UNIVERSITY OF MARYLAND, PHYSICS DEPARTMENT

PHYS 272, HW 4 Solutions

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1 PROBLEM A

1.1 PART I

Since the flux is a scalar quantity, you can simply compute the flux passing through each side of the cube and add up all the contributions to obtain the total flux. The surface integral is really easy to find, so there's no need to parametrize:

$$flux_1 = \int_0^1 \int_0^1 dxdy(\hat{z}) \cdot (z\hat{x} - 3\hat{y}) = 0$$
$$flux_2 = \int_0^1 \int_0^1 dxdy(\hat{z}) \cdot (z\hat{x} - 3\hat{y}) = 0$$
$$flux_3 = \int_0^1 \int_0^1 dydz(\hat{x}) \cdot (z\hat{x} - 3\hat{y}) = 1/2$$
$$flux_4 = \int_0^1 \int_0^1 dydz(-\hat{x}) \cdot (z\hat{x} - 3\hat{y}) = -1/2$$
$$flux_5 = \int_0^1 \int_0^1 dxdz(\hat{y}) \cdot (z\hat{x} - 3\hat{y}) = -3$$
$$flux_6 = \int_0^1 \int_0^1 dxdz(-\hat{y}) \cdot (z\hat{x} - 3\hat{y}) = 3$$

Now, add up all the contributions:

$$\sum_i flux_i = 0$$

Where i goes from 1 to 6.

1.2 PART II

The vector orthonormal to the sphere is just $\hat{r}$, so, when you take the dot product with the vector field $\hat{r}$ the result is just one. Then the flux is simply this dot product (since its constant) times the surface integral for a sphere, which is just $4\pi R^2$. Formally:

$$flux = \int_{surf} \hat{\mathbf{n}} \cdot \mathbf{v} = \int_0^{2\pi} \int_0^{\pi} d\phi d\theta R^2 sin(\phi) \cdot 1 = 4\pi R^2$$

Notice that this is not a constant vector field, which is why the flux depends on the radius of the sphere.

1.3 Part iii

$$\nabla \cdot \mathbf{v} = \frac{\partial}{\partial x} z + \frac{\partial}{\partial y}(-3) + \frac{\partial}{\partial z} 0 = 0$$

1.4 Part iv

See Fig 1.1

$$\nabla \cdot \mathbf{v} = \frac{\partial}{\partial x} x + \frac{\partial}{\partial y} y + \frac{\partial}{\partial z} z = 3$$

1.5 Part v

See Fig 1.2

$$\nabla \cdot \mathbf{v} = \frac{\partial}{\partial x}(-x) + \frac{\partial}{\partial y}(-y) + \frac{\partial}{\partial z}(-z) = -3$$

1.6 Part vi

See Fig 1.3

$$\nabla \cdot \mathbf{v} = \frac{\partial}{\partial x}(x) + \frac{\partial}{\partial y}(-y) + \frac{\partial}{\partial z}(0) = 0$$

1.7 Part vii

See Fig 1.4

$$\nabla \cdot \mathbf{v} = \frac{\partial}{\partial x} y + \frac{\partial}{\partial y}(-x) + \frac{\partial}{\partial z} 0 = 0$$

1.8 Part viii

See Fig 1.5

$$\nabla \cdot \mathbf{v} = \frac{\partial}{\partial x} x^2 + \frac{\partial}{\partial y}(-y^2) + \frac{\partial}{\partial z} 0 = 2x - 2y$$

2 Problem B

2.1 Part I

$$\nabla \cdot \mathbf{v} = \frac{\partial}{\partial x}x + \frac{\partial}{\partial y}y + \frac{\partial}{\partial z}0 = 2$$

By Gauss Theorem:

$$\begin{gathered} flux = \int d^3\mathbf{r}\nabla\cdot\mathbf{v} \\ = 2\int_0^1\int_0^1\int_0^1 dxdydz \\ = 2 \end{gathered}$$

Now, find the flux using the dumb method: compute the flux passing through each side of the square and sum all the contributions. The surfaces here are simple enough that you need not parametrize.

$$\begin{gathered} flux_1 = \int_0^1\int_0^1 dxdy(\hat{z})\cdot(x\hat{x}+y\hat{y}) = 0 \\ flux_2 = \int_0^1\int_0^1 dxdy(\hat{z})\cdot(x\hat{x}+y\hat{y}) = 0 \\ flux_3 = \int_0^1\int_0^1 dydz(\hat{x})\cdot(0\hat{x}+y\hat{y}) = 0 \\ flux_4 = \int_0^1\int_0^1 dydz(\hat{x})\cdot(1\hat{x}+y\hat{y}) = 1 \\ flux_5 = \int_0^1\int_0^1 dxdz(\hat{y})\cdot(x\hat{x}+0\hat{y}) = 0 \\ flux_6 = \int_0^1\int_0^1 dxdz(\hat{y})\cdot(x\hat{x}+1\hat{y}) = 1 \end{gathered}$$

So, the total flux:

$$\sum_i flux_i = 2$$

Where i goes from 1 to 6. Surprise, surprise! Math works.

3 Problem C (graded)

3.1 Part I

The electric field generated by a point charge will be symmetric in space at every point. So, it makes sense to choose a gaussian surface that is also symmetric in space at every point, i.e. a sphere:

$$\begin{gathered} \int da\cdot E = \frac{Q}{\epsilon_0} \\ E\cdot 4\pi r^2 = \frac{Q}{\epsilon_0} \\ E = \frac{Q}{4\pi\epsilon_0 r^2} \end{gathered}$$

Where Q is the total charge.

3.2 Part II

The only contribution to the electric field will come from the radial component. Now, consider a cylindrical gaussian surface. Notice that the electric field will have the same magnitude at every point in the surface of the cylinder! So, this is a good choice. Now, consider the charged line has a homogenous charge distrubution, which means that the total charge Q will be $Q = \lambda L$ where λ is the charge per length, and L is just the length of the line:

$$\int da \cdot E = \frac{Q}{\epsilon_0}$$
$$E \cdot 2\pi r L = \frac{\lambda L}{\epsilon_0}$$
$$E = \frac{\lambda}{2\pi r \epsilon_0}$$

And, just as we expect, the electric field will only depend on the radial distance to the line. For a more intuitive approach, you can consider the radial distance to be the "z-component" of the electric field, then the component perpendicular to the z-axis will just cancel out everywhere because the line is infinite. The third spatial component is alligned such that it is always zero.

3.3 Part III

For the symmetry corresponding to this problem, it is better to choose your gaussian surface to be a cylinder with its radial component alligned parallel to the sheet of charge. Notice that this is a good choice, because, once the only component that will contribute to the electric field is that perpendicular to the sheet (becuase the parallel components will simply cancel out, since the sheet is taken to be infinite). The total charge will simply be the charge per unit area times the area: $Q = \sigma A$, where Q is the total charge, σ is the charge per area, and A is the area.

$$EA = \frac{\sigma A}{\epsilon_0}$$
$$E = \frac{\sigma}{\epsilon_0}$$

But notice that this is the electric field enclosed by the gaussian surface, i.e. the sum of the field both, above and below the sheet of charge. So, the field at any point away from the sheet of charge would just be half of this:

$$E = \frac{\sigma}{2\epsilon_0}$$

3.4 Part IV

Notice that outside the sphere ($r > R$, where R is the radius of the sphere), the electric field will look exactly the same as a point charge, since it is symmetrical everywhere in space. So:

$$E = \frac{Q}{4\pi\epsilon_0 r^2}$$

Inside the sphere the situation becomes a bit more interesting. Notice that the total charge enclosed by the gaussian surface will simply be $Q[\frac{r^3}{R^3}]$, think about it for 2 seconds and convice yourself its true (think of the ratios of volumes, $\frac{V_{enc}}{V_{sphere}} = \frac{4\pi r^3/3}{4\pi R^3/3}$). So:

$$E \cdot 4\pi r^2 = \frac{Qr^3}{\epsilon_0 R^3}$$
$$E = \frac{rQ}{4\pi\epsilon_0 R^3}$$

3.5 Part V

Notice that the charge distribution ρ is (i) continuous, i.e. it has a value at every point in space, and (ii) decays to zero as $r \to \infty$. So, a spherical gaussian surface should do the trick. Remember that ρ is simply a charge distribution (that is, dQ/dV, where dV is simply the unit volume), so, to obtain the total charge enclosed, simply integrate over all the enclosed space:

$$E \cdot 4\pi r^2 = \frac{1}{\epsilon_0} \int dV \rho_0 e^{-r/a}$$
$$\text{where } dV = r^2 sin(\phi) d\theta d\phi dr$$
$$E \cdot 4\pi r^2 = \frac{1}{\epsilon_0} \int_0^r \int_0^\pi \int_0^{2\pi} d\theta d\phi dr' r'^2 sin(\phi) \rho_0 e^{-r'/a}$$

Where r' is just a integration dummy variable. You can leave it as r, it really doesn't matter so long as you indicate that you are integrating up to an arbitrary point r in space enclosing the gaussian surface. If you have solved up to this point you will obtain full credit.

To solve the integral, there are a few methods you could use. You may integrate by parts, or you can differentiate inside the integral, as I am about to explain. Bear with me for a moment and suppose that you can take a to be a variable. Even better, make a new variable $b = -1/a$. So

$$\int dr r^2 e^{rb} = \int dr \frac{d^2}{db^2} e^{rb}$$

Now, because you are integrating with respect to r, you can simply "pull out" the differential operator:

$$\int dr r^2 e^{rb} = \int dr \frac{d^2}{db^2} e^{rb} = \frac{d^2}{db^2} \int dr e^{rb}$$

And now you are left with an integral that is much, much easier to solve! After you have solved the integral, you can simply differentiate with respect to b twice and obtain the final result. Or you can simply use Wolfram Alpha or Mathematica (which I personally prefer, just make sure you actually know how to do the integrals)...Either way, you should obtain the following result:

$$E \cdot 4\pi r^2 = \frac{2\pi}{\epsilon_0}[4a^3 - 2ae^{-r/a}(2a + 2ar + r^2)]$$
$$E = \frac{1}{2\epsilon_0 r^2}[4a^3 - 2ae^{-r/a}(2a^2 + 2ar + r^2)]$$

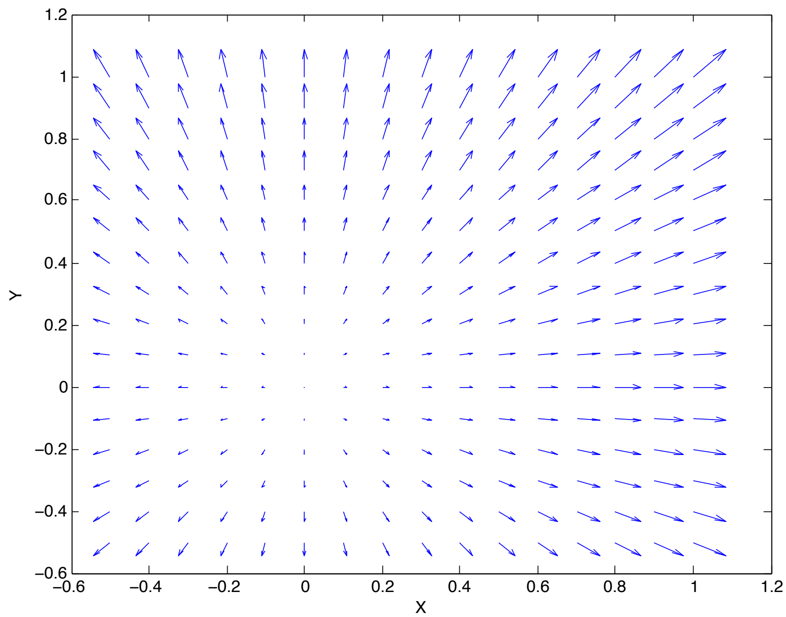

Figure 1.1: Vector field corresponding to Part IV

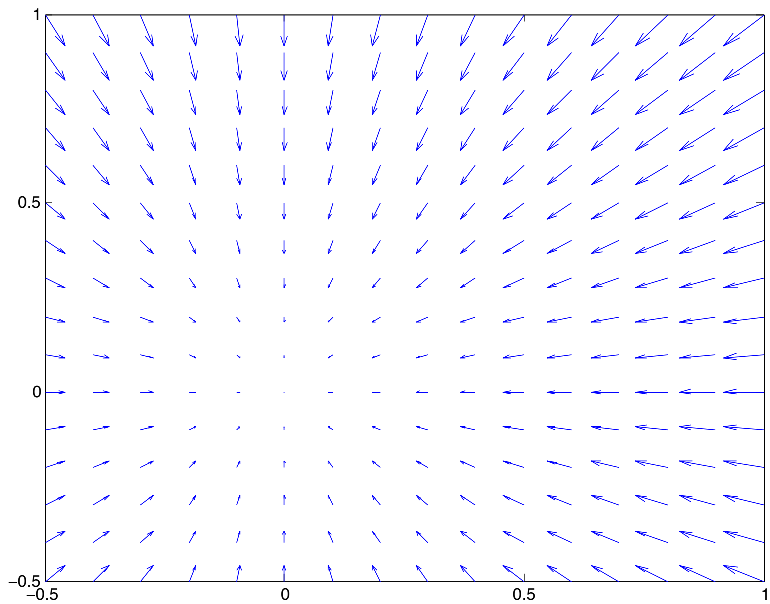

Figure 1.2: Vector field corresponding to Part V

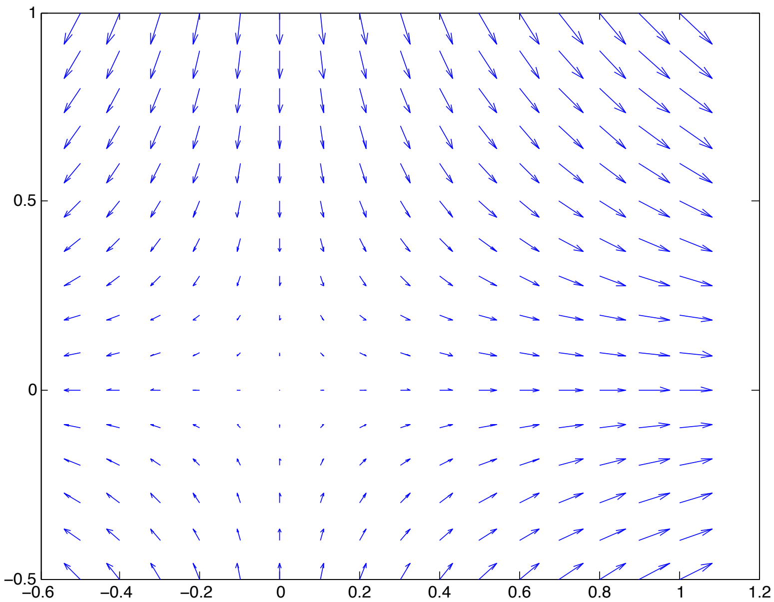

Figure 1.3: Vector field corresponding to Part VI

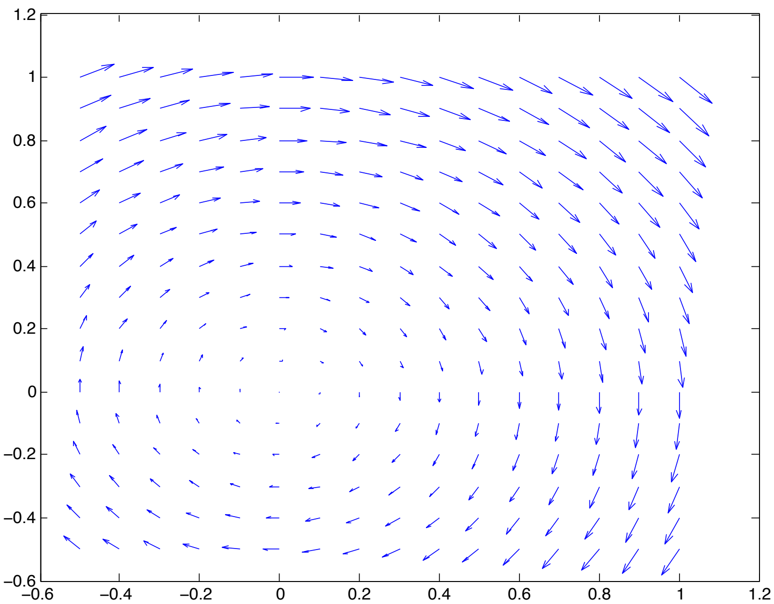

Figure 1.4: Vector field corresponding to Part VII

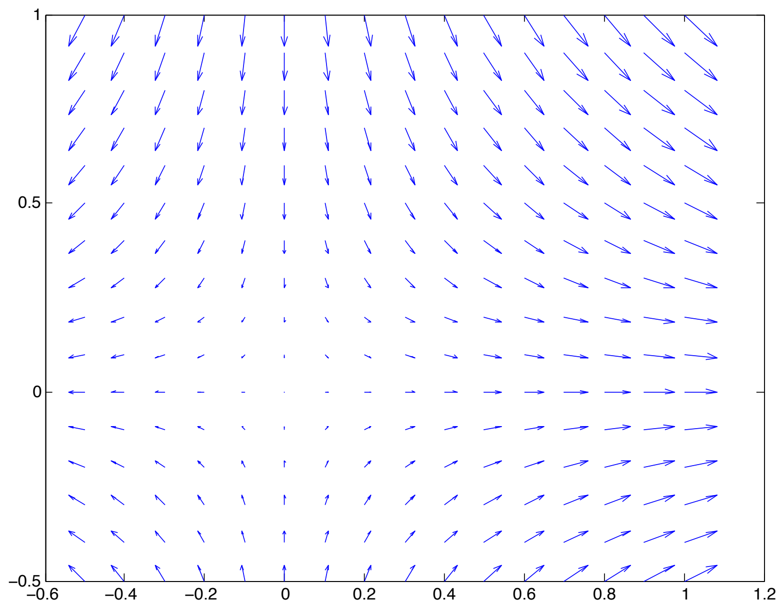

Figure 1.5: Vector field corresponding to Part VIII