PHY 272: FIELDS
PROBLEM SET 7
due February 18, **before** *class*

I. CURL

We finish our quick tour of vector calculus this week with the concept of **curl**. We know that, given a vector field $\mathbf{v}$ and a closed oriented path C we can compute the line integral

$$\oint \mathbf{v}.d\mathbf{l}. \tag{1}$$

This kind of integral is called the *circulation* of $\mathbf{v}$ around the path C. Notice that the sign of the circulation depends on the orientation of the path.

i) To sharpen our intuition, look at the figure below and write down whether the circulation is positive, negative of zero.

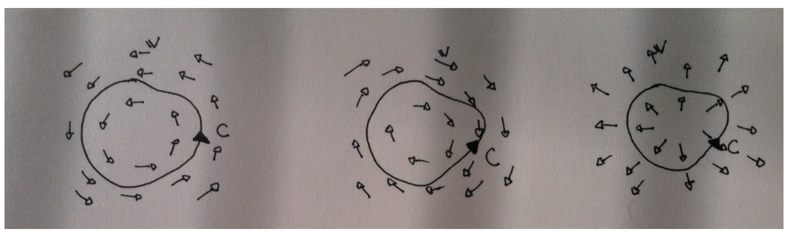

FIG. 1. Pardon the shadow of my venetian blinds (the weird gray vertical stripes).

ii) A recurrent theme in our class is that most fancy looking integrals reduce to a simple multiplication. Consider the vector field

$$\mathbf{v}(\mathbf{r}) = \frac{y\hat{\mathbf{x}} - x\hat{\mathbf{y}}}{x^2+y^2+z^2}. \tag{2}$$

($\mathbf{r}$ is the vector with coordinates x, y and z, that is, $\mathbf{v} = x\hat{\mathbf{x}} + y\hat{\mathbf{y}} + z\hat{\mathbf{z}}$). Compute the circulation of this field around a circular loop of radius R, lying on the xy plane and oriented counterclockwise as seen from the $z > 0$ region (I know, this phrase requires parsing, it is part of the exercise). Now, here is the catch: you are not allowed to use the full machinery we develop in homework 2 -part B; you should instead plot this field and realize you can get the answer with one multiplication.

The same way that the concept of flux had an infinitesimal version (the divergence), the concept of circulation has an infinitesimal version too, the **curl**. It is defined as following. Consider a little close circuit C lying on the xy plane. The z-component of the curl of the vector field is given by

$$(\nabla \times \mathbf{v})_z = (\nabla \times \mathbf{v}).\hat{\mathbf{z}} = \lim_{area \to 0} \frac{1}{area} \oint_C \mathbf{v}.d\mathbf{l}, \tag{3}$$

where "area" is the area of the surface bounded by C. Similarly, for the other components we consider little circuit perpendicular to the y and x directions

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$$(\nabla \times \mathbf{v})_z = (\nabla \times \mathbf{v}).\hat{\mathbf{z}} = \lim_{area \to 0} \frac{1}{area} \oint_C \mathbf{v}.d\mathbf{l}. \tag{4}$$

Keep in mind: we take the curl of a **vector field** and the result is another **vector field**!

One way to visualize the curl of a vector field is to imagine that the vector field represents the velocity of the water on a river. The curl gives the angular velocity of a little boat floating on it and being carried by the current.

It's frequently impractical to compute the curl of a vector field using the definition above so, just like we did with the divergence, we will derive a formula for it in cartesian coordinates. We will discuss its derivation in class (and it's also explained in Purcell&Morin) so I'll just quote the result here:

$$\begin{aligned}\nabla \times \mathbf{v} &= \left(\frac{\partial v_z}{\partial y} - \frac{\partial v_y}{\partial z}\right)\hat{\mathbf{x}} + \left(\frac{\partial v_x}{\partial z} - \frac{\partial v_z}{\partial x}\right)\hat{\mathbf{y}} + \left(\frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y}\right)\hat{\mathbf{z}} \\ &= \left(\hat{\mathbf{x}}\frac{\partial}{\partial x} + \hat{\mathbf{y}}\frac{\partial}{\partial y} + \hat{\mathbf{z}}\frac{\partial}{\partial z}\right) \times (v_x\hat{\mathbf{x}} + v_y\hat{\mathbf{y}} + v_z\hat{\mathbf{z}}), \end{aligned} \tag{5}$$

where v_x, v_y, v_z are the x, y, z components of $\mathbf{v}$.

iii) Sketch the field and compute the curl of the vector field $\mathbf{v} = y\hat{\mathbf{x}} - x\hat{\mathbf{y}}$

iv) Sketch the field and compute the curl of the vector field $\mathbf{v} = -y\hat{\mathbf{x}} + x\hat{\mathbf{y}}$

v) Sketch the field and compute the curl of the vector field $\mathbf{v} = y\hat{\mathbf{x}} + x\hat{\mathbf{y}}$

vi) Sketch the field and compute the curl of the vector field $\mathbf{v} = x\hat{\mathbf{x}} + y\hat{\mathbf{y}} + z\hat{\mathbf{z}}$

II. CYLINDRICAL CAPACITOR

Consider a cylindrical capacitor where the two plates are two cylindrical surfaces of radii R_1 and R_2, and length L. Assume $L \gg R_1, R_2$ so you can neglect "edge effects" (in practice this means you can compute the electric field as if the capacitor were infinite). Compute its capacitance.

III. SPHERICAL CAPACITOR

Consider a capacitor where the two plates are two concentric spherical surfaces of radii R_1 and R_2.

i) Compute its capacitance.

ii) Take $R_1 = 1m$ and $R_2 = 0.5m$. How much energy can be stores in this capacitor if we assume that the largest electric field it can sustain without electrical breakdown is 3×10^6 V/m (that is the value at which air becomes partially conductive).