PHYS 272, HW 2 Solutions

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February 12, 2014

1 PROBLEM A

1.1 PART I

For partial differentiation you can simply think of the function you're differentiating as consisting of constants and a single variable for each variable you're differentiating. This way, when you're differentiating with respect to x, you hold y and z constant. Similarly, whilst differentiating with respect to y you can think of x and z as constants and so on.

$$f(x,y,z) = \frac{x}{x^2+y^2+z^2}$$
$$\frac{\partial f}{\partial x} = \frac{-x^2+y^2+z^2}{(x^2+y^2+z^2)^2}$$
$$\frac{\partial f}{\partial y} = \frac{-2xy}{(x^2+y^2+z^2)^2}$$
$$\frac{\partial f}{\partial z} = \frac{-2xz}{(x^2+y^2+z^2)^2}$$
$$\text{So, } \nabla f = \frac{-x^2+y^2+z^2}{(x^2+y^2+z^2)^2}\hat{x} + \frac{-2xy}{(x^2+y^2+z^2)^2}\hat{y} + \frac{-2xz}{(x^2+y^2+z^2)^2}\hat{z}$$

1.2 PART II

If the direction of greatest increase is given by ∇f, then the direction of least change will be given by a vector perpendicular to ∇f. Particularly, any vector perpendicular to ∇f will give the direction of minimal change. Therefore, the direction of least change will be given by a plane perpendicular to ∇f. We can find this by using the definition of dot product, that is, if the dot product of two vectors is zero, then the vectors are orthogonal. So:

$$\frac{\partial f}{\partial x}(x-x_0) + \frac{\partial f}{\partial y}(y-y_0) + \frac{\partial f}{\partial z}(z-z_0) = 0$$

Where the vector $(x-x_0)\hat{x}+(y-y_0)\hat{y}+(z-z_0)\hat{z}$ denotes a vector that must pass through the point $(x_0, y_0, z_0) = (1,4,3)$, and the dot product is equal to zero to ensure orthogonality. So, the plane:

$$\tfrac{-x^2+y^2+z^2}{(x^2+y^2+z^2)^2}(x-1)+\tfrac{-2xy}{(x^2+y^2+z^2)^2}(y-4)+\tfrac{-2xz}{(x^2+y^2+z^2)^2}(z-3)=0$$

Gives the direction of least increase, just as any vector lying along the plane.

2 Problem B

2.1 Part III

The function:

$$r(t) = R[cos(t)\hat{x}+sin(t)\hat{y}]$$

describes a circle of radius R for all $0 < t < 2\pi$.

2.2 Part IV

$$\tfrac{d}{dt}r(t) = R[-sin(t)\hat{x}+cos(t)\hat{y}]$$

2.3 Part V

$$\begin{gathered}\int_C \mathbf{F}\cdot d\mathbf{l}\\ =\int_{t_i}^{t_f}\mathbf{F}\cdot\tfrac{d\mathbf{r}}{dt}dt\\ =\int_0^{2\pi}Ax\hat{x}\cdot[-Rsin(t)\hat{x}+Rcos(t)\hat{y}]dt\\ =-\int_0^{2\pi}AR^2cos(t)sin(t)dt\end{gathered}$$

Notice I used $x = Rcos(t)$, then we can apply the chain rule by replacing $u = sin(t)$ and $du = cos(t)dt$, so:

$$\int_C \mathbf{F}\cdot d\mathbf{l} = -R^2A\tfrac{sin^2(t)}{2}|_0^{2\pi} = 0$$

3 Problem C

3.1 Part VI

$$\nabla f = (\tfrac{\partial}{\partial x}\hat{x}+\tfrac{\partial}{\partial y}\hat{y}+\tfrac{\partial}{\partial z}\hat{z})\tfrac{Ax^2}{2} = Ax\hat{x}$$

3.2 Part VII

$$\begin{gathered}f(B)-f(A)=\tfrac{A}{2}x^2|_{-R}^{R}=0\\ \int_C\nabla\mathbf{f}\cdot d\mathbf{l}=\int_{t_i}^{t_f}\nabla\mathbf{f}\cdot\tfrac{d\mathbf{r}}{dt}dt\\ =-\int_0^{\pi}AR^2cos(t)sin(t)dt=0\end{gathered}$$

As expected.

4 Problem D

4.1 Part I

This is just a constant field.

4.2 Part II

See Fig 4.1.

4.3 Part III

See Fig 4.2.

4.4 Part IV

See Fig. 4.3

5 Problem E

By Coulomb's law, the electric field any point located at a distance r from a charge q is given as:

$$\mathbf{E} = \frac{q\hat{r}}{4\pi\epsilon_0 r^2}$$

The field produced by two charges is just the sum of the field produced by each individual charge. So:

$$\mathbf{E} = \Sigma_i \frac{q_i \hat{r}_i}{4\pi\epsilon_0 r_i^2}$$

Where $r_1 = \mathbf{r} - 0\hat{x} = \mathbf{r}(x,y)$ and $r_1 = \mathbf{r} - \hat{x} = \mathbf{r}(x-1,y)$

So: $\mathbf{E} = \frac{1}{4\pi\epsilon_0}[\frac{q_1\hat{r}_1}{r_1^2} + \frac{q_2\hat{r}_2}{r_2^2}]$

$$= \frac{1}{4\pi\epsilon_0}[\frac{1C(x\hat{x}+y\hat{y})}{(x^2+y^2)^{3/2}} + \frac{-2C[(x-1)\hat{x}+y\hat{y}]}{((x-1)^2+y^2)^{3/2}}]$$

Here, $\hat{r}_i$ denotes the unit vector, which is simply the vector itself divided by its magnitude. The two dimensional sketch of the field lines is just the usual dipole electric field. Recall electric field lines always point from positive to negative.

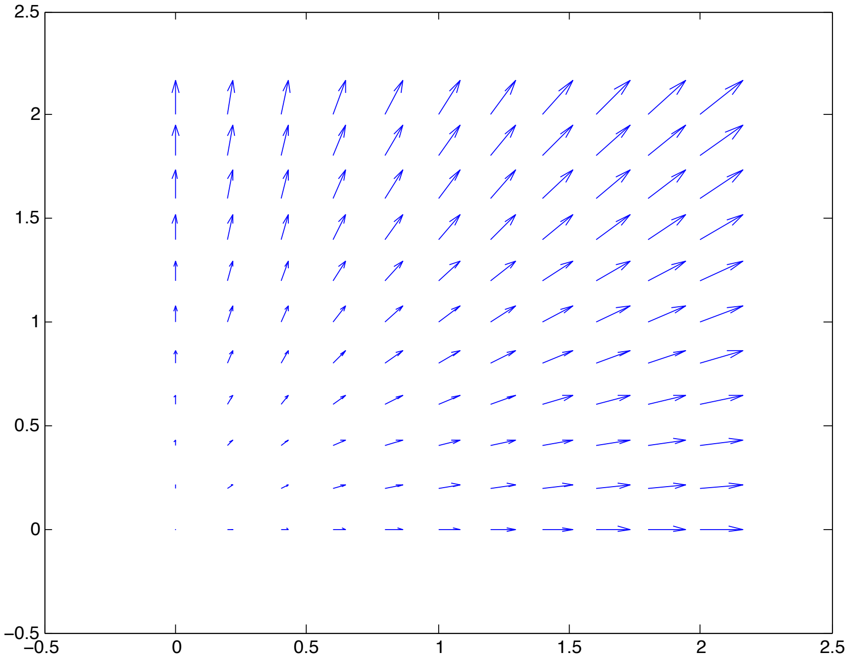

Figure 4.1: Vector field corresponding to Part II

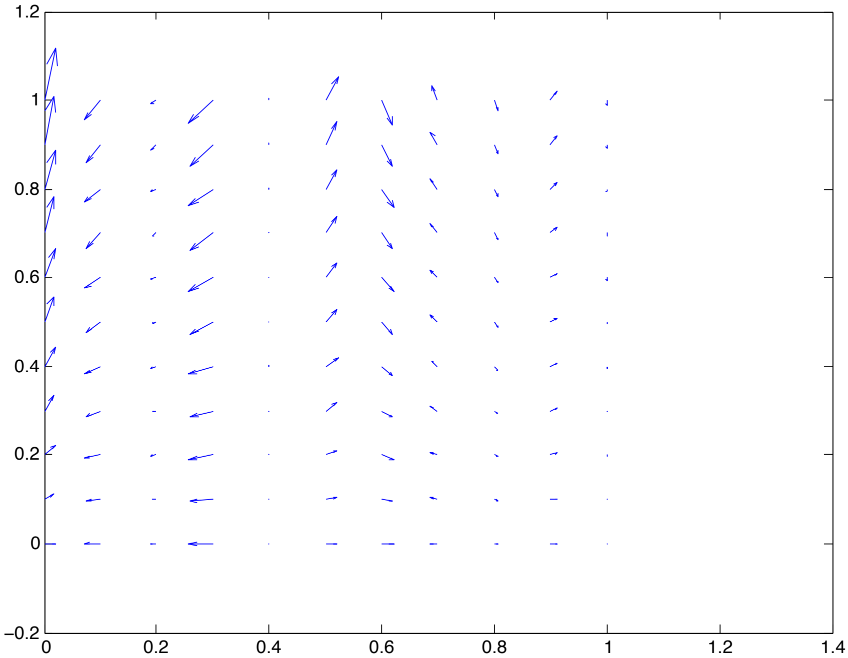

Figure 4.2: Vector field corresponding to Part III

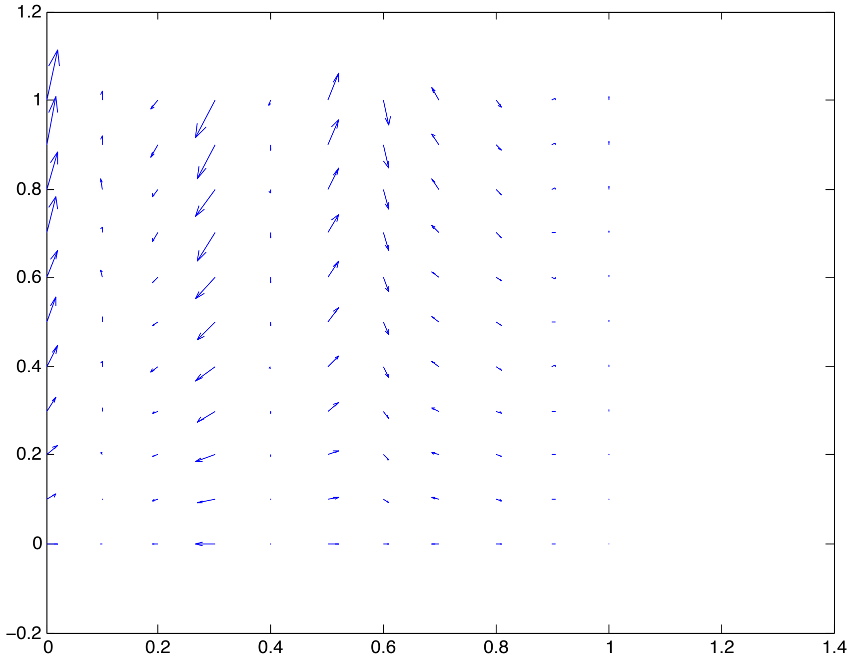

Figure 4.3: Vector field corresponding to Part IV