UNIVERSITY OF MARYLAND, PHYSICS DEPARTMENT

PHYS 272, HW 8 Solutions

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1 PROBLEM I

For any scalar f:

$$\begin{aligned}\nabla\times\nabla f &= \nabla\times\left(\frac{\partial}{\partial x}f\hat{x}+\frac{\partial}{\partial y}f\hat{y}+\frac{\partial}{\partial z}f\hat{z}\right)\\ &= \left[\frac{\partial}{\partial y}\frac{\partial}{\partial z}f-\frac{\partial}{\partial z}\frac{\partial}{\partial y}f\right]\hat{x}\\ &-\left[\frac{\partial}{\partial x}\frac{\partial}{\partial z}f-\frac{\partial}{\partial z}\frac{\partial}{\partial x}f\right]\hat{y}\\ &+\left[\frac{\partial}{\partial x}\frac{\partial}{\partial y}f-\frac{\partial}{\partial y}\frac{\partial}{\partial x}f\right]\hat{z}\end{aligned} \tag{1.1}$$

By chain rule $\frac{\partial}{\partial i}\frac{\partial}{\partial j}f=\frac{\partial}{\partial j}\frac{\partial}{\partial i}f$ for some generic scalar function f and generic coordinates i, j. So:

$$\begin{aligned}\nabla\times\nabla f &= 0\hat{x}+0\hat{y}+0\hat{z}\\ &= 0\end{aligned} \tag{1.2}$$

Proved.
For a generic vector $\mathbf{A}=A_x\hat{x}+A_y\hat{y}+A_z\hat{z}$:

$$\begin{aligned}\nabla\cdot\nabla\times\mathbf{A} &= \nabla\cdot\left(\left[\frac{\partial}{\partial y}A_z-\frac{\partial}{\partial z}A_y\right]\hat{x}-\left[\frac{\partial}{\partial x}A_z-\frac{\partial}{\partial z}A_x\right]\hat{y}+\left[\frac{\partial}{\partial x}A_y-\frac{\partial}{\partial y}A_x\right]\hat{z}\right)\\ &= \frac{\partial}{\partial x}\left[\frac{\partial}{\partial y}A_z-\frac{\partial}{\partial z}A_y\right]-\frac{\partial}{\partial y}\left[\frac{\partial}{\partial x}A_z-\frac{\partial}{\partial z}A_x\right]+\frac{\partial}{\partial z}\left[\frac{\partial}{\partial x}A_y-\frac{\partial}{\partial y}A_x\right]\\ &= 0\end{aligned} \tag{1.3}$$

In the last step, you can just rearrange terms and make use of the chain rule. It is then easy to see that terms cancel out and the final result is just zero. Proved.

2 Problem II

Consider the toroidal solenoid with a constant cross section all the way around. Choose the coordinate system such that the center of the solenoid sits at the origin. To compute the magnetic field **B** at some arbitrary point **P**, consider how individual infinitesimal segments of wire $d\mathbf{l}$ contribute to the field. Because the solenoid has rotational symmetry, you can pick a coordinate such that **P** sits in the $x-z$ plane. Then, by the Biot-Savart law, the infinitesimal piece of wire $d\mathbf{l}$ will contribute:

$$d\mathbf{B}=\frac{\mu_0 I d\mathbf{l}\times\mathbf{r}}{4\pi r^3} \tag{2.1}$$

Where $\mathbf{r}$ is the vector that points from $d\mathbf{l}$ to **P** with magnitude $||\mathbf{r}||=r$, then:

$$d\mathbf{l}\times\mathbf{r}=(r_z dl_y-r_y dl_z)\hat{x}-(r_z dl_x-r_x dl_z)\hat{y}+(r_y dl_x-r_x dl_y)\hat{z} \tag{2.2}$$

Because of the symmetry of the torus, we can find another infinitesimal contribution to the magnetic field that is exactly the same, but reflected across the $x-z$ plane. Denote this by $d\mathbf{B}'$ So:

$$d\mathbf{B}+d\mathbf{B}'=\frac{\mu_0 I}{2\pi}(r_x dl_z-r_z dl_x)\hat{y} \tag{2.3}$$

Therefore, for an arbitrary point **P**, each bit of current contributing to $\mathbf{B}(\mathbf{P})$ has another contribution from somewhere else whose summed contribution only has a component along the theta unit vetor $\hat{\theta}$. So, when we add up all the contributions, we end up with just a $\hat{\theta}$ component: at each point $\mathbf{B}=B_\theta\hat{\theta}$.

Now you can construct a simple amperian loop as follows:

$$\begin{aligned}B2\pi r &= \mu N I\\ B &= \frac{\mu N I}{2\pi r}\end{aligned} \tag{2.4}$$

3 Problem III (graded)

Pick your axis such that the magnetic field is perfectly aligned along a single component: $\mathbf{B} = B_z\hat{z}$ (the same result will hold indenpent of the coordinate you choose to align the field.) Once the electron enters the magnetic field, it will experience an acceleration that is described by the Lorentz force law (I believe in previous solutions I referred to the Lorentz force law as the Coulomb force law, apologies for any confusion):

$$\begin{aligned}\mathbf{F} &= q(\mathbf{E}+\mathbf{v}\times\mathbf{B})\\ &= q(v_x\hat{x}+v_y\hat{y}+v_z\hat{z})\times B_z\hat{z}\\ &= q(v_y\hat{x}-v_x\hat{y})B_z\end{aligned} \tag{3.1}$$

Then, by Newton's second law:

$$\begin{aligned}\mathbf{F} &= m\mathbf{a}\\ &= qB_z(v_y\hat{x}-v_x\hat{y})\\ &= \frac{m\mathbf{v}^2}{r}\end{aligned} \tag{3.2}$$

Therefore the electron undergoes circular motion in the $x-y$ plane, and move at constant velocity along the $z-$axis (so the motion looks like a helix). Denote the resultant velocity in the $x-y$ plane $\mathbf{v}' = v_x\hat{x}+v_y\hat{y}$ (recall that the circular motion only takes place in the $x-y$ plane, so we are not interested in what happens in the $z-$axis). The resultant frequency of motion is then:

$$f = \frac{||\mathbf{v}||}{2\pi r} = \frac{||q\mathbf{B}||}{2\pi m} = \frac{eB_z}{2\pi m} \tag{3.3}$$