Re-cap of complex Fourier Series

$$f(x) = \sum_{n=-\infty}^{+\infty} c_n e^{in\pi x/L}$$ for some set of coefficients $\{c_n\}$.

Sum from $-\infty$ to $+\infty$!!

This Fourier representation of $f(x)$ works as long as
① $f(x)$ is periodic with period $2L$ (or if you never evaluate $f(x)$ outside the $-L$ to L interval)
② $f(x)$ is ~~a~~ "square integrable"

But the right hand side is a complicated sum of many complex functions $e^{in\pi x/L}$. Each of these functions has a real & imaginary part. How can they represent a real $f(x)$ function?

Answer: For every function $e^{in\pi x/L}$, there is also a function $e^{-in\pi x/L}$. These two functions add together and cancel their imaginary components, as long as $c_n = c^*_{(-n)}$.

For example, consider the case $n = 6$.
Then we have 2 terms:

$$\ldots + c_6 e^{i6\pi x/L} + \ldots + c_{(-6)} e^{-i6\pi x/L} + \ldots$$

Now if $c_6 = c^*_{(-6)}$, then we can write

$$c_6^* = c_{(-6)}$$

Then the sum is

$$\ldots + c_6 e^{i6\pi x/L} + \ldots + \underbrace{c_6^* e^{-i6\pi x/L}}_{\left(c_6 e^{i6\pi x/L}\right)^*} + \ldots$$

$$\ldots + c_6 e^{i6\pi x/L} + \ldots + \left(c_6 e^{i6\pi x/L}\right)^* + \ldots$$

Now we can see that we are adding a complex number to its own complex conjugate:

Let $z_6 \equiv c_6 e^{i6\pi x/L}$ = a complex #

Then $z_6^* = \left(c_6 e^{i6\pi x/L}\right)^*$

What happens when you add a number to its complex conjugate? The imaginary part cancels:

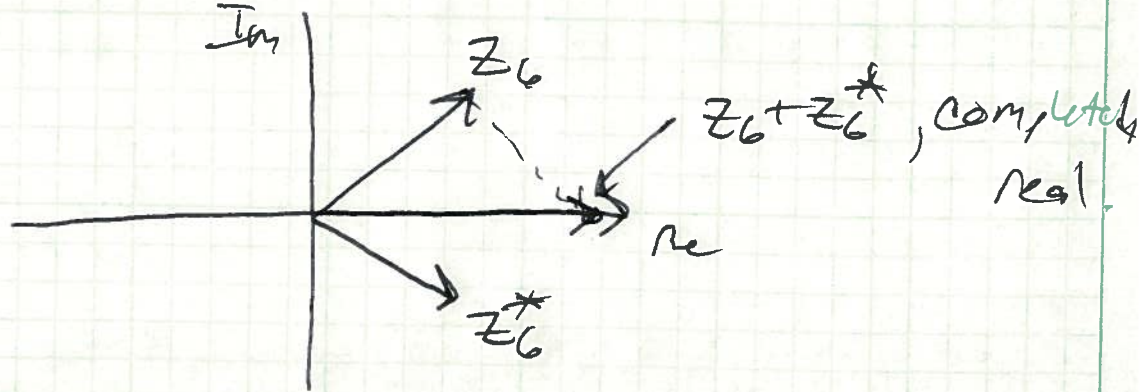
$z_6 = a_6 + i b_6$ ← real & imaginary parts of z_6

$z_6^* = a_6 - i b_6$

Then $z_6 + z_6^* = 2a_6$ ← completely real.

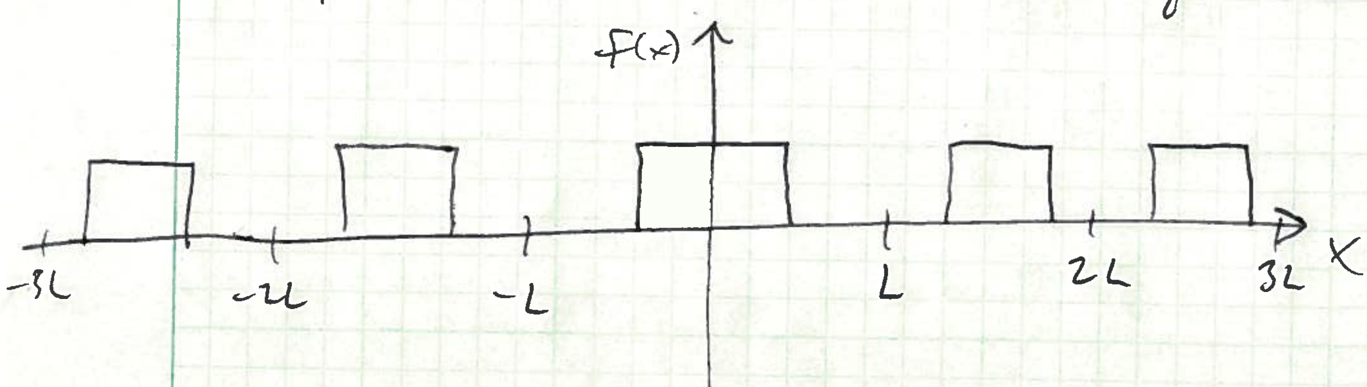
AMPAD™

Geometrically we add two vectors in the complex plane:



This will happen for every ~~term~~ pair of terms in the infinite sum as long as $\boxed{c(n) = c^*(-n)}$.

In the previous lecture we calculated the complex Fourier series for this square wave:

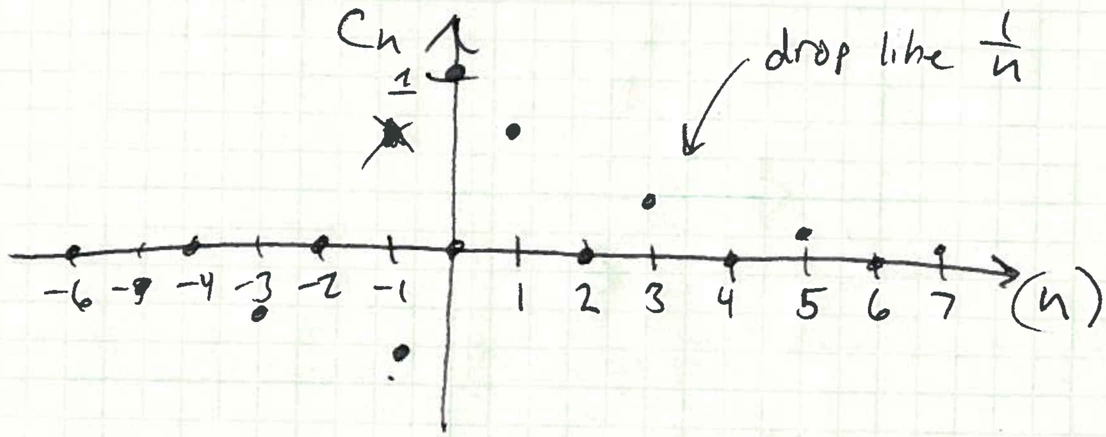


The result was

$$c_n = \frac{2}{n\pi}(-1)^{(n-1)/2} \quad , \text{ odd } (n) \text{ only}$$

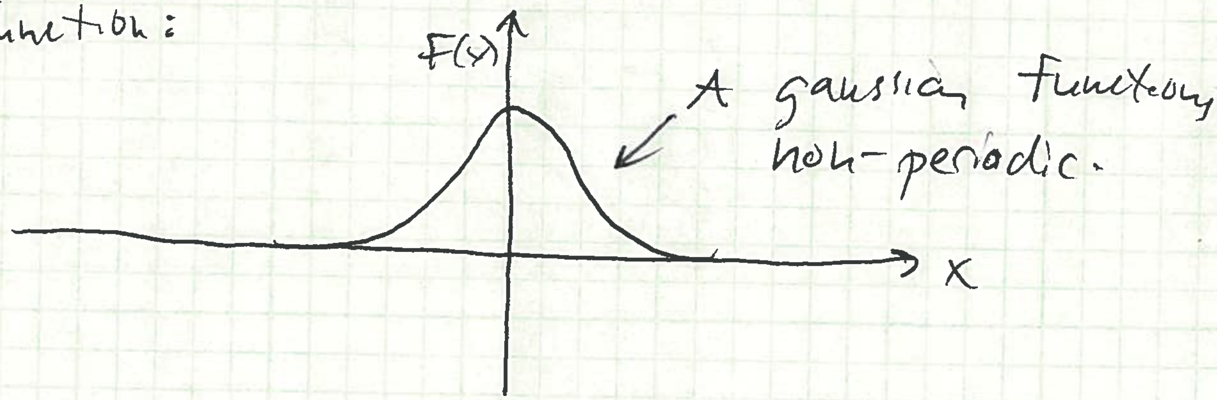
$$c_0 = 1 .$$

These coefficients are a function of a discrete variable, (n). We can plot them:



A periodic function $F(x)$ can be written as a Fourier series. The coefficients $\{C_n\}$ can be plotted like a discrete function of a discrete variable (n).

But suppose we have a non-periodic function that we want to represent as a Fourier Series, and we want to represent it all the way to $\pm\infty$. For example, suppose we want to represent a Gaussian function:



Can we do a Fourier Series for this function?

Answer: Yes. Our Fourier Series is periodic with period $2L$. To represent a non-periodic function, we simply ~~allow~~ let L go to infinity. After all, a non-periodic function is just like a periodic one with an infinite period.

Here's how it works. Start with our complex Fourier series:

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{in\pi x/L}$$

Now let Δn = the change in (n) between terms. Since $n = -2, -1, 0, 1, 2, \ldots$, $\Delta n = 1$. So we can write

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{in\pi x/L} \underbrace{(\Delta n)}_{1} = \sum_{n=-\infty}^{\infty} \left(\frac{L}{\pi}\right) c_n e^{in\pi x/L} \left(\frac{\pi \Delta n}{L}\right)$$

Now define $k \equiv \left(\frac{n\pi}{L}\right)$ and $\Delta k = \left(\frac{\pi \Delta n}{L}\right)$.

Also define $A(k) = \left(\sqrt{2\pi}\right)\left(\frac{L c_n}{\pi}\right)$

AMPAD™

Then

$$f(x) = \sum_{n=-\infty}^{\infty} \left(\frac{A(k)}{\sqrt{2\pi}}\right) e^{in\pi x/L} (\Delta k)$$

$$e^{in\pi x/L} = e^{ikx}$$

$$f(x) = \sum_{n=-\infty}^{\infty} \frac{A(k)}{\sqrt{2\pi}} e^{ikx} (\Delta k)$$

This is still the same Fourier series. It is periodic with period $2L$. We've simply rewritten it in a different format.

Example: What is $A(k)$? It is simply the coefficient function that we plotted earlier. So for the square wave (periodic), when

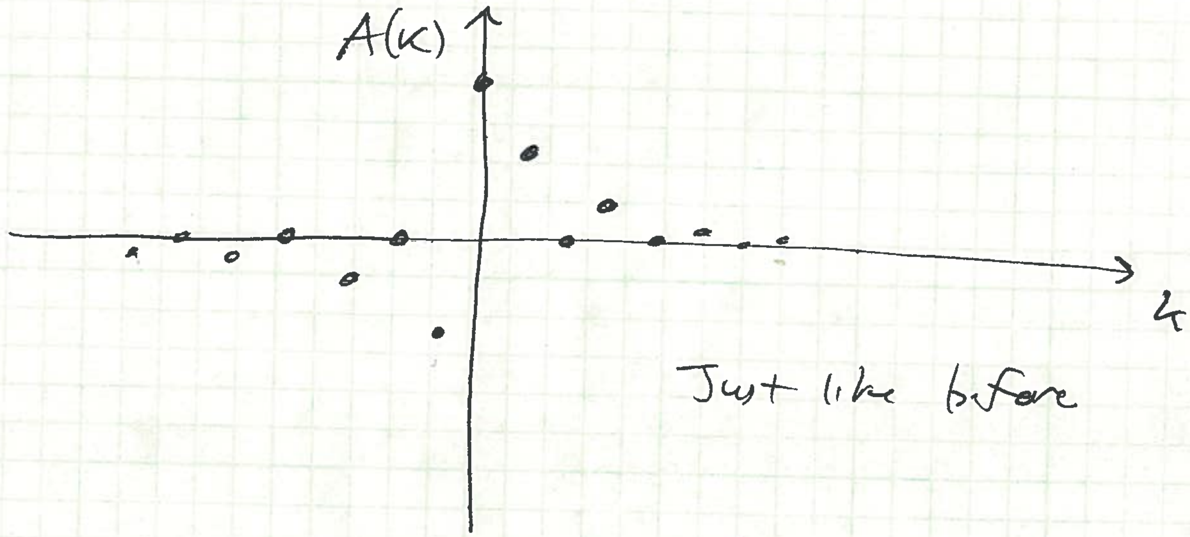
$$c_n = \left(\frac{2}{n\pi}\right)(-1)^{(n-1)/2}, \text{ odd } (n) \text{ only},$$

For this case we have

$$A(k) = \sqrt{2\pi}\, \frac{2L}{n\pi^2} (-1)^{(n-1)/2}, \text{ odd } n \text{ only}$$

$$= (\sqrt{2\pi})\, \frac{2\cancel{L}}{k\pi} (-1)^{(n-1)/2}, \text{ odd } n \text{ only}$$

So $A(k)$ plotted looks like



Just like before

Now we let $L \to \infty$, so we can represent a non-periodic function:

$$\cancel{F(x) = \sum_{k=-\infty}^{\infty} \frac{A(k)}{\sqrt{2\pi}} e^{ikx} \Delta k}$$

$$F(x) = \lim_{L \to \infty} \sum_{n=-\infty}^{\infty} \frac{A(k)}{\sqrt{2\pi}} e^{ikx} \Delta k$$

Since $\Delta k = \frac{\pi \Delta n}{L} = \frac{\pi}{L}$, as $L \to \infty$, $\Delta k \to dk$.

And k becomes a continuous variable, rather than a discrete variable, and the infinite sum becomes an integral:

$$F(x) = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} A(k) e^{ikx} dk$$

or $$\boxed{F(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dk \, A(k) e^{ikx},}$$ k - continuous

This is the Fourier representation of a non-periodic function. In this expression, the function $f(x)$ is ~~a~~ written as a sum of basis vectors $\{e^{ikx}\}$. In the discrete Fourier series, the basis vector were ~~a~~ discrete: $\{e^{in\pi x/L}\}$. For this continuous Fourier series, the ~~ee~~ basis vectors themselves are continuous, because k is continuous.

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \underbrace{A(k)}_{\text{A continuum of expansion coefficients}} \underbrace{e^{ikx}}_{\text{a continuum of basis vectors}} \, dk$$

~~the~~

Just like any Fourier series, the question is what are the correct coefficients to represent my function $f(x)$? For the discrete Fourier series, the coefficients were discrete, but for this continuous Fourier series, the expansion coefficients are a continuous function $A(k)$.

So how should we calculate $A(k)$ for a particular function $f(x)$?

Well, for a discrete Fourier Series we used Fouriers Trick:

$$C_n = \frac{1}{2L}\int_{-L}^{L} f(x)\, e^{-in\pi x/L}\, dx$$

Now re-write it as $k \equiv \frac{n\pi}{L}$

$$\underbrace{C_n\left(\frac{L}{\pi}\right)\sqrt{2\pi}}_{A(k)} = \frac{\sqrt{2\pi}}{2\pi}\int_{-L}^{L} f(x)\, e^{-ikx}\, dx$$

$$A(k) = \frac{1}{\sqrt{2\pi}}\int_{-L}^{L} f(x)\, e^{-ikx}\, dx$$

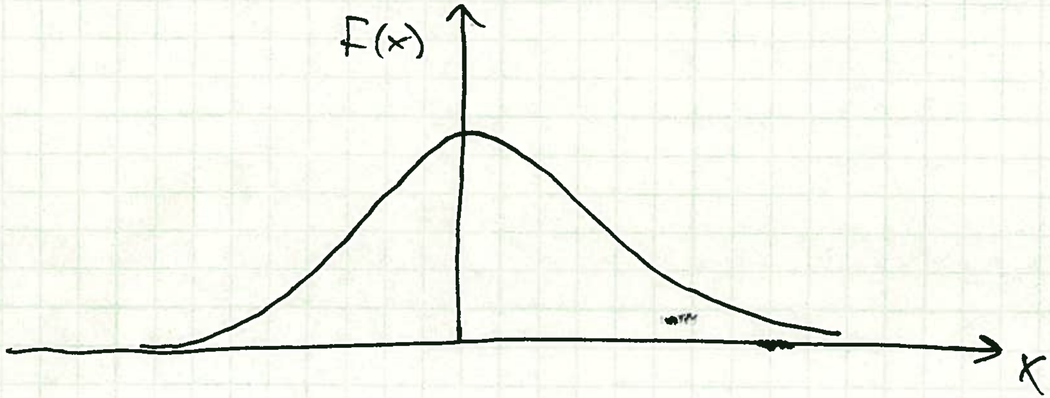
Now let $L \to \infty$:

$$\boxed{A(k) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)\, e^{-ikx}\, dx} \leftarrow \boxed{\text{Plancherel's Theorem}}$$

Just another case of Fouriers Trick.

Example Fourier Transform

Let $f(x) = e^{-ax^2}$ (a gaussian)



Maybe this represents the shape of an infinitely long string at $t = \emptyset$.

Question: What is the Fourier Transform $A(k)$?

Answer: $A(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \left(e^{-ax^2} \right) e^{-ikx} \, dx$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-(ax^2 + ikx)} \, dx$$

Complete the square: Define $s \equiv \sqrt{a} \left(x + \frac{ik}{2a} \right)$

Then $s^2 = ax^2 - \frac{k^2}{4a} + ikx$, and $ds = \sqrt{a} \, dx$

$$A(k) = \frac{1}{\sqrt{2\pi}} \frac{1}{\sqrt{a}} e^{-k^2/4a} \int_{-\infty}^{\infty} e^{-(ax^2 + ikx - k^2/4a)} \left(\sqrt{a} \, dx \right)$$

$$= \frac{e^{-k^2/4a}}{\sqrt{2\pi a}} \underbrace{\int_{-\infty}^{\infty} e^{-s^2} \, ds}_{\sqrt{\pi}}$$

Progressive or Traveling waves

For a string fixed at $x=\emptyset$ & $x=L$, the normal modes are

$$y_n(x,t) = A_n \sin\left(\frac{n\pi x}{L}\right) e^{i\omega_n t} \quad , \omega_n = \sqrt{\frac{T}{\mu}}\,\frac{n\pi}{L}$$

The general solution is a sum of normal modes:

$$y(x,t) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi x}{L}\right) e^{i\omega_n t}$$

Each normal mode satisfies the boundary conditions:

$$y(x=0,t) = \emptyset$$
$$y(x=L,t) = \emptyset$$

string is fixed at $x=\emptyset$ & $x=L$ for all time

Check:

$$y_n(x=\emptyset,t) = A_n \sin\left(\frac{n\pi(\emptyset)}{L}\right) e^{i\omega_n t} = \emptyset$$

and

$$y_n(x=L,t) = A_n \sin\left(\frac{n\pi L}{L}\right) e^{i\omega_n t}$$
$$= A_n \sin(n\pi) e^{i\omega_n t} = \emptyset$$

Since each normal mode satisfies the boundary condition, the general solution is guaranteed to satisfy the boundary condition, since its just a sum of normal modes.

Strictly speaking we should write the normal modes like this:

$$y_n(x,t) = \begin{cases} 0, & x < 0 \\ A_n \sin\left(\frac{n\pi x}{L}\right) e^{i\omega_n t}, & 0 \le x \le L \\ 0, & x > L \end{cases}$$

since the string only exists between 0 and L.

We can re-write our normal modes this way:

$$y_n(x,t) = A_n \sin\left(\frac{n\pi x}{L}\right) e^{i\omega_n t}$$

take the real part

$$= A_n \sin\left(\frac{n\pi x}{L}\right) \cos(\omega_n t)$$

trig identity

$$= A_n \frac{1}{2}\left[\sin\left(\left(\frac{n\pi}{L}\right)x - \omega_n t\right) + \sin\left(\left(\frac{n\pi}{L}\right)x + \omega_n t\right)\right]$$

substitute: $\omega_n = \sqrt{\frac{T}{\mu}}\frac{n\pi}{L}$

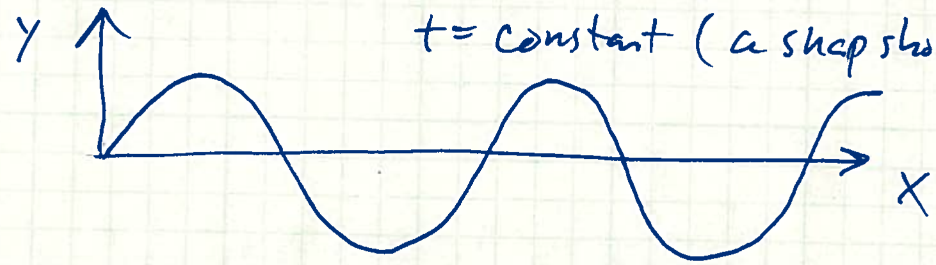
$$y_n(x,t) = \frac{A_n}{2}\left[\sin\left(\left(\frac{n\pi}{L}\right)\left(x - \sqrt{\frac{T}{\mu}}t\right)\right) + \sin\left(\left(\frac{n\pi}{L}\right)\left(x + \sqrt{\frac{T}{\mu}}t\right)\right)\right]$$

Simplify notation:

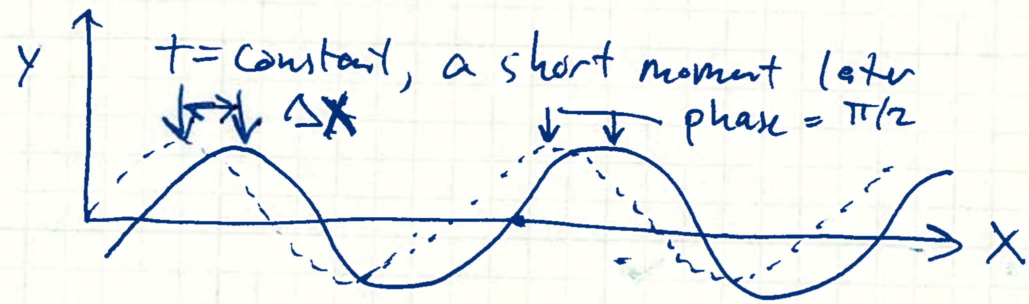
Let $k \equiv \frac{n\pi}{L}$ = "wave number"

$$y_n(x,t) = \frac{A_n}{2}\left[\sin\left(k\left(x - \sqrt{\frac{T}{\mu}}t\right)\right) + \sin\left(k\left(x + \sqrt{\frac{T}{\mu}}t\right)\right)\right]$$

Here the normal mode is written as a sum of two sine functions. Each sine function is known as a travelling wave (or progressive wave). At any moment in time, each sine function looks like a perfect sine wave:



As time goes forward, the sine wave moves to the right or left:



How far has the wave advanced? Let's see how the phase changes:

$$\text{phase} = k\left(x - \sqrt{\tfrac{T}{\mu}}\,t\right) = \text{constant in time}$$

How can we "ride" follow the wave as time goes forward? We must hold the phase constant:

$$\Delta(\text{phase}) = \emptyset = k\left(\Delta x - \sqrt{\tfrac{T}{\mu}}\,\Delta t\right)$$

$$\Delta x = \sqrt{\frac{T}{\mu}}\,\Delta t$$

$$\frac{\Delta x}{\Delta t} = \sqrt{\frac{T}{\mu}}$$

$$\frac{\Delta x}{\Delta t} = \text{units of velocity.}$$

Let's check:

$$\sqrt{\frac{T}{\mu}} = \sqrt{\frac{\text{Newtons}}{\cancel{\text{mass}}\ \text{kg/m}}} = \sqrt{\frac{\text{kg}\cdot\text{m}\cdot\text{m}}{\text{s}^2\cdot\text{kg}}}$$

$$= \frac{\text{m}}{\text{s}} \checkmark$$

yes, units of velocity.

So the point of constant ~~velocity~~ phase advances at a speed of

$$v = \sqrt{\frac{T}{\mu}} = \text{"phase velocity"}$$

The phase velocity is the speed at which a peak or trough moves in a travelling wave.

So our normal mode solution is

$$y_n(x,t) = \frac{A_n}{2} \sin\left[\left(k\underbrace{(x-vt)}_{\text{rightward}}\right) + \sin\left(k\underbrace{(x+vt)}_{\text{leftward}}\right)\right]$$

It's a sum of two travelling waves, one moving to the right, and one moving to the left.

Suppose we consider each travelling wave by itself:

$$y(x,t) = \frac{A_n}{2} \sin\left[k(v-vt)\right]$$

~~Is~~ Does this travelling wave satisfy the equation of motion all by itself?

Let's check: The equation of motion is the wave equation:

$$\frac{\partial^2 y}{\partial x^2} = \left(\frac{\mu}{T}\right)\frac{\partial^2 y}{\partial t^2}$$

If $y = \frac{A_n}{2}\sin\left(k(x-vt)\right)$, Then

$$\frac{\partial^2 y}{\partial x^2} = \frac{A_n}{2}(-k^2)\sin\left(k(x-vt)\right)$$

and $$\frac{\partial^2 y}{\partial t^2} = \frac{A_n}{2}(-v^2k^2)\sin\left(k(x-vt)\right)$$

∴ To satisfy the wave equation we must have

$$\frac{A_n}{2}(-k^2)\sin\left(k(x-vt)\right) \stackrel{?}{=} \left(\frac{\mu}{T}\right)\frac{A_n}{2}(-v^2k^2)\sin\left(k(x-vt)\right).$$

$$k^2 \stackrel{?}{=} \frac{\mu}{T}v^2k^2$$

$$v^2 \stackrel{?}{=} \frac{T}{\mu}$$ ✓ Yes, This is the definition of v.

So the travelling wave does satisfy the wave equation.

Since $T/\mu = v^2$, we should really write the wave equation as

$$\boxed{\frac{\partial^2 y}{\partial x^2} = \frac{1}{v^2}\frac{\partial^2 y}{\partial t^2}}$$ wave equation.

The nice thing about this form of the equation is that we can explicitly see the velocity of travelling waves in the equation itself.

One peculiar aspect of the travelling waves is that they satisfy the wave equation, but they cannot be written as a linear combination of ~~normal modes~~ our typical normal modes:

$$\text{travelling wave} \stackrel{?}{=} \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi x}{L}\right) e^{i\omega_n t} \quad ??$$

No. There is no way to find a set of coefficients $\{A_n\}$ to make a travelling wave. Why not?

The answer: ~~Ex~~ The normal mode solution written above is for a string fixed between $x=0$ and $x=L$. ~~So~~ But each travelling wave, by itself, violates these boundary conditions:

$$\left.\begin{aligned} y(x=0,t) &= 0 \\ y(x=L,t) &= 0 \end{aligned}\right\}$$ travelling wave does not satis

But for an <u>infinitely</u> long string, with <u>no boundaries</u>, the travelling wave is perfectly allowed.

So consider an infinitely long string, with <u>no boundary</u>. An acceptable solution is

$$y(x,t) = A\sin\left(k(x-vt)\right)$$

or $$y(x,t) = A\sin(kx-\omega t)$$

when $\omega = kv$. or $v = \dfrac{\omega}{k}$.

What is the "spatial period" of this wave?

"spatial period" = "wavelength" $= \lambda$.
= change in x which changes the phase by 2π.

phase $= kx-\omega t = 2\pi$ when $x=\lambda$.

$k\lambda = 2\pi + \omega t$

Hold t fixed at $t=\emptyset$:

$k\lambda = 2\pi$

$$\boxed{\lambda = \frac{2\pi}{k}} \text{ or } \boxed{k = \frac{2\pi}{\lambda}}$$

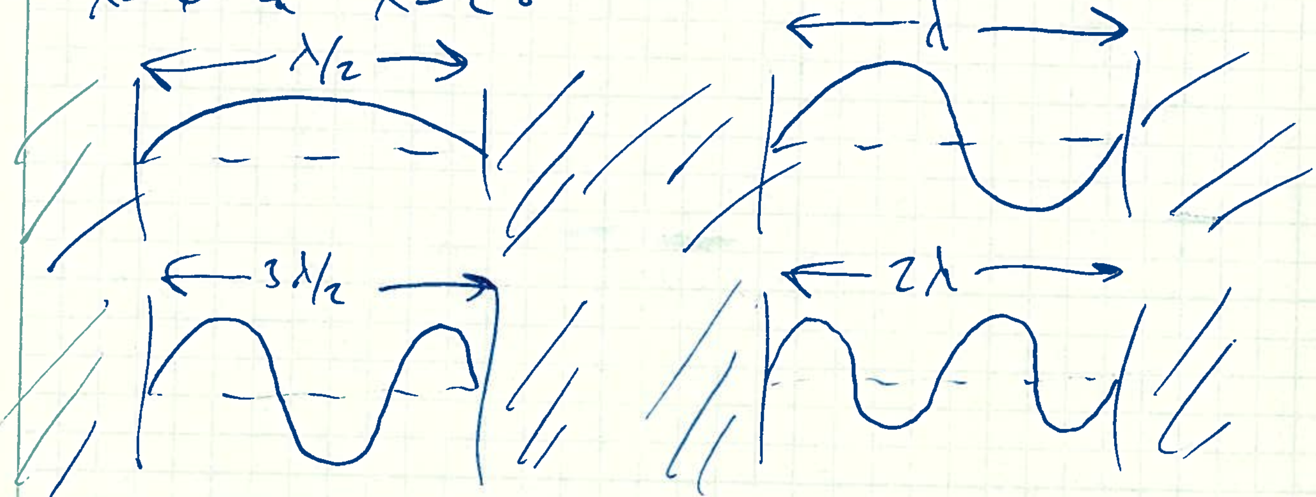
Similarly, $\omega = \dfrac{2\pi}{T}$, T = time period
= time to advance the phase by 2π.

If we also define $\boxed{f \equiv \frac{1}{T}}$, then

$$\boxed{\omega = 2\pi f}$$

What values of λ (or k) are allowed?

For a string fixed at $x = \emptyset$ & $x = L$, only ~~ce~~ certain values of λ are allowed, because the string displacement must go to zero at $x = \emptyset$ & $x = L$:



For these boundary conditions to be satisfied, we must have

$$n\left(\frac{\lambda}{2}\right) = L$$

$$\text{or} \quad \lambda = \frac{2L}{n}$$

For fixed string.

$$\text{or} \quad k = \frac{2\pi}{\lambda} = \frac{2\pi}{(2L/n)} = \boxed{\frac{n\pi}{L}}$$

Only a discrete set of k are allowed because of the boundary conditions.

We can call the discrete set of k values

$$k_n = \frac{n\pi}{L}.$$

Also, since k & ω are related by

$$\omega = kv,$$

If k is discrete (k_n), then ω must also be discrete:

$$\omega_n = k_n v = \left(\frac{n\pi}{L}\right) v$$

?

And what is v? It is determined by the equation of motion:

$$\frac{\partial^2 y}{\partial x^2} = \frac{1}{v^2}\frac{\partial^2 y}{\partial t^2}$$

For the continuous string, the equation of motion is

$$\frac{\partial^2 y}{\partial x^2} = \left(\frac{\mu}{T}\right)\frac{\partial^2 y}{\partial t^2}$$

So $v = \sqrt{\frac{T}{\mu}}$, and $\boxed{\omega_n = \sqrt{\frac{T}{\mu}}\,\frac{n\pi}{L}}$

The boundary conditions "pick out" a discrete set of k & ω values which are allowed.

This is why the general solution for the fixed string is a discrete sum over discrete normal modes:

$$y_n(x,t) = \sum_{n=1}^{\infty} \sin\left(\frac{n\pi x}{L}\right) e^{i\omega_n t} \qquad \left(\frac{n\pi}{L} = k_n\right)$$

$$= \sum_{n=1}^{\infty} \sin(k_n x)\, e^{i\omega_n t}$$

discrete sum over discrete k & discrete ω.

It's the boundary conditions.

But what if we get rid of the boundary conditions? Then any value of k is allowed, as long as its frequency satisfies $\boxed{\omega = vk}$

ω and k must have this relationship in order to satisfy the wave equation.

With no boundary conditions, k becomes continuous, and any value is allowed, as long as its frequency is given by $\omega = kv$.

So the general solution with no boundary conditions is no longer a discrete sum over discrete normal modes. Instead it is a continuous sum over a continuum of allowed k values. It is a Fourier Transform, instead of a Fourier Series:

$$y(x,t) = \Bigg[\underbrace{\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} A(k)\, e^{ikx}\, dk}_{\text{continuous sum over a continuum of allowed } k \text{ values.}}\Bigg] e^{i\omega t}$$

or $$y(x,t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} dk\, A(k)\, e^{i(kx+\omega t)}$$