PHY 272: FIELDS
PROBLEM SET 4
due February 25, **before** *class*

A. Divergence

We considered before surface integrals of the type:

$$\int_S da\ \hat{\mathbf{n}}.\mathbf{v} \tag{1}$$

which represented how much of the vector field $\mathbf{v}$ "passed through" the surface S, namely, the "flux" of the vector field $\mathbf{v}$ through the surface S. What if we consider a *closed* surface like a spherical surface ? Closed surfaces have the property that they enclose a volume of which they are the boundary. Think of a spherical surface enclosing the solid sphere inside it.

Suppose we orient a closed surface so the unit normal vector $\hat{\mathbf{n}}$ points outwards. In some parts of the surface the flux can be positive, the vector field aligns with the outward normal and seems to be "coming out" of the surface. In others parts the flux can be negative, the vector anti-aligns with the normal and looks like it is "going inside" the surface. Summing all these contributions we get the overall flux of the vector field and, if positive, it means that the field is mostly "springing" from the interior and, if negative, it means that the field is "disappearing" inside it. (Make little drawing of these situations as you follow the discussion).

i) Compute the flux of the vector field $\mathbf{v} = z\hat{\mathbf{x}} - 3\hat{\mathbf{y}}$ through the surface of a cube $0 < x < 1, 0 < y < 1, 0 < z < 1$.

ii) Compute the flux of the vector field $\mathbf{v} = (x\hat{\mathbf{x}}+y\hat{\mathbf{y}}+z\hat{\mathbf{z}})/\sqrt{x^2+y^2+z^2} = \hat{\mathbf{r}}$ through a sphere of radius R centered at the origin.

We want now to invent a measure of how much a vector field "spring out" or "disappears into" a certain *point* in space. For that, let us define the *divergence* of a vector field at a given point as

$$\nabla.\mathbf{v} = \lim_{V\to 0} \frac{1}{V}\oint_S da\ \hat{\mathbf{n}}.\mathbf{v}. \tag{2}$$

Let me explain the notation. The symbol $\oint$ is the integral sign for a *closed* surface. V is the volume of the region contained by the surface S. $\nabla.\mathbf{v}$ is called the divergence of the vector field $\mathbf{v}$ at the point in the center of the volume and is a number, not a vector. Since $\mathbf{v}$ has a divergence in every point of space, the divergence of a vector field is a scalar field. The reason for the funny notation $\nabla.$ will be explained momentarily but sometimes the divergence is denoted by $div\mathbf{v}$. Finally, someone proved that the shape of the surface S doesn't not matter as long as it is small enough and we can calculate the divergence with any shaped-surface and obtain the same result.

The definition of divergence above is simple to understand but difficult to use in practice. So, we will prove now a formula that is useful for when the vector field is given in terms of cartesian coordinates $\mathbf{v}(\mathbf{r}) = v_x\hat{\mathbf{x}} + v_y\hat{\mathbf{y}} + v_z\hat{\mathbf{z}}$, where v_x, v_y and v_z are functions of x, y and z. Take the little volume to be a cube with one vertex at coordinates (x, y, z) and sides dx, dy and dz. The contribution to the flux from the two sides orthogonal to the x axis is

$$\begin{aligned}(\hat{\mathbf{x}}.\mathbf{v}(x+dx,y,z) - \hat{\mathbf{x}}.\mathbf{v}(x,y,z))\ \underbrace{dydz}_{\text{area of a face}} &\approx \left(v_x(x,y,z) + dx\frac{\partial v_x(x,y,z)}{\partial x} - v_x(x,y,z)\right)\ dydz \\ &= \frac{\partial v_x(x,y,z)}{\partial x}\ \underbrace{dxdydz}_{\text{volume of cube}}\ ,\end{aligned} \tag{3}$$

where we used the fact that the normal vector to the two faces of the cube we considered are $\hat{\mathbf{n}} = \hat{\mathbf{x}}$ and $\hat{\mathbf{n}} = -\hat{\mathbf{x}}$. The contributions of the two faces perpendicular to the y axis and the two perpendicular to the z axis are computed similarly and the final result is

$$\nabla.\mathbf{v} = \lim_{V\to 0} \frac{1}{V}\oint_S da\ \hat{\mathbf{n}}.\mathbf{v} = \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z}. \tag{4}$$

Notice that this expression for the divergence in cartesian coordinates can be thought out as

$$\nabla.\mathbf{v} = \underbrace{\left(\frac{\partial}{\partial x}\hat{\mathbf{x}} + \frac{\partial}{\partial y}\hat{\mathbf{y}} + \frac{\partial v_z}{\partial z}\hat{\mathbf{z}}\right)}_{\nabla}.(v_x\hat{\mathbf{x}} + v_y\hat{\mathbf{y}} + v_z\hat{\mathbf{z}}) \tag{5}$$

therefore explaining the notation $\nabla.\mathbf{v}$ for the divergence.

iii) Compute the divergence of the vector field $\mathbf{v} = z\hat{\mathbf{x}} - 3\hat{\mathbf{y}}$.

iv) Make a rough plot (only the $z = 0$ two dimensional plane to make it easier) of the vector field $\mathbf{v} = x\hat{\mathbf{x}} + y\hat{\mathbf{y}} + z\hat{\mathbf{z}}$ and compute its divergence.

v) Make a rough plot (only the $z = 0$ two dimensional plane to make it easier) of the vector field $\mathbf{v} = -x\hat{\mathbf{x}} - y\hat{\mathbf{y}} - z\hat{\mathbf{z}}$ and compute its divergence.

vi) Make a rough plot (only the $z = 0$ two dimensional plane to make it easier) of the vector field $\mathbf{v} = x\hat{\mathbf{x}} - y\hat{\mathbf{y}}$ and compute its divergence.

vii) Make a rough plot (only the $z = 0$ two dimensional plane to make it easier) of the vector field $\mathbf{v} = y\hat{\mathbf{x}} - x\hat{\mathbf{y}}$ and compute its divergence.

viii) Make a rough plot (only the $z = 0$ two dimensional plane to make it easier) of the vector field $\mathbf{v} = x^2\hat{\mathbf{x}} - y^2\hat{\mathbf{y}}$ and compute its divergence.

By the way, there is an analogue expression to compute the divergence of a field given in cylindrical, spherical or any other coordinate system. Maybe we will discuss that later. Maybe not.

B. Divergence theorem (or Gauss theorem)

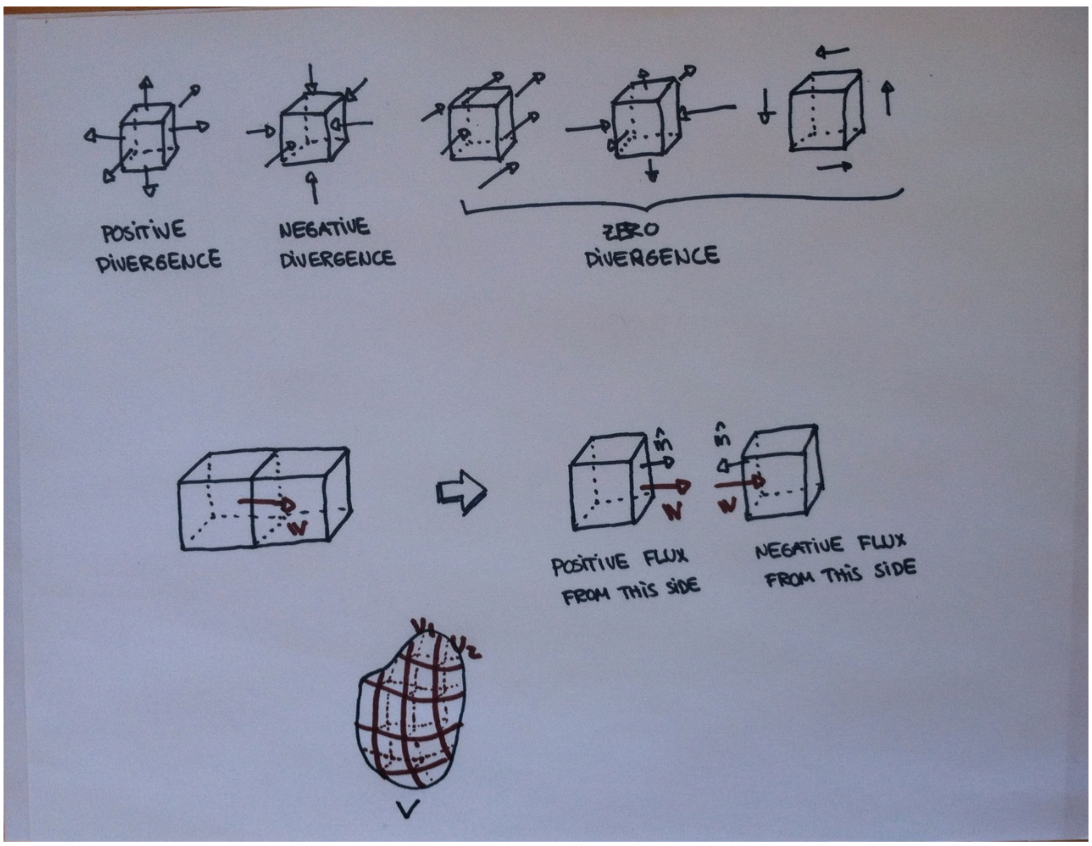

FIG. 1. Figures for the divergence and Gauss' theorem discussion.

Take a closed surface S. Now divide its interior into many little regions of volume $V_1, V_2, \cdots$, as I tried to depict at the bottom of the figure above. The surface of each of these regions we will call $S_1, S_2, \cdots$. Consider now the sum of the flux of a vector field $\mathbf{v}$ through all these little surfaces with the normal vector chosen to point outwards. It turns out that the sum of the fluxes over all the little surfaces equals the flux through the big surface S

$$\sum_i \oint_{V_i} da\ \hat{\mathbf{n}}.\mathbf{v} = \oint_S da\ \hat{\mathbf{n}}.\mathbf{v}. \tag{6}$$

The reason for this is that every piece of the little surfaces is either a part of the surface S (if the little surface happens to be on the boundary of the volume) or is shared with another little surface (if it is in the interior of the volume). But the parts in the interior cancel because if the vector $\mathbf{v}$ is going in some little volume it will be going out of another volume. The contribution $da\hat{\mathbf{n}}.\mathbf{v}$ to one surface integral gets cancelled by another contribution $da\ (-\hat{\mathbf{n}}).\mathbf{v}$ of the neighboring surface (look at the figure above). Only the surfaces at the boundary of the big volume V contribute to the sum.

The contribution of every little surface, if small enough, is just the volume V_i times the divergence of the field $\mathbf{v}$ and their sum turns into an integral of the divergence:

$$\sum_i \oint_{V_i} da\ \hat{\mathbf{n}}.\mathbf{v} = \oint_S da\ \hat{\mathbf{n}}.\mathbf{v} \approx \sum_i V_i \nabla.\mathbf{v} \approx \int_V d^3\mathbf{r}\ \nabla.\mathbf{v}. \tag{7}$$

This is the divergence theorem (or Gauss' theorem or, better yet, Gauβ's theorem). One more time, now with feeling:

$$\int_V d^3\mathbf{r}\ \nabla.\mathbf{v} = \oint_S da\ \hat{\mathbf{n}}.\mathbf{v}, \tag{8}$$

where S is the boundary of V. Notice that this is a possible way to generalize the fundamental theorem of calculus:

$$\int_a^b dx\ \frac{df}{dx} = f(b) - f(a). \tag{9}$$

They both have the form "integral of the derivative of a thing over a region" = "integral of the thing over the boundary of the region". In the case of the fundamental theorem, the boundary of the segment $[a, b]$ are the two points a and b and the sign for $-f(a)$ comes from orienting the segment $[a, b]$.

Now let us see if you understand what this means by working out one example

i) Consider the vector field $\mathbf{v} = x\hat{\mathbf{x}} + y\hat{\mathbf{y}}$. Compute $\nabla.\mathbf{v}$ and integrate it over the cube $0 < x < 1, 0 < y < 1, 0 < z < 1$. Now compute the flux of $\mathbf{v}$ through the same cube. Are they the same?

C. Gauss law

Use symmetry arguments and the Gauss law to find the electric fields of the following charge distributions (you are not allowed to use the Coulomb law here):

i) a point charge

ii) an infinite, straight, homogeneously charged wire

iii) an infinite homogeneously charged plane

iv) a solid sphere with total charge Q (field inside and outside the sphere)

iv) the continuos charge distribution $\rho(r, \theta, \phi) = \rho_0 e^{-r/a}$, where r is the radial distance from the origin (this is the charge distribution of the electron moving around the proton in a hydrogen atom.