Mechanical Impedance.

Recall the definition of impedance from our study of oscillating electrical circuits:

$$\vec{V} \equiv \vec{I} Z$$

Z = impedance = a complex number which tells us the ratio of peak voltage to peak current:

$$|Z| = \frac{|V|}{|I|}$$

If Z is complex, then there is also a phase shift between $\vec{V}$ and $\vec{I}$.

Roughly speaking, a large impedance means that there will be a small current produced by a given voltage. Conversely, ~~a~~ if the impedance is small, then the same voltage will produce a large current

We can define an analogous quantity for mechanical systems like a continuous string. We define

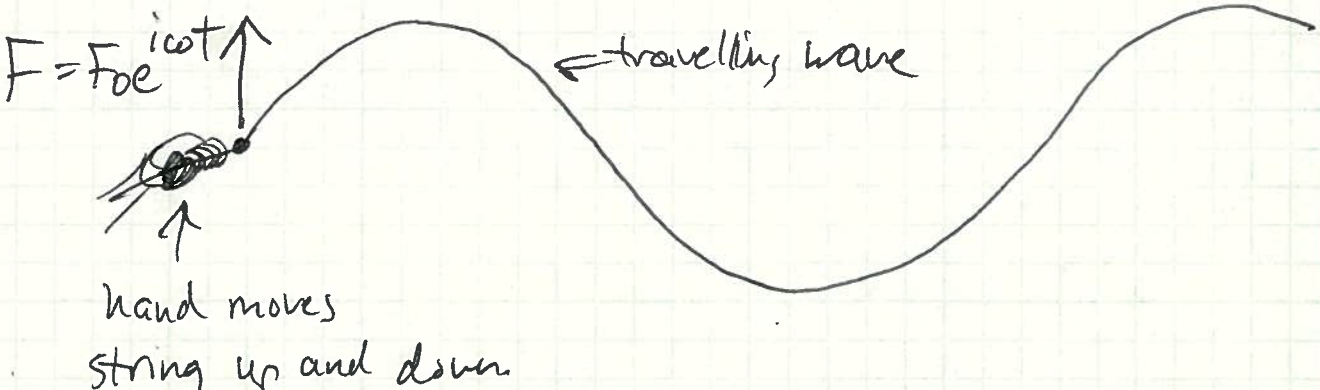
$$Z_{\text{mechanical}} \equiv \frac{|\text{transverse force}|}{|\text{transverse velocity}|} = \frac{|F|}{|V_T|}$$

$$\text{or} \quad |F| \equiv |V_T| Z = |\dot{y}| Z$$

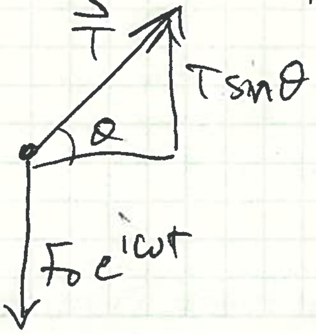
With this definition, a large impedance implies a small ~~traves~~ transverse velocity, and a small impedance implies a large transverse velocity (for the same transverse force.)

Calculation of the impedance of a continuous string.

Consider ~~an~~ a ~~the~~ long string being driven up and down by an oscillating force at some point on its length. This generates a travelling wave:



At the location where the force is applied, the force must be responsible for maintaining the tension in the string (otherwise there can be no travelling wave.) For example, when the tension is positive, the force is negative:



$$F_0 e^{i\omega t} = -T\sin\theta \approx -T\tan\theta = -T\underbrace{\left(\frac{\partial y}{\partial x}\right)}_{\substack{\uparrow \\ \text{slope of The} \\ \text{string.}}}$$

The travelling wave means that the slope of the string behaves sinusoidally:

$$y(x,t) = Ae^{-i(kx-\omega t)}$$

$$\text{so} \quad \frac{\partial y}{\partial x} = -ikAe^{-i(kx-\omega t)}$$

$$\therefore \quad F_0 e^{i\omega t} = -T\frac{\partial y}{\partial x} = (-T)\left(-ikAe^{-i(kx-\omega t)}\right)$$

$$= ikTAe^{-i(k\underset{\substack{\uparrow \\ x=0 \\ \text{when the} \\ \text{force is applied}}}{x}-\omega t)}$$

$$F_0 e^{i\omega t} = ikTAe^{i\omega t}$$

$$A = \frac{F_0}{ikT} = \frac{F_0}{i\left(\frac{\omega}{V_p}\right)T} = \frac{F_0 \overset{\swarrow \text{phase velocity}}{V_p}}{i\omega T}$$

$$\text{Therefore,} \quad y(x,t) = Ae^{-i(kx-\omega t)} = \frac{F_0 V_p}{i\omega T}e^{-i(kx-\omega t)}$$

The transverse velocity is $\frac{\partial y}{\partial t}$:

$$\dot{y}(x,t) = \frac{F_0 v_p}{i\omega T}(i\omega)\, e^{-i(kx-\omega t)}$$

$$= \frac{F_0}{(T/v_p)}\, e^{-i(kx-\omega T)}$$

Then the mechanical impedance is

$$Z = \frac{|\vec{F}|}{|\dot{y}|} = \frac{T \leftarrow \text{Tension}}{v_p \leftarrow \text{phase velocity}}$$

Also, $\sqrt{\frac{T}{\underset{\uparrow\ \text{mass density}}{\rho}}} = v_p$, so $v_p^2 \rho = T$.

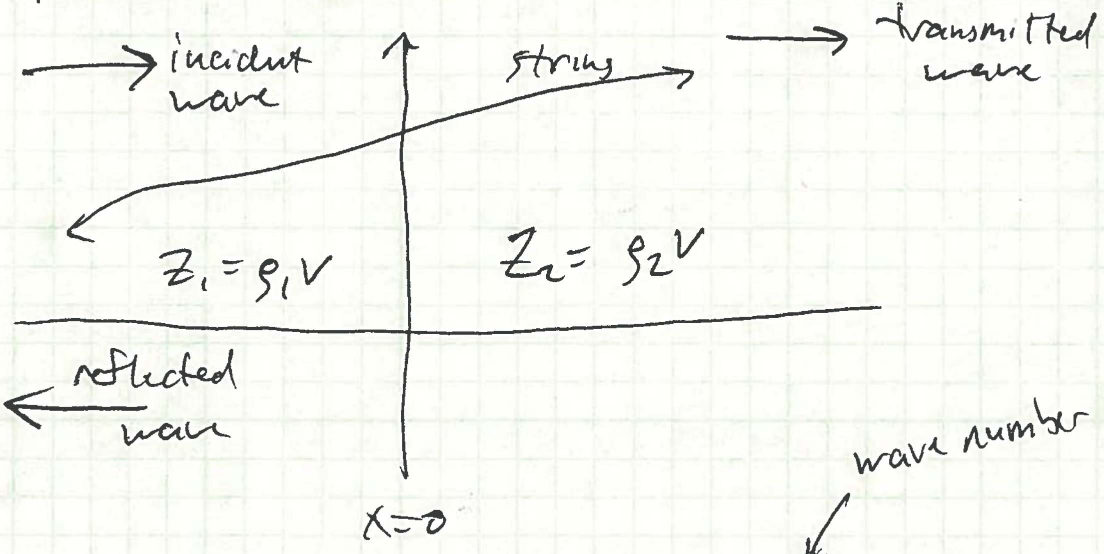
Therefore $$\boxed{Z = \frac{T}{v_p} = \frac{v_p^2 \rho}{v_p} = \underset{\uparrow}{\rho v_p}}$$

mechanical impedance of string is the phase velocity times the mass density

Reflection & Transmission at a boundary.

Consider a string where the mass density suddenly changes:

What will happen as a travelling wave approaches this discontinuity from the left? In general we can expect that part of the wave will be transmitted to the right, and part will be reflected to the left.



wave number k_1

incident wave : $y_i(x,t) = A_1 e^{-i(k_1 x - \omega t)}$

reflected wave : $y_r(x,t) = B_1 e^{-i(-k_1 x - \omega t)}$

travelling to the left

transmitted wave: $y_t(x,t) = A_2 e^{-i(k_2 x - \omega t)}$

$\uparrow$ wave number in medium 2

Boundary Conditions:

① At $x=0$, the rope is continuous, so

$$y_i + y_r = y_t \quad (\text{at } x=0)$$

② The slope $\frac{\partial y}{\partial x}$ must be finite ~~at~~ and continuous at $x=0$

Condition ① gives

$$A_1 e^{-i(k_1 x - \omega t)} + B_1 e^{-i(-k_1 x - \omega t)} = A_2 e^{-i(k_2 x - \omega$$

At $x=0$:

$$A_1 e^{i\omega t} + B_1 e^{i\omega t} = A_2 e^{i\omega t}$$

$$\boxed{A_1 + B_1 = A_2} \quad ①$$

Condition ② gives

$$\frac{\partial}{\partial x}(y_i + y_r) = \frac{\partial}{\partial x}(y_t) \quad \text{at } x=0:$$

$$-ik_1 A_1 e^{-i(k_1 x - \omega t)} + ik_1 B_1 e^{-i(-k_1 x - \omega t)}$$

$$= -ik_2 A_2 e^{-i(k_2 x - \omega t}$$

$$-ik_1A_1e^{-i(k_1x-\omega t)} + ik_1B_1e^{-i(-k_1x-\omega t)} = -ik_2A_2e^{-i(k_2x-\omega t}$$

At $x=0$ this is

$$-k_1A_1 + k_1B_1 = -k_2A_2$$

We can re-write this in terms of the mechanical impedance:

$$k_1 = \frac{\omega}{\underset{\overset{\nwarrow}{\text{phase velocity in medium 1}}}{V_{p1}}} \quad , \quad k_2 = \frac{\omega}{\underset{\overset{\uparrow}{\text{phase velocity in medium 2}}}{V_{p2}}}$$

Note that ω is the same in both ~~media~~ regions 1 & 2. Otherwise the string would get out-of-phase with itself and be discontinuous at $x=0$.

Continuing, substitute for k_1 & k_2:

$$-\frac{\omega}{V_{p1}}A_1 + \frac{\omega}{V_{p1}}B_1 = -\frac{\omega}{V_{p2}}A_2$$

Now multiply by the tension (T): (and multiply by -1)

$$\omega\underbrace{\left(\frac{T}{V_{p1}}\right)}_{Z_1}(A_1-B_1) = \omega\underbrace{\left(\frac{T}{V_{p2}}\right)}_{Z_2}(A_2)$$

$$\boxed{A_1 - B_1 = \frac{Z_2}{Z_1} A_2} \qquad ②$$

Also recall Eq. ①:

$$\boxed{A_1 + B_1 = A_2} \qquad ①$$

These are the two boundary conditions.

Now we can determine the amplitude of the reflected wave (B_1) in terms of the amplitude of the incoming wave (A_1):

Subract ② from ①:

$$2B_1 = A_2\left(1 - \frac{Z_2}{Z_1}\right) = A_2\left(\frac{Z_1 - Z_2}{Z_1}\right)$$

$$B_1 = \frac{1}{2}A_2\left(\frac{Z_1 - Z_2}{Z_1}\right)$$

What is A_2? It's the amplitude of the transmitted wave. We can find it in terms of A_1:

Add ① & ②:

$$2A_1 = A_2\left(1 + \frac{Z_2}{Z_1}\right) = A_2\left(\frac{Z_1 + Z_2}{Z_1}\right)$$

$$\therefore \boxed{\frac{A_2}{A_1} = \frac{2Z_1}{Z_1 + Z_2}}$$

This tells us how large the transmitted wave will be, in terms of the impedances and the amplitude of the incoming wave.

Now we can find B_1:

$$B_1 = \frac{1}{2}A_2\left(\frac{Z_1-Z_2}{Z_1}\right) = \frac{1}{2}\underbrace{\left(\frac{2Z_1A_1}{Z_1+Z_2}\right)}_{A_2}\left(\frac{Z_1-Z_2}{Z_1}\right)$$

$$\boxed{\frac{B_1}{A_1} = \frac{Z_1-Z_2}{Z_1+Z_2}}$$

↑

This tells us how large the reflected wave will be in terms of the impedances and the amplitude of the incoming wave.

So our final results are

$$\boxed{\frac{A_2}{A_1} = \text{"transmission coefficient of amplitude"} = \frac{2Z_1}{Z_1+Z_2}}$$

$$\boxed{\frac{B_1}{A_1} = \text{"reflection coefficient of amplitude"} = \frac{Z_1-Z_2}{Z_1+Z_2}}$$

Two extreme examples

① Suppose that $Z_1 = Z_2$, so that there is no boundary. We call this situation "impedance matching". Then

$$\frac{A_2}{A_1} = \frac{2Z_1}{\underset{\substack{\uparrow \\ Z_2=Z_1}}{Z_1+Z_2}} = \frac{2Z_1}{2Z_1} = 1 \leftarrow \text{100\% transmitted amplitude}$$

and

$$\frac{B_1}{A_1} = \frac{Z_1-Z_2}{Z_1+Z_2} = 0 \text{ when } Z_1=Z_2 \leftarrow \text{0\% reflected amplitude}$$

This makes sense: if there is no boundary, then there should be no reflection.

② Suppose that the mass density of region 2 is very, very large. For example, suppose region 2 becomes infinitely heavy, like a brick wall. Then we have

$Z_2 = \rho v \to \infty$

The impedance of the wall is infinite.

Then $Z_2 \to \infty$, so

$$\frac{A_2}{A_1} = \frac{2Z_1}{Z_1+Z_2} \to 0 \text{ as } Z_2 \to \infty.$$

Then is 0% transmission of amplitude into the wall.

Conversely,

$$\frac{B_1}{A_1} = \frac{Z_1-Z_2}{Z_1+Z_2} \longrightarrow -1 \text{ when } Z_2 \to \infty.$$

We get 100% reflection, but the amplitude changes sign, when the incoming wave strikes a wall.

Phys 273

Exam 2 Review

Normal Mode: In a multi-particle system, a normal mode is a type of motion where all particles oscillate at the same frequency.

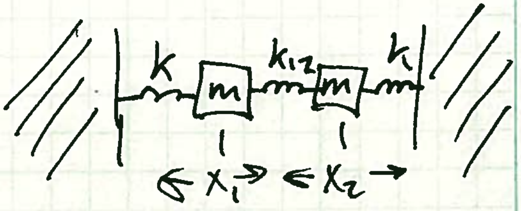
- The number of normal modes is equal to the number of particles.
- Each normal mode goes at its own frequency.
- The general solution is a sum over normal modes:

$$\vec{x}(t) = \sum_{n=1}^{N} a_n \vec{q}_n e^{i\omega_n t}$$

where $\vec{q}_n = (q_{1n}, q_{2n}, \ldots, q_{Nn}) = n^{th}$ normal mode

= "normal mode eigenvector"

2 coupled oscillators $(N=2)$



Two normal modes:

$\vec{q}_1 = (1,1)$ = "symmetric mode"

$\vec{q}_2 = (1,-1)$ = "anti-symmetric mode"

Then $\vec{x}(t) = a_1(1,1)e^{i\omega_1 t} + a_2(1,-1)e^{i\omega_2 t}$

$\downarrow$

$(x_1(t), x_2(t)) = a_1(1,1)e^{i\omega_1 t} + a_2(1,-1)e^{i\omega_2 t}$

Eqs. of Motion

$m\ddot{x}_1 + (k+k_{12})x_1 - k_{12}x_2 = 0$

$m\ddot{x}_2 + (k+k_{12})x_2 - k_{12}x_1 = 0$

For this system we called $\omega_1 = \omega_{small} = \omega_s$

and $\omega_2 = \omega_{Large} = \omega_L$

The frequencies are $\omega_1 = \omega_s = \sqrt{\frac{k}{m}}$

$$\omega_2 = \omega_L = \sqrt{\frac{k+2k_{12}}{m}}$$

Explicitly the solution is

$$x_1(t) = a_1 e^{i\omega_1 t} + a_2 e^{i\omega_2 t}$$

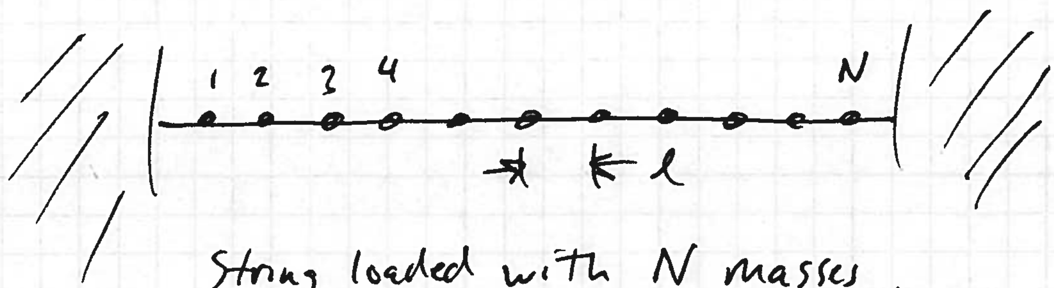
$$x_2(t) = a_1 e^{i\omega_1 t} - a_2 e^{i\omega_2 t}$$

or, taking the real part,

$$x_1(t) = a_1 \cos(\omega_1 t) + a_2 \cos(\omega_2 t)$$

$$x_2(t) = a_2 \cos(\omega_1 t) - a_2 \cos(\omega_2 t)$$

N-coupled oscillators – The loaded string.



String loaded with N masses, each of mass (m).

string tension = T.

Transverse oscillations: Each mass has y-displacement (y_p).

String is fixed at each end, so $y_{p=0} = 0$ and $y_{p=N+1} = 0$. $\Big\}$ boundary conditions

Equation of motion for mass (p):

$$\ddot{y}_p + 2\omega_0^2 y_p - \omega_0^2 (y_{p+1} + y_{p-1}) = 0$$

wher $\omega_0^2 = T/ml$.

Normal mode solutions:

$$\vec{q}_n = \left(\sin\left(\frac{n\pi}{N+1}\right), \sin\left(\frac{2n\pi}{N+1}\right), \sin\left(\frac{3n\pi}{N+1}\right), \ldots, \sin\frac{Nn\pi}{N+1} \right)$$

$$\omega_n = 2\omega_0 \sin\left(\frac{n\pi}{2(N+1)}\right)$$

We also wrote the $\vec{q}_n$ vectors on a particle-by-particle basis as

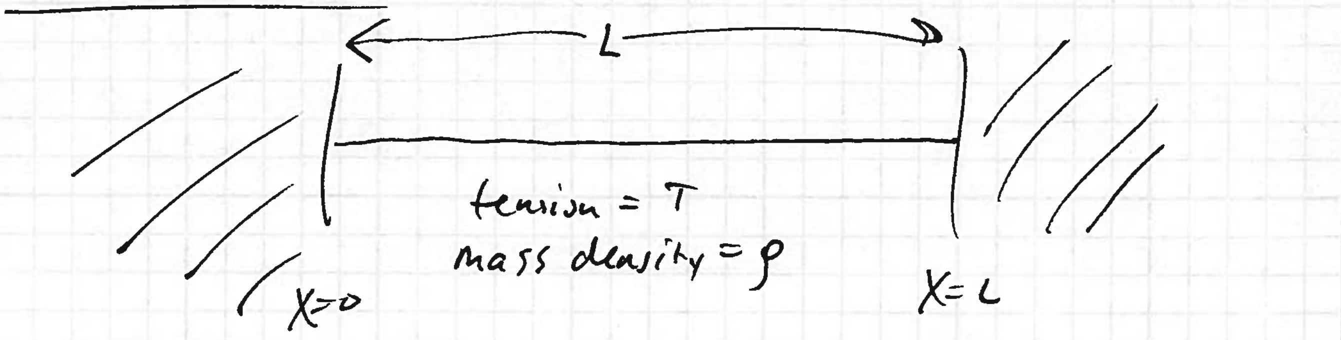
$$A_{pn} = \sin\left(\frac{pn\pi}{N+1}\right)$$

which mass

which normal mode

The solution is the same for longitudinal oscillations, except the motion is in the x direction, rather than the y direction

Continuous Systems – string fixed at $x=0$ & $x=L$



Equation of Motion:

$$\boxed{\frac{\partial^2 y}{\partial x^2} = \frac{\rho}{T}\frac{\partial^2 y}{\partial t^2}}$$

Classical Wave Equation.

Normal Mode Solution

$$y_n(x) = c_n \sin\left(\frac{n\pi x}{L}\right)$$

$$\omega_n = \sqrt{\frac{T}{\rho}}\ \frac{n\pi}{L}\ ,\ n = 1, 2, 3, \ldots \infty.$$

General Solution:

$$y(x,t) = \sum_{n=1}^{\infty} c_n \sin\left(\frac{n\pi x}{L}\right) e^{i\omega_n t}$$

Fourier's Trick

Once the normal modes and normal frequencies of a system are known, the only thing that remains is to find the expansion coefficients to describe the system at $t=0$. We use Fourier's Trick to do this.

For the loaded string, Fourier's Trick says

$$a_i = \frac{\vec{y}_0 \cdot \vec{q}_i}{|\vec{q}_i|^2}$$

where $\vec{y}_0$ is the set of initial positions $(y_1(t=0), y_2(t=0), \ldots\ldots)$

~~Four~~ For the continuous string fixed at $x=0$ and $x=L$, Fourier's Trick says

$$c_n = \frac{2}{L}\int_0^L y(x,t=0)\,\sin\left(\frac{n\pi x}{L}\right)dx$$

Fourier's Trick depends upon the fact that the eigen vectors are orthogonal:

$$\int_0^L \sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{m\pi x}{L}\right)dx = \frac{L}{2}\delta_{nm}$$ for the continuous string

and $$\sum_{j=1}^{N} \sin\left(\frac{jn\pi}{N+1}\right)\sin\left(\frac{jm\pi}{N+1}\right) = \left(\frac{N+1}{2}\right)\delta_{nm}$$ for the discrete loaded string.

Mathematics of Fourier Series & Fourier Transform

Any periodic function with period $2L$ can be written

$$F(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left[a_n\cos\left(\frac{n\pi x}{L}\right) + b_n\sin\left(\frac{n\pi x}{L}\right)\right]$$

where $$a_n = \frac{1}{L}\int_{-L}^{L} F(x)\cos\left(\frac{n\pi x}{L}\right)dx$$

$$b_n = \frac{1}{L}\int_{-L}^{L} F(x)\sin\left(\frac{n\pi x}{L}\right)dx.$$

This same thing can be written in complex notation:

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{in\pi x/L}$$

where $c_n = \frac{1}{2L}\int_{-L}^{L} F(x)\, e^{-in\pi x/L}\, dx$ ← Fourier's Trick

The two forms can be converted into each other:

$$a_n = c_n + c_{(-n)}$$
$$b_n = i(c_n - c_{(-n)})$$
$$a_0 = c_0 + c_0 = 2c_0$$

and

$$c_n = \begin{cases} \frac{1}{2}(a_{(-n)} + ib_{(-n)}) & , \text{ for } n < 0 \\ \frac{1}{2}a_0 & , \text{ for } n = 0 \\ \frac{1}{2}(a_n - ib_n) & , \text{ for } n > 0 \end{cases}$$

The complex form is more compact and elegant. It also generalizes to the case where $F(x)$ is <u>no longer periodic</u> (period $L \to \infty$):

$$F(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} dk\, A(k)\, e^{ikx}$$ (non-periodic $F(x)$)

$$A(k) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} F(x)\, e^{-ikx}\, dx$$ ⇐ "Plancherel's Theorem"

But it's really just another example of Fourier's Trick.

Travelling waves - Continuous String with no boundaries

If we have a continuous string with no boundary conditions (no walls), then we can have travelling wave solutions

$$y(x,t) = A \sin(kx - \omega t)$$

$k = \frac{2\pi}{\lambda}$, λ = wavelength.

$\omega = 2\pi f = \frac{2\pi}{T}$

The peaks and troughs move forward at the "phase velocity"

$$V_{phase} = v = \frac{\omega}{k} = \lambda f$$

The equation of motion is still the wave equation:

$$\frac{\partial^2 y}{\partial x^2} = \frac{\rho}{T} \frac{\partial^2 y}{\partial t^2}$$

And the travelling waves satisfy this as long as

$v = \sqrt{\frac{T}{\rho}}$ = phase velocity.

So we could write the Eq. of Motion as

$$\frac{\partial^2 y}{\partial x^2} = \frac{1}{v^2} \frac{\partial^2 y}{\partial t^2}$$

The general solution for a continuous string with no boundaries is a sum over all travelling waves. But since any wavelength is allowed, we have to sum over a continuum of k-values:

$$y(x,t=0) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} dk\, A(k)\, e^{ikx}$$

where $$A(k) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} dx\, F(x)\, e^{-ikx}$$

Then, as time goes forward, each normal mode (e^{ikx}) gets its own phase factor ($e^{i\omega t}$)

$$y(x,t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} dk\, A(k)\, e^{i(kx+\omega t)}$$

Review

Mechanical impedance of a string:

$$Z = \frac{\text{transverse force}}{\text{transverse velocity}}$$

$$Z = \frac{T \leftarrow \text{Tension}}{v_p \leftarrow \text{phase velocity}}$$

$$= \underset{\substack{\uparrow \\ \text{mass density}}}{\rho}\;\underset{\substack{\uparrow \\ \text{phase velocity}}}{v_p}$$

Reflection & Transmission at an impedance boundary

$$\frac{\text{transmitted amplitude}}{\text{incoming amplitude}} = \frac{A_2}{A_1} = \frac{2 Z_1}{Z_1 + Z_2}$$

$$\frac{\text{reflected amplitude}}{\text{incoming amplitude}} = \frac{B_1}{A_1} = \frac{Z_1 - Z_2}{Z_1 + Z_2}$$

Energy Transport by a wave

$$\text{Power} = \text{Energy Transmission} = \frac{1}{2}\,\underset{\substack{\uparrow \\ \text{impedance of string}}}{Z}\;\underset{\substack{\uparrow \\ \text{frequency}}}{\omega^2}\;\underset{\substack{\uparrow \\ \text{Amplitude of wave}}}{A^2}$$