UNIVERSITY OF MARYLAND, PHYSICS DEPARTMENT

PHYS 272, HW 3 Solutions

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1 PROBLEM A

1.1 PART I

The orthonormal vector (orthonormal means unit vector perpendicular to...) to a square in the $x-y$ plane will be given by $\hat{n}=\hat{z}$. The area of a square its simply its length square: $area = 1$. So:

$$flux = \mathbf{\hat{n}}\cdot\mathbf{v}\times area = (2\hat{x}-3\hat{y}+\hat{z})\cdot(0\hat{x}+0\hat{y}+\hat{z})\times 1 = 1$$

1.2 PART II

You want the vector $\mathbf{\hat{n}}$ to be perpendicular to the sphere at every point. So:

$$flux = [\frac{x\hat{x}+y\hat{y}+z\hat{z}}{(x^2+y^2+z^2)^{3/2}}\cdot\frac{x\hat{x}+y\hat{y}+z\hat{z}}{(x^2+y^2+z^2)^{1/2}}]\times 4\pi R^2$$
$$= \frac{x^2+y^2+z^2}{(x^2+y^2+z^2)^2}\times 4\pi R^2$$

But notice that $x^2+y^2+z^2=R^2$ for a sphere of radius R.

$$flux = 4\pi$$

You can think of the flux as "the amount of vectors" that are passing through the surface (using a naive approach, the "amount of lines" passing through the surface.) So, the flux will be the **same**, independently of the radius of the sphere. Notice that this is only true in this case because the vector field is not changing.

1.3 Part III

The parametric equation $r(u,v) = u\hat{x} + v\hat{y} + 3\hat{z}$ has a constant component in the $z-$axis, i.e. it cannot take any other values. So, given the interval in which u and v are defined, it follows that the equation describes an infinite plane that goes from $y = -1$ and $x = 0$ to infinity.

1.4 Part IV

The parametric equation $r(u,v) = cos(u)\hat{x} + sin(u)\hat{y} + 2v\hat{z}$ has the same form as the parametric equation for a circle in the $x - y$ plane, however, it can also take vales from $0 \rightarrow 1$ in the $z-$ plane. The v dependence is indepentent of the u dependence and its linear in z, therefore, the equation describes a cylinder. Notice that if the equation was dependent on just a single variable, i.e. $r(u) = cos(u)\hat{x} + sin(u)\hat{y} + 2u\hat{z}$, then the equation would describe a helix!

1.5 Part V

Same logic that we used when parametrizing a circle, except that in this case, the radius will not be fixed, it will be defined on an interval, so the parametric equation for a disc will be:

$$r(u,v) = vcos(u)\hat{x} + vsin(u)\hat{y} + 0\hat{z}$$

For $0 < u < 2\pi$, $0 < v < R$

1.6 Part VI

Notice that the equation of a sphere of radius R in spherical coordinates is simply $\rho = R$. The conversions formulas for converting Cartesian coordinates onto spherical coordinates are: $x = \rho sin(u)cos(v), y = \rho sin(u)sin(v), z = \rho cos(u)$ for $0 \leq u \leq \pi$ and $0 \leq v \leq 2\pi$. So, the parametric equation becomes:

$$r(u,v) = Rsin(u)cos(v)\hat{x} + Rsin(u)sin(v)\hat{y} + Rcos(u)\hat{z}$$

For the previously defined intervals of u and v.

1.7 Part VII

Disk:

$$\frac{d\mathbf{r}}{du} = -vsin(u)\hat{x} + vcos(u)\hat{y} + 0\hat{z}$$
$$\frac{d\mathbf{r}}{dv} = cos(u)\hat{x} + sin(u)\hat{y} + 0\hat{z}$$
$$\frac{d\mathbf{r}}{du} \times \frac{d\mathbf{r}}{dv} = -v[sin^2(u) + cos^2(u)]\hat{z} = -v\hat{z}, \text{ since } sin^2(u) + cos^2(u) = 1$$
$$||\frac{d\mathbf{r}}{du} \times \frac{d\mathbf{r}}{dv}|| = v$$
$$\hat{\mathbf{n}}_{disk} = -\hat{z}$$

Which makes sense, considering we are looking for an orthonormal vector to a disk in the $x - y$ plane.
Sphere:

$$\frac{d\mathbf{r}}{du} = Rcos(u)cos(v)\hat{x} + Rcos(u)sin(v)\hat{y} - Rsin(u)\hat{z}$$
$$\frac{d\mathbf{r}}{dv} = -Rsin(u)sin(v)\hat{x} + Rsin(u)cos(v)\hat{y} + 0\hat{z}$$
$$\frac{d\mathbf{r}}{du} \times \frac{d\mathbf{r}}{dv} = [R^2sin^2(u)cos(v)]\hat{x} - [-R^2sin^2(u)sin(v)]\hat{y} + [R^2cos(u)sin(u)(sin^2(v) + cos^2(v))]\hat{z}$$
$$= [R^2sin^2(u)cos(v)]\hat{x} + [R^2sin^2(u)sin(v)]\hat{y} + [R^2cos(u)sin(u)]\hat{z}$$
$$||\frac{d\mathbf{r}}{du} \times \frac{d\mathbf{r}}{dv}|| = [R^4sin^4(u)(cos^2(v) + sin^2(v)) + R^4cos^2(u)sin^2(u)]^{1/2}$$
$$= [R^4sin^2(u)(sin^2(u) + cos^2(u))]^{1/2}$$
$$= R^2sin(u)$$
$$\hat{\mathbf{n}}_{sphere} = \frac{sin(u)R^2[sin(u)cos(v)\hat{x} + sin(u)sin(v)\hat{y} + cos(u)\hat{z}]}{R^2sin(u)}$$
$$= sin(u)cos(v)\hat{x} + sin(u)sin(v)\hat{y} + cos(u)\hat{z}$$

1.8 Part VIII

Change the expression for the disk $r(u,v)$ to $r(x,y)$, where $x = vcos(u)$ and $y = vsin(u)$. Then you will need to change the limits to express them in terms of x and y. Here you have the surface is a disk, so x and y will be defined over the intervals $0 \le x \le R$ and $0 \le y \le R$

$$flux = \int_0^R \int_0^R dxdy[-x\hat{x} - y\hat{y}] \cdot [y\hat{x} - x\hat{y} + xy\hat{z}]$$
$$= \int_0^R \int_0^R dxdy[-xy + xy]$$
$$= 0$$

2 Problem B

This problem is easiest to do if you think of the cylinder as consisting of a bunch of (infinitesimal) rings of radius R, each carrying charge dQ. If you are trying to find the electric field at an arbitrary distance D from the center of a given ring, then the x and y components are gonna cancel themselves out due to the symmetry of the cylinder (think about it, each x and y component of the electric field will have an equal and opposite counterpart). In this argument, it is implied that the z axis is aligned along the central axis of the cylinder.

From Coulomb's law, we know that each ring will contribute:

$$d\mathbf{E_z} = \frac{1}{4\pi\epsilon_0}\frac{dQ\hat{r}}{r^2}$$

You can obtain the magnitude E_z of the vector $\mathbf{dE_z}$ by integrating over the ring. You should obtain:

$$dE_z = \frac{1}{4\pi\epsilon_0}\frac{DdQ}{(R^2+D^2)^{3/2}}$$

Where $dQ = QdD/h$. So, you can just add up the contributions from each infinitesimal ring (integrate on both sides):

$$E_z = \int_{surface} \frac{1}{4\pi\epsilon_0}\frac{DdQ}{(R^2+D^2)^{3/2}} = \frac{Q}{4\pi\epsilon_0 h}\int_0^D \frac{D'dD'}{(R^2+D'^2)^{3/2}}$$

If you have solved up to this point, you will receive full credit. The rest is just math. The integral is much easier to solve by making a u substitution. Let $u = R^2 + D^2$, then $du = 2DdD$. After integration, you should obtain the final result:

$$E_z = \frac{Q}{4\pi\epsilon_0 h}[\frac{1}{R} - \frac{1}{(R^2+D^2)^{1/2}}]$$

At a point far away, D goes to infinity, so the term $\frac{1}{(R^2+D^2)^{1/2}}$ goes to zero, leaving us with the result $E_z = \frac{Q}{4\pi\epsilon_0 h}\frac{1}{R}$, as expected (it looks like a point charge).

3 Problem C

Choose your coordinate system such that the rod lies along the $x-$axis and z is always zero. Let the charge distribution per unit length (charge density) along the rod be $\lambda = \frac{dQ}{dx}$.

Now, the following expression will give you the $y-$component of the electric field generated by an infinitesimally small segment of the rod, dx:

$$dE_y = \frac{dx\lambda cos(\theta)}{4\pi\epsilon_0 r^2}$$

Where θ is the angle that the electric field forms with the $y-$axis and r is the distance from the infinitesimal point dx to the point where we are measuring the electric field. Take integrals on both sides and let R be the distance along y from the rod to the field point, such that $r^2 = x^2 + R^2$, then:

$$E_y = \int dx \frac{\lambda cos(\theta)}{4\pi\epsilon_0 r^2}$$
$$= \int_0^L \frac{dx\lambda}{4\pi\epsilon_0(x^2+R^2)} \frac{R}{(x^2+R^2)^{1/2}}$$
$$= \frac{\lambda}{4\pi\epsilon_0 R} \frac{L}{(L^2+R^2)^{1/2}}$$

Same argument for x

$$dE_x = dEsin(\theta), \text{ so:}$$
$$E_x = \int dx \frac{\lambda sin(\theta)}{4\pi\epsilon_0 r^2}$$
$$= \int_0^L dx \frac{\lambda x}{4\pi\epsilon_0(x^2+R^2)^{3/2}}$$
$$= \frac{\lambda}{4\pi\epsilon_0}[\frac{-1}{(L^2+R^2)^{1/2}} + \frac{1}{R}]$$

So, the final result will be:

$$E = E_x\hat{x} + E_y\hat{y}$$
$$= \frac{\lambda}{4\pi\epsilon_0 R}\frac{L}{(L^2+R^2)^{1/2}}\hat{y} + \frac{\lambda}{4\pi\epsilon_0}[\frac{-1}{(L^2+R^2)^{1/2}} + \frac{1}{R}]\hat{x}$$