UNIVERSITY OF MARYLAND, PHYSICS DEPARTMENT

PHYS 272, HW 6 Solutions

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1 PROBLEM A

$$\mathbf{r}_1 - \mathbf{r}_2 = 3\hat{x} - 1\hat{y} - 6\hat{z}$$
$$|\mathbf{r}_1 - \mathbf{r}_2| = [3^2 + 1^2 + 6^2]^{1/2} = (46)^{1/2}$$
$$\mathbf{R} = \frac{\mathbf{r}_1 - \mathbf{r}_2}{|\mathbf{r}_1 - \mathbf{r}_2|^3} = \frac{3}{(46)^{3/2}}\hat{x} - \frac{1}{(46)^{3/2}}\hat{y} - \frac{6}{(46)^{3/2}}\hat{z}$$

2 PROBLEM B

In the initial state, only the shell of radius $r = 2a$ is charged. Denote its charge by q_i. The charge in the $r = a$ shell is zero.

After some time has passed, the inital charge will be distributed among the two spherical shells. Denote q_{f1} as the final charge of the $r = a$ shell, and q_{f2} as the final charge in the $r = 2a$ shell. Then, by charge conservation:

$$q_i = q_{f1} + q_{f2}$$

Just to be clear:

Initial state: $Q_{total} = q_i$

Final state: $Q_{total} = q_{f1} + q_{f2}$

For the conducting shell of radius a and evenly distributed charge q_{f2} (after some time has passed and the shells have reached electrostatic equilibrium), the electric potential at the surface is:

$$\phi_1 = \frac{q_{f1}}{4\pi\epsilon_0 a}$$

And it will have a corresponding surface charge distribution:

$$\sigma_1 = \frac{q_{f1}}{4\pi a^2}$$

So, the potential energy for the $r = a$ shell will be:

$$\begin{aligned} U_1 &= \frac{1}{2}\int_{shell} dA\sigma_1\phi_1 \\ &= \sigma_1\phi_1 \int_{shell} dA \\ &= \frac{1}{2}\frac{q_{f1}}{4\pi a^2}\frac{q_{f1}}{4\pi\epsilon_0 a}4\pi a^2 \\ &= \frac{q_{f1}^2}{8\pi\epsilon_0 a} \end{aligned}$$

Where dA is the area element and we take the integral over the surface of the spherical shell (since the charge is distributed over the area of the shell, this integral will simply yield the known surface area of a sphere).

Similarly, the sphere with radius $2a$ has an electric potential $\phi_2 = \frac{q_{f2}}{8\pi\epsilon_0 a}$, surface charge density $\sigma_2 = \frac{q_{f2}}{16\pi a^2}$, and thus a potential energy:

$$U_2 = \frac{q_{f2}^2}{16\pi\epsilon_0 a}$$

You want to find q_{f2} and you know that the total charge corresponds to the initial charge, q_i. Rewrite $q_{f1} = q_i - q_{f2}$. Now you can write the total potential energy as a function of q_{f2}:

$$\begin{aligned} U_{total} = U_1 + U_2 &= \frac{q_{f1}^2}{8\pi\epsilon_0 a} + \frac{q_{f2}^2}{16\pi\epsilon_0 a} \\ &= c[q_{f2}^2 + 2q_{f1}^2] \\ &= c[3q_{f2}^2 + 2q_i^2 - 4q_i q_{f2}] \end{aligned}$$

Were we define $c \equiv \frac{1}{16\pi\epsilon_0 a}$. And replace $q_{f1} = q_i - q_{f2}$.

You may find the final state with minimal energy (when electrostatic equilibrium is reached) by taking the derivative of the expression for the total energy with respect to q_{f2} and set it equal to zero:

$$\frac{\partial U_{total}}{\partial q_{f2}} = c\frac{\partial}{\partial q_{f2}}[3q_{f2}^2 + 2q_i^2 - 4q_i q_{f2}] = 0$$

So, the energy will be minimum when

$$q_{f2} = \frac{2}{3}q_i$$

3 PROBLEM C

Recall the discussion for an infinite charged plane. Now, when you have two charged infinite planes you can simply superimpose the electric field found for each plane, when taken individually. Then it follows that the electric field between the two planes will be uniform, orthogonal to the plane and will go from the positively charged plane to the negatively charged plane:

$$\mathbf{E} = \frac{\sigma}{\epsilon_0}\hat{r}$$

Where $\hat{r}$ is the direction orthogonal to both planes. Everywhere else the individual contributions from each plane will cancel out.

We know that $\mathbf{E} = -\nabla\phi$. So, integrate over the direction of the electic field to find ϕ:

$$\phi = \int_0^r drE = \frac{\sigma}{\epsilon_0}r$$

Recall that here E is just the magnitude of the electric field ($\mathbf{E} = E\hat{r}$). Everywhere else the potential is zero. Also note that here the point $r = 0$ is set at the positive plane.

4 PROBLEM D (GRADED)

Draw a cylindrical Gaussian surface of radius r, centered around the same axis as the cylinder. By Gauss' Law:

$$\int daE = \frac{Q_{enc}}{\epsilon_0}$$

Where the Gaussian surface is $\int da = 2\pi rL$, and L is the length of the Gaussian cylinder (bear with me for a bit). Q_{enc} due to the charged cylinder when $r > R$ will be $Q_{enc} = \sigma A = 2\pi\sigma RL$. Where A is the surface area of the cylinder (since the Gaussian fully encloses the cylinder). The charge enclosed due to the axial charge will be $Q_{enc} = \lambda L = -2\pi\sigma RL$. So, the total charge enclosed will be:

$$Q_{enc} = 2\pi\sigma RL - 2\pi\sigma RL = 0$$

And the electric field will be zero.

$$E(r > R) = 0$$

And the electric potential:

$$\phi(r > R) = -\int_\infty^r dr'E(r') = 0$$

For the case when $r < R$, the charge cylinder does not contribute to the total charge enclosed. So the electric field is

$$E(r < R) = -\frac{R\sigma}{r\epsilon_0}$$

And the electric potential:

$$\begin{aligned}
\phi(r < R) &= -\int_\infty^r dr' E(r') \\
&= \int_\infty^R dr' E(r') - \int_R^r dr' E(r') \\
&= 0 - \int_R^r dr' E(r') \\
&= \frac{\sigma R}{\epsilon_0} \int_R^r \frac{dr'}{r'} \\
&= \frac{\sigma R}{\epsilon_0} log(r) - log(R) \\
&= \frac{\sigma R}{\epsilon_0} log(r/R)
\end{aligned}$$