Z_0}{Z_L + Z_0}$$

As $Z_L \to \infty$, the reflected current wave does have a 180° phase shift, (unlike the voltage wave), because no current can flow through an infinite impedance.