PHY 272: FIELDS

PROBLEM SET 2

due February 11, **before** *class*

A. Gradient

You are familiar with the concept of the derivative of a function of one variable. Its magnitude measures how the function f changes as x is varied. The sign tells us if f is increasing or decreasing as x increases. We can generalize this concept for a function of many variables. Take, for instance a function defined in three dimensions $f(x, y, z)$. Another name for a function like that is a *scalar field* as f is a rule giving a number (scalar) for every point of the three-dimensional space. The partial derivatives $df/dx, df/dy$ and df/dz give the rate of change of f as x (or y or z) is varied.

i) Just to make sure you know what I mean by partial derivatives compute the three partial derivatives for the function:

$$f(x, y, z) = \frac{x}{x^2 + y^2 + z^2} \tag{1}$$

Keep in mind that sometimes a scalar function is presented not as a function of the three coordinates but as a function of the position vector $\mathbf{r} = x\hat{\mathbf{x}} + y\hat{\mathbf{y}} + z\hat{\mathbf{z}}$. For instance, the function f above could be described equally well by $f(\mathbf{r}) = \hat{\mathbf{x}}.\mathbf{r}/|\mathbf{r}|^2$ (Can you see why?).

What would be the rate of change of f as we change the point $\mathbf{r}$ along an arbitrary direction ? If this direction is along one of the axis, we know the answer, the rate of change is given by the partial derivative df/dx, df/dy or df/dz. But what of it is along some other direction? The answer is more simply stated by defining the *gradient vector field*:

$$\nabla f(x, y, z) = \frac{df}{dx}\hat{\mathbf{x}} + \frac{df}{dy}\hat{\mathbf{y}} + \frac{df}{dz}\hat{\mathbf{z}}. \tag{2}$$

The rate of change as the point (x, y, z) is varied along the direction of the **unit** vector $\hat{\mathbf{n}}$ is

$$\text{rate of change of f along } \hat{\mathbf{n}} = \hat{\mathbf{n}}.\nabla f = n_x\frac{df}{dx} + n_y\frac{df}{dy} + n_z\frac{df}{dz}. \tag{3}$$

Notice that the direction of the vector ∇f is the one along which f increases the most and $-\nabla f$ the direction along which f decreases the most.

ii) For the scalar field $f(x, y, z) = \frac{x}{x^2+y^2+z^2}$ above, what is the direction f changes the least at the point $\mathbf{r} = \hat{\mathbf{x}} + 4\hat{\mathbf{y}} + 3\hat{\mathbf{z}}$ (namely, $(x, y, z) = (1, 4, 3)$) ?

B. Line Integrals

Suppose a force $\mathbf{F}$, parallel to the x axis, acts on a body that moves along the x axis from l_1 to l_2. The *work* done by this force on the body is equal to $F(l_2 - l_1)$. If the force $\mathbf{F}$ makes an angle θ with the x axis the work would be $F(l_2 - l_1)\cos\theta = \mathbf{F}.\widehat{\Delta l}$ with $\widehat{\Delta l} = \mathbf{F}.(l_2 - l_1)\hat{\mathbf{x}}$.

But what if the path was not a straight line and/or the force changes along the path? Well, we can break out the path into infinitesimal segments $d\mathbf{l}$ and compute the work done on each little segment as $dW = \mathbf{F}.d\mathbf{l}$. Then we sum the contribution of each segment

$$W = \int_C \mathbf{F}.d\mathbf{l}. \tag{4}$$

The C under the integral sign indicates that the sum should be over the curve C. How do we actually compute these integrals? In general you should follow these steps:

a) parametrize the curve: Find a vector function of one parameter $\mathbf{r}(t)$ such that when t is varied, $\mathbf{r}(t)$ touches every point on the path. For instance, if the path is a straight line going from $\hat{\mathbf{x}}$ to $2\hat{\mathbf{x}}$ we can parametrize it as $\mathbf{r}(t) = (1 + t)\hat{\mathbf{x}}$ with $0 < t < 1$. If the path is a straight line from $\hat{\mathbf{y}}$ to $\hat{\mathbf{x}} + \hat{\mathbf{y}}$ you can parametrize it as $\mathbf{r}(t) = 0.5t\hat{\mathbf{x}} + \hat{\mathbf{y}}$ with $0 < t < 2$. Can you parametrize:

iii) a circle on the $x - y$ plane of radius R centered on the origin?

b) compute the tangent: the derivative $d\mathbf{r}(t)/dt$ gives a vector tangent to the path (for every value of t). The infinitesimal piece of path is given by $d\mathbf{l} = d\mathbf{r}/dt dt$.

iv) compute $d\mathbf{r}/dt$ for the parametrization of the circle above.

c) compute the scalar product and integrate: the line integral is given by

$$\int_C \mathbf{F}.d\mathbf{l} = \int_{t_i}^{t_f} \mathbf{F}.\frac{d\mathbf{r}}{dt}\; dt, \tag{5}$$

where t_i and t_f are the initial and final values of the parameter t. The integral on the right is just a regular integral of a function of one variable you learned in kindergarten.

v) compute the line integral $\int_C \mathbf{F}.d\mathbf{l}$ where C is the circle you parametrized before and $\mathbf{F}$ is the vector field

$$\mathbf{F}(\mathbf{r}) = Ax\ \hat{\mathbf{x}}, \tag{6}$$

where A is a constant.

C. The gradient theorem: a generalization of the fundamental theorem of calculus

There is an important relation for line integrals when the vector field $\mathbf{F}$ happens to be the gradient of an scalar field: $\mathbf{F} = \nabla f$.

$$\int_C \nabla f.d\mathbf{l} = f(B) - f(A), \tag{7}$$

where A and B are the initial and final points of the path C. If you understand the meaning of the gradient and of the line integral this theorem should be obvious.: $\nabla f.d\mathbf{l}$ is the rate of change of the function f along the direction tangent to the curve. As we move along the curve adding the increases of f (and decreases) we end up with the total tally of the change of f along the path, namely, the difference between the final and initial value of f, $f(B) - f(A)$.

vi) verify that $\mathbf{F} = Ax\ \hat{\mathbf{x}} = \nabla f$ for $f = Ax^2/2$.

vii) compute both sides of eq. 7 in the case when the path C is **half** the circle you parametrized above, going from $R\hat{\mathbf{x}}$ to $-R\hat{\mathbf{x}}$.

D. Visualizing vector fields

Make a (two-dimensional) plot of the following vector fields

i) $\mathbf{v} = 3\hat{\mathbf{x}} - 2\hat{\mathbf{y}}$ (this is a constant field, independent of the coordinates x and y)

ii) $\mathbf{v} = \mathbf{r} = x\hat{\mathbf{x}} + y\hat{\mathbf{y}}$

iii) $\mathbf{v} = \frac{\hat{\mathbf{r}}}{r^2}$, where $\mathbf{r} = x\hat{\mathbf{x}} + y\hat{\mathbf{y}}$

iv) $\mathbf{v} = \frac{\mathbf{r}}{r^3} - \frac{\mathbf{r}-\hat{\mathbf{x}}}{(\mathbf{r}-\hat{\mathbf{x}})^3}$, where $r^2 = |\mathbf{r}|^2 = x^2 + y^2$.

This time, and this time only, I want you to do this by hand. That means I'm looking only for the qualitative shape of the vector field, not for precise plotting.

E. Electric field of two charges

One $1C$ charge is located at $\mathbf{r} = 0$ and one -2 C charge is located one meter away at $\mathbf{r} = \hat{\mathbf{x}}$. Find the electric field everywhere and make a (two- dimensional) sketch of the field lines.