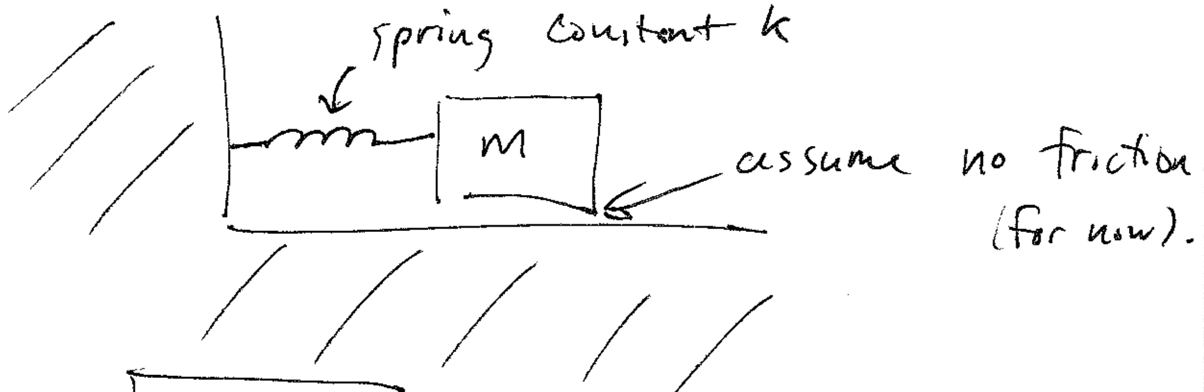
Simple Harmonic Oscillator (SHO)

Consider a mass on a spring (horizontal)



Force = $\boxed{F = -kx}$, as long as we choose $x = \emptyset$ to be the location when the spring exerts no force.

"Hooke's Law" (empirical).

minus sign means that the force tries to restore the equilibrium position

Equation of motion (Newton's 2nd Law)

$$F = ma$$

$$-kx = m\ddot{x} \qquad \ddot{x} \equiv \frac{d^2x}{dt^2} = a$$

$$\boxed{\ddot{x} + \frac{k}{m}x = \emptyset}$$ ← SHO equation of motion.

Solution: $\boxed{x(t) = A\cos(\omega t + \delta)}$

Constants:

(1) A = "amplitude" = units of meters (in SI)
= maximum displacement

(2) ω = "angular frequency" = units of Hz (sec^{-1})

(3) δ = phase = units of radians (unitless).

<u>Note</u>: The argument of the cosine function should be in <u>radians</u>.

<u>Note</u>: The cosine function returns a number between -1 and $+1$, which is <u>unitless</u>.

<u>Question</u>: Why does the solution have <u>3</u> constants?

<u>Answer</u>: This equation of motion is <u>2^{nd} order in time</u>. (This just means that a 2^{nd} time derivative appears in the equation.)

> Any 2^{nd} order Differential Equation requires <u>2</u> initial conditions. These 2 initial conditions will determine 2 of the 3 constants.

For example, specifying the position and velocity at $t = 0$ will determine A & δ.

But what about ω? Can we change ω by choosing different initial conditions?

Answer: No. ω is determined by k and m.

Substitute solution into the Eq. of motion.

$$\ddot{x} + \frac{k}{m}x = 0 \quad \text{where } x(t) = A\cos(\omega t+\delta)$$

$$\ddot{x}(t) = -A\omega^2\cos(\omega t+\delta)$$

$$-A\omega^2\cos(\omega t+\delta) + \left(\frac{k}{m}\right)\left(A\cos(\omega t+\delta)\right) = 0.$$

$$\left(\frac{k}{m} - \omega^2\right)A\cos(\omega t+\delta) = 0$$

How can this equation ~~be~~ be satisfied?

Option 1: $A = 0 \Rightarrow$ This means that $x(t) = 0$. (the trivial solution.)

Option 2: $\boxed{\omega = \sqrt{\frac{k}{m}}}$ ← Simple Harmonic Oscillator Frequency.

All masses on springs vibrate at a frequency of $\omega = \sqrt{k/m}$, no matter what the initial conditions are.

We call this the "natural frequency", and we use the symbol ω_0:

$$\boxed{\omega_0 \equiv \sqrt{\frac{k}{m}}}$$

Energy considerations

Potential energy is stored in the spring:

$$PE(x) = U(x) = -\int_\emptyset^x F(\underbrace{x')\,dx'}_{\text{integration variable}} = \int_\emptyset^x kx'\,dx'$$

$$= \tfrac{1}{2}kx^2$$

$$\boxed{U(x) = \tfrac{1}{2}kx^2} \text{ for a spring.}$$

Kinetic energy:

$$KE = \tfrac{1}{2}mv^2 = \tfrac{1}{2}m\dot{x}^2$$

As a function of time:

$$U(t) = U(x(t)) = \tfrac{1}{2}k\left[A\cos(\omega_0 t+\delta)\right]^2 = \tfrac{1}{2}kA^2\cos^2(\omega t+\delta)$$

$$KE(t) = \tfrac{1}{2}m\left[-A\omega_0 \sin(\omega_0 t+\delta)\right]^2 = \tfrac{1}{2}m\omega_0^2 A^2 \sin^2(\omega t+\delta)$$

$$\text{Total energy} = KE(t) + U(t)$$

$$= \tfrac{1}{2}\underbrace{m\omega_0^2}_{k} A^2 \sin^2(\omega_0 t+\delta) + \tfrac{1}{2}kA^2\cos^2(\omega_0 t+\delta)$$

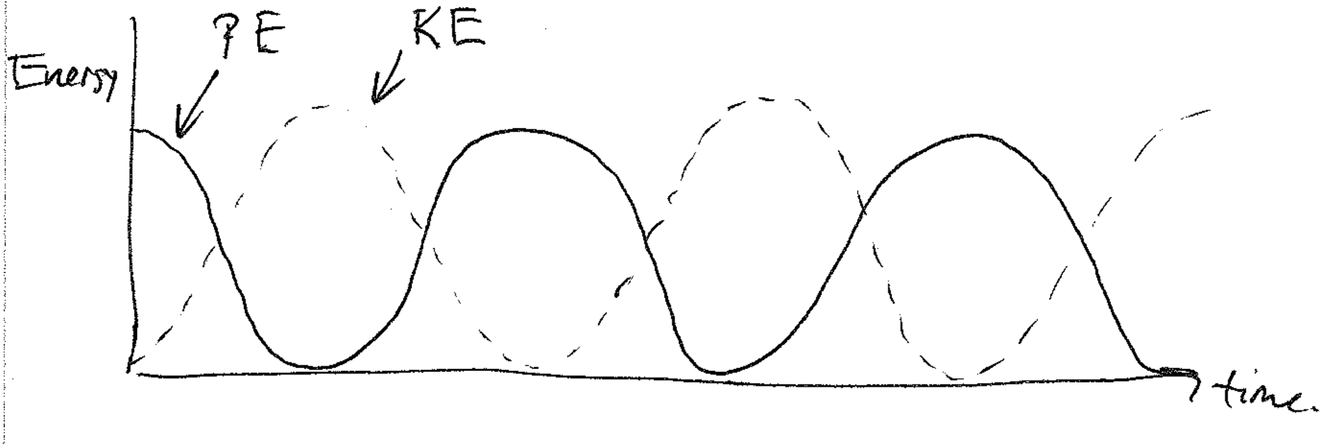
$$= \tfrac{1}{2}kA^2\left(\underbrace{\sin^2(\omega_0 t+\delta) + \cos^2(\omega_0 t+\delta)}_{1}\right)$$

$$= \tfrac{1}{2}kA^2 = \text{constant.}$$

Energy is conserved.

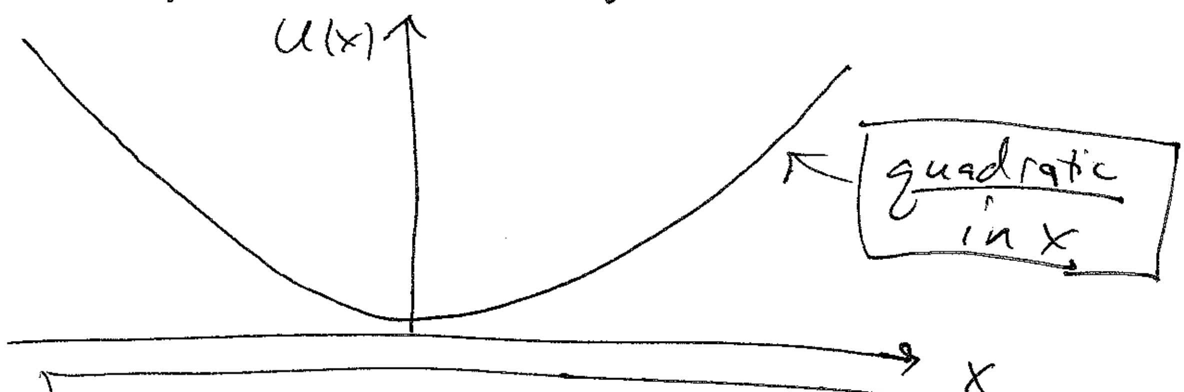
(Surprise, Surprise)

Energy is being exchanged between kinetic and potential:



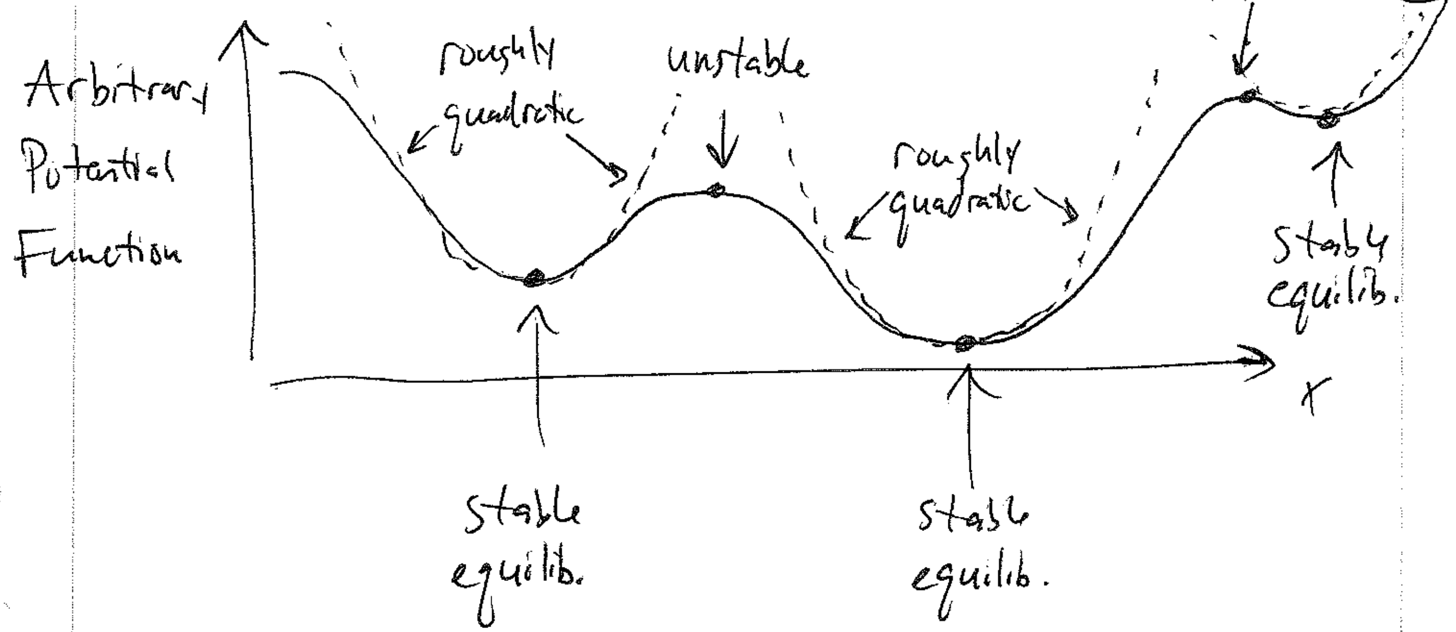
Harmonic Potential and Small Oscillations

For a spring $PE = U(x) = \frac{1}{2}kx^2$, where x is displacement from equilibrium.



$\Rightarrow$ A quadratic potential gives rise to simple harmonic motion.

This is very useful, because almost any potential function will be approximately quadratic near a stable equilibrium:



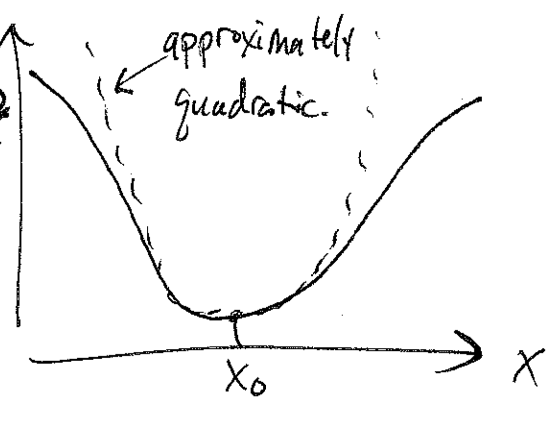
$\Rightarrow$ small oscillations about stable equilibria will be approximately simple harmonic motion.

To be <u>formal</u>, expand the potential function as a Taylor series about the stable equilibrium:

$$U(x_0+a) = U(x_0) + U'(x_0)a + \tfrac{1}{2}U''(x_0)a^2 + \ldots$$

equilibrium position (x_0)

displacement from equilibrium (a)



What does this expansion look like for a mass on a spring?

For a spring, $U(x) = \frac{1}{2}kx^2$

$x_0 =$ equilibrium position $= \emptyset$

$U(x_0) = \emptyset$

$U'(x) = kx$

$\boxed{U'(x_0) = \emptyset}$ ← This must always be true at an equilibrium position, because this is the negative of the force, and force $= \emptyset$ at equilibrium.

$U''(x) = k$

and $x = x_0 + a = a$ (because $x_0 = \emptyset$).

$\therefore$ For a spring, $\underset{\substack{\uparrow \\ x_0 = \emptyset}}{U(x_0+a)} = U(a) = \emptyset + (\emptyset)a + \frac{1}{2}ka^2$ + no other terms.

$\therefore \boxed{U(a) = \frac{1}{2}ka^2}$ ← exact result for a perfect spring.

Frequency of small oscillations:

we know that a perfect spring creates a SHO with angular frequency

$$\omega_0 = \sqrt{\frac{k}{m}}$$

and $K = U''(x_0)$, so

$$\omega_0 = \sqrt{\frac{U''(x_0)}{m}}$$

So, for a generic potential, we will have approximately simple harmonic motion near a stable equilibrium with natural frequency

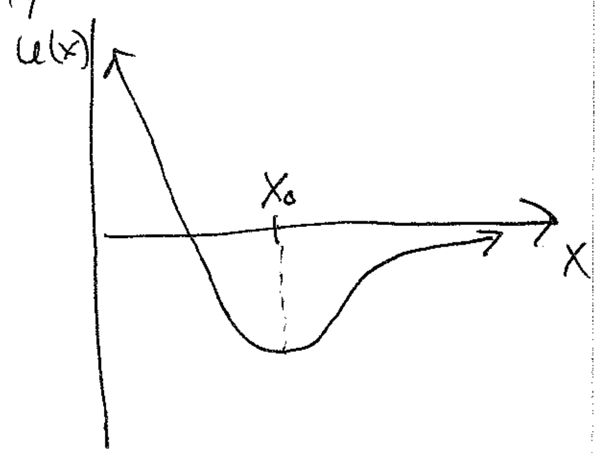
$$\boxed{\omega_0 \approx \sqrt{\frac{U''(x_0)}{m}}}$$

← Frequency of small oscillations near stable equilibrium.

Example: Diatomic Molecule

Consider 2 atoms bound together in a molecule, one heavy, and one light. The potential experienced by the light atom is roughly

$$U(x) = \frac{-a}{x^6} + \frac{b}{x^{12}}$$



The Force is

$$F = -\frac{dU}{dx} = \frac{1}{x^7}\left(\frac{12b}{x^6} - 6a\right)$$

Equilibrium position: $F(x_0) = \emptyset$.

$$6a = \frac{12b}{x_0^6} \implies x_0 = \left(\frac{2b}{a}\right)^{\frac{1}{6}}$$

Frequency of small oscillations:

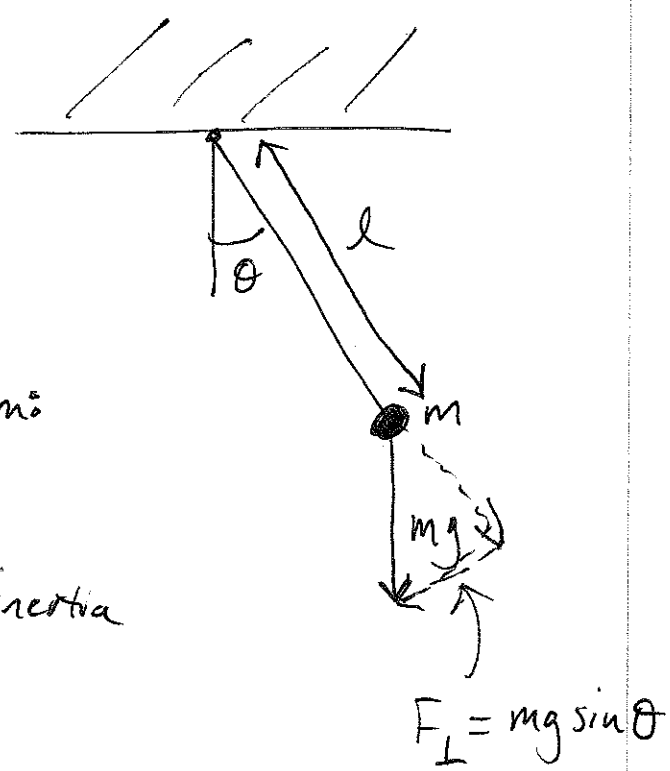
$$U''(x_0) = -\frac{42a}{x_0^8} + \frac{156b}{x_0^{14}} = \frac{72b}{(2b/a)^{7/3}}$$

$$x_0 = \left(\frac{2b}{a}\right)^{\frac{1}{6}}$$

Therefore

$$\boxed{\omega_0 = \sqrt{\frac{U''(x_0)}{m}} = \sqrt{\frac{72b}{m(2b/a)^{7/3}}}}$$

Plane Pendulum



Old-Fashioned method:

torque $= -F_\perp l = -mgl\sin\theta$

Newton's 2nd Law in angular form:

torque $= I\ddot{\theta}$

moment of inertia $= ml^2$

$$-mgl\sin\theta = ml^2\ddot{\theta}$$

$$\boxed{\ddot{\theta} + \frac{g}{l}\sin\theta = 0}$$ Equation of Motion.

This equation is <u>not</u> the simple harmonic oscillator equation because of the $\sin(\theta)$ factor. But for small θ,

$$\sin\theta \approx \theta \text{ , so that}$$

$$\boxed{\ddot{\theta} + \frac{g}{l}\theta \approx \emptyset} \quad \text{for small } \theta$$

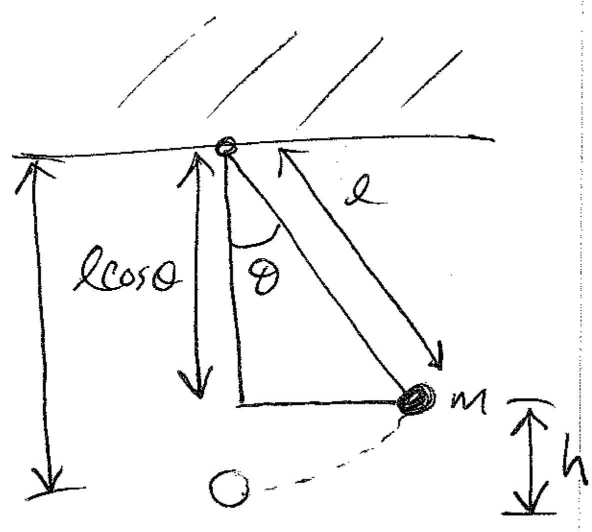
Now we can read off the frequency for small oscillations:

$$\boxed{\omega_0 = \sqrt{g/l}} \quad \text{for small oscillations.}$$

More sophisticated analysis of the plane pendulum:

Potential energy:

$$U(h) = mgh$$
$$= mg(l - l\cos\theta)$$
$$= mgl(1-\cos\theta)$$

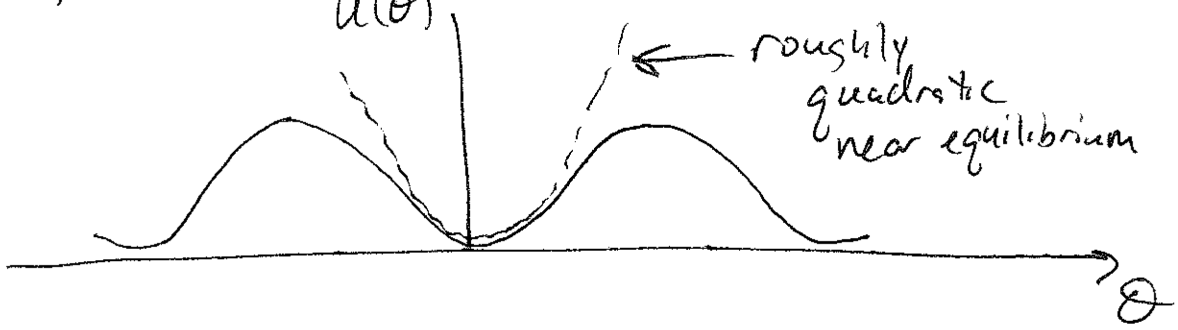


Then $\dfrac{d^2U}{d\theta^2} = mgl\cos\theta$

and $\dfrac{d^2U}{d\theta^2}(\theta=\emptyset) = mgl$

Then $\omega_0 = \sqrt{\dfrac{\frac{d^2U}{d\theta^2}(\theta=\emptyset)}{\underbrace{I}_{ml^2}}} = \sqrt{\dfrac{mgl}{ml^2}} = \sqrt{\dfrac{g}{l}}$

same result!


$u(\theta)$
roughly quadratic near equilibrium
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