Physics 142/270 Class Notes

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Contents

I. ELECTROSTATICS

1. Chapter 23 (Serway) / 26 (Knight)

- 2 kinds of charge: $\pm$
- analogous to magnetic poles: N/S (magnetic charge cannot exist individually)

$\mathbf{F}_{12} = k_e \frac{q_1 q_2}{r_{12}^2} \hat{\mathbf{r}}_{12}$	**Coulomb's Law**
$\mathbf{r}_{12} \equiv \mathbf{r}_1 - \mathbf{r}_2$	separation vector
$\hat{\mathbf{r}} \equiv \frac{\mathbf{r}}{\lvert r \rvert}$	unit vector
$k_e \equiv \frac{1}{4\pi\epsilon_0}$	Coulomb constant
$\epsilon_0 = 8.85 \times 10^{-12} \left[\frac{C^2}{Nm^2}\right]$	permittivity of free space

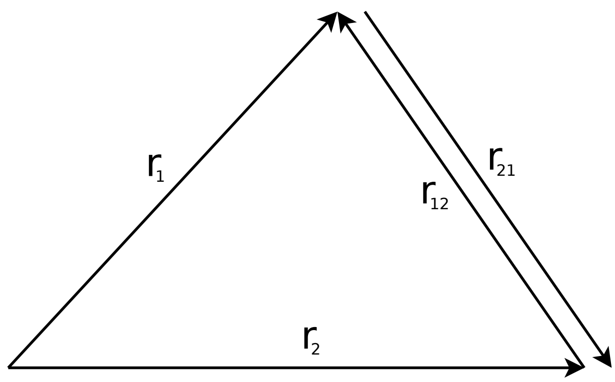

FIG. 1: separation vector

Compare to Newton's theory of universal gravitation.

$\mathbf{F}_{12} = -G \frac{m_1 m_2}{r_{12}^2} \hat{\mathbf{r}}_{12}$	**Newton's Law**
$G \equiv 6.67 \times 10^{-11} \left[\frac{Nm^2}{kg^2}\right]$	gravitational constant

Because the force due to gravity is proportional to your own mass...

$$\begin{array}{l|l} \mathbf{F}_i = m_i\,\mathbf{g}_i & \text{gravitational force} \\ \mathbf{g}_i \equiv \sum_{j\neq i} \frac{\mathbf{F}_{ij}}{m_i} = \sum_{j\neq i} -G\frac{m_j}{r_{ij}^2}\hat{\mathbf{r}}_{ij} & \textbf{local acceleration} \text{ due to gravity} \end{array}$$

...the quantity $\mathbf{g}_i$ is independent of m_i and is only location dependent ($\mathbf{r}_i$).

The analogy continues

$$\begin{array}{l|l} \mathbf{F}_i = q_i\,\mathbf{E}_i & \text{electrostatic force} \\ \mathbf{E}_i \equiv \sum_{j\neq i} \frac{\mathbf{F}_{ij}}{q_i} = \sum_{j\neq i} k_e\frac{q_j}{r_{ij}^2}\hat{\mathbf{r}}_{ij} & \textbf{Electric Field} \end{array}$$

The electric field $\mathbf{E}_i$ is independent of q_i and is only location dependent ($\mathbf{r}_i$). Therefore $\mathbf{r}_i$ is simply where we evaluate the field and q_j are the source charges at locations $\mathbf{r}_j$.

2. Chapter 23 (Serway) / 27 (Knight)

Let's denote the field at a location $\mathbf{E}_i(\mathbf{r}_i)$ as simply $\mathbf{E}(\mathbf{r}_f)$ and the source charge q_j as simply q with location $\mathbf{r}_q$.

$$\begin{array}{r|l} \mathbf{E}(\mathbf{r}_f) = \sum_q k_e\frac{q}{r_{fq}^2}\hat{\mathbf{r}}_{fq} & \text{discrete charge distribution} \\ \mathbf{E}(\mathbf{r}_f) = \int k_e\frac{dq}{r_{fq}^2}\hat{\mathbf{r}}_{fq} & \text{continuous charge distribution} \\ \rho \equiv \frac{dq}{dV} & \text{volume charge density} \\ \eta \equiv \frac{dq}{dA} & \text{surface charge density} \\ \lambda \equiv \frac{dq}{d\ell} & \text{linear charge density} \end{array}$$

3. Chapter 25 (Serway) / 29, 30 (Knight)

Energy is a consequence of Newton's second law with conservative forces, $\mathbf{F}(\mathbf{r}) = -\boldsymbol{\nabla}\,U(\mathbf{r})$, where U is the potential energy, $U(\mathbf{r}) = -\int \mathbf{dr}\cdot\mathbf{F}(\mathbf{r})$.

$$m\frac{\mathbf{dv}}{dt} = -\boldsymbol{\nabla}\,U(\mathbf{r}) \tag{I.1}$$
$$m\frac{\mathbf{dv}}{dt}\cdot\mathbf{dr} = -\boldsymbol{\nabla}\,U(\mathbf{r})\cdot\mathbf{dr} \tag{I.2}$$
$$m\,\mathbf{v}\cdot\mathbf{dv} = -\boldsymbol{\nabla}\,U(\mathbf{r})\cdot\mathbf{dr} \tag{I.3}$$
$$\frac{1}{2}mv^2\Big|_{v_0}^{v} = -U(\mathbf{r})\big|_{\mathbf{r}_0}^{\mathbf{r}} \tag{I.4}$$
$$\frac{1}{2}mv^2 + U(\mathbf{r}) = Energy \tag{I.5}$$

Here we also have the fact that $\mathbf{F}_q \propto q$.

$\mathbf{F} = q\,\mathbf{E}$		electrostatic force
$U(\mathbf{r}) = -q \int \mathbf{dr} \cdot \mathbf{E}$		potential energy
$V(\mathbf{r}) = \frac{U(\mathbf{r})}{q}$	$= -\int \mathbf{dr} \cdot \mathbf{E}$	electric potential

Don't confuse electric potential energy with electric potential.

For the Coulomb force...

$\mathbf{F}_{12} = k_e \frac{q_1 q_2}{r_{12}^2} \hat{\mathbf{r}}_{12}$	electrostatic force
$U_{12} = k_e \frac{q_1 q_2}{r_{12}}$	potential energy
$V_{12} = k_e \frac{q_2}{r_{12}}$	electric potential

Analogously for Newtonian gravity...

$\mathbf{F}_{12} = -G \frac{m_1 m_2}{r_{12}^2} \hat{\mathbf{r}}_{12}$	gravitational force
$U_{12} = -G \frac{m_1 m_2}{r_{12}}$	potential energy
$??? = -G \frac{m_2}{r_{12}}$	nothing you have covered

For charge distributions...

$V(\mathbf{r}_f) = \sum_q k_e \frac{q}{r_{fq}}$	discrete charge distribution
$V(\mathbf{r}_f) = \int k_e \frac{dq}{r_{fq}}$	continuous charge distribution

4. Chapter 24 (Serway) / 28 (Knight)

$\Phi_J \equiv \int \mathbf{J} \cdot d\mathbf{A}$	general flux relation for current $\mathbf{J}$

Let $\mathbf{J} = \rho\,\mathbf{v}$ where ρ is the volume density of the substance the current is transporting and $\mathbf{v}$ is the transport velocity. Let Q be the amount of substance, e.g. mass or charge, such that $Q = \int dV\,\rho$.

Integral	Differential	
$\oint \mathbf{J} \cdot \mathbf{dA} = -\frac{d}{dt} Q$	$\boldsymbol{\nabla} \cdot \mathbf{J} = -\frac{d}{dt}\rho$	continuity equation
$\oint \mathbf{J} \cdot \boldsymbol{d\ell} = 0$	$\boldsymbol{\nabla} \times \mathbf{J} = 0$	irrotational (no vorticity)

Example: $\mathbf{J}$ is a water current and $\mathbf{A}$ is the cross sectional area of a pipe. $\int \mathbf{J} \cdot \mathbf{dA}$ calculates the amount of water per unit time which crosses the boundary. $\oint \mathbf{J} \cdot \mathbf{dA}$ detects sources or sinks (leaks) of water.

The electric field is analogous to a current, but very abstractly.

Integral	Differential	
$\oint \mathbf{E} \cdot \mathbf{dA} = \frac{1}{\epsilon_0} q$	$\boldsymbol{\nabla} \cdot \mathbf{E} = \frac{1}{\epsilon_0} \rho$	**Gauss' Law**
$\oint \mathbf{E} \cdot \boldsymbol{d\ell} = 0$	$\boldsymbol{\nabla} \times \mathbf{E} = 0$	symmetry assumption (for now)

Positive and negative electric charge serves as sources and sinks of electric field. The electric field is analogous to a current and electric charge has an associated current, do not confuse the two.

This set of equations can reproduce Coulomb's Law. One could also construct an analogous set of equations for Newtonian gravity.

II. MAGNETISM

A. The Magnetic Field

1. *Chapter 26 (Serway) / 27 (Knight)*

Bar magnets interact with each other analogously to how electric dipoles interact with each other. Let us study electric dipoles in depth. If a dipole is very small, the surrounding field is effectively uniform in the neighborhood of the dipole.

$\boldsymbol{\tau} = \boldsymbol{\mu}_E \times \mathbf{E}$	torque on electric dipole

2. *Chapter 29 (Serway) / 33 (Knight)*

Bar magnets interact with each other analogously to how electric dipoles interact with each other. Let us therefore say that a small bar magnet (or magnetic dipole $\boldsymbol{\mu}_B$) has a magnetic field $\mathbf{B}$ analogous to the electric field $\mathbf{E}$ of a small electric dipole $\boldsymbol{\mu}_E$.

$\boldsymbol{\tau} = \boldsymbol{\mu}_B \times \mathbf{B}$	torque on magnetic dipole

But the interaction between electric and magnetic stuff is very different.

$\mathbf{F}_B = q\,\mathbf{v} \times \mathbf{B}$	magnetic force on charge
$\mathbf{F}_{E+B} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B})$	**Lorentz force law**

The magnetic force on charge is motion dependent, like friction, and cannot be generated from a scalar potential. But unlike friction it is not dissipative.

$$\frac{mv^2}{r} = qvB \quad \Big| \quad \text{cyclotron motion}$$

One can consider a continuous line of moving charge

$$\mathbf{dF} = dq\,\mathbf{v} \times \mathbf{B} \tag{II.1}$$

$$\mathbf{dF} = I\,\boldsymbol{d\ell} \times \mathbf{B} \tag{II.2}$$

where $I = \frac{dq}{dt}$ is the linear current. If everything is uniform $\mathbf{F} = I\,\boldsymbol{L} \times \mathbf{B}$. One can then consider a flat loop of current in a uniform magnetic field

$$\boldsymbol{\mu}_B = I\,\mathbf{A} \quad \Big| \quad \text{magnetic moment of current loop}$$

Also important/interesting: the **Hall Effect**, **velocity selector**, **mass spectrometer**.

B. Sources of the Magnetic Field

1. Chapter 30 (Serway) / 33 (Knight)

One can consider the interaction between a moving charge and magnetic field to determine the direction of field generated by the moving charge (Newton's third law). It's easiest to start with parallel currents...

$$\mathbf{B}_q(\mathbf{r}_f) \propto q\,\mathbf{v}_q \times \hat{\mathbf{r}}_{fq} \tag{II.3}$$

Furthermore, if a loop of charge has a dipole moment with magnetic field falling off like $1/r^3$, then the moving charge magnetic field should fall off like $1/r^2$ much like it's electric field.

$$\mathbf{B}_q(\mathbf{r}_f) = \frac{\mu_0}{4\pi}\frac{q}{r_{fq}^2}\,\mathbf{v}_q \times \hat{\mathbf{r}}_{fq} \quad \Big| \quad \textbf{Biot-Savart Law}$$

and we can also make the substitution $dq\,\mathbf{v} = I\,\boldsymbol{d\ell}$.

Coulomb's law was equivalent to Gauss' law + the lack of vorticity. What is the Biot-Savart Law equivalent to?

Integral	Differential	
$\oint \mathbf{B} \cdot \mathbf{dA} = 0$	$\boldsymbol{\nabla} \cdot \mathbf{B} = 0$	No magnetic charge
$\oint \mathbf{B} \cdot \boldsymbol{d\ell} = \mu_0 I$	$\boldsymbol{\nabla} \times \mathbf{B} = \mu_0 \mathbf{J}$	**Ampere's Law**

The combined laws are those of electro-magnetostatics (all currents must be net charge zero).

Nice examples:

- Infinitely long wire
- Coaxial cable
- Solenoid
- Force between parallel wires

C. Induction

1. *Chapter 31 (Serway) / 34 (Knight)*

Consider pushing a bar magnet through a loop of current. Depending on the orientation, the magnet may resist entrance or be sucked into the loop. If we perform work or get work out of this system, then energy must be going to or coming from somewhere.

Let's assume the magnet is fairly fixed in its properties and therefore the energy goes into or comes from the loop. A magnetic field will not do anything useful around the loop, so it must be an electric field around the loop which can accelerate charge and therefore do work. So a voltage difference around the loop (line integral of electric field) must be created or used to push and pull the magnet. The voltage around the loop must be related to magnetic charge approaching or retreating from the loop - or equivalently changing the number of magnetic field lines which penetrate the loop.

Integral	Differential	
$\oint \mathbf{E} \cdot \boldsymbol{d\ell} = -\frac{\partial}{\partial t} \int \mathbf{B} \cdot \mathbf{dA}$	$\boldsymbol{\nabla} \times \mathbf{E} = -\frac{\partial}{\partial t}\mathbf{B}$	**Faraday's Law**

$-\oint \mathbf{E} \cdot \boldsymbol{d\ell}$ is effectively the voltage around the loop if we make a tiny snip to insert a lightbulb/load. Otherwise we can create a short and possibly melt the wire.

The minus sign here is **Lenz's Law**; it can be reasoned from conservation of energy as follows: Push a bar magnet into or pull a bar magnet out of a loop with no initial current. A Faraday-like law would then induce a current into the loop - the electric field would push positive charge in that direction. The induced current generate a magnetic field and associated dipole moment. This dipole moment must be pointed in a particular direction in accord with conservation of energy.

Nice examples:

- Sliding Rod
- AC Generator
- Inductor
- Transformer

III. LINEAR CIRCUIT COMPONENTS

A. Capacitors and Dielectrics

1. *Chapter 29, 30 (Knight)*

A capacitor is any 2-terminal component which obeys the relation

$$\Delta V = \frac{1}{C} Q \quad \Big| \quad \textbf{Capacitor Voltage}$$

when charge $+Q$ is stored on one terminal and $-Q$ on the other; the two terminals are completely insulated from each other. The electric field lines point from the + terminal to the - terminal and therefore the voltage increases from the - terminal to the + terminal. Examples of capacitors which can be derived via Gauss' Law:

- Parallel plates
- Concentric cylinders
- Concentric spheres

Real world capacitors are like parallel plates folded or wound up... and almost always with some dielectric material between the plates.

Current flow through the capacitor: A capacitor in a circuit will attempt to maintain a net charge zero (or as close to that as it can) via the electrostatic force between the two plates. And excluding sparks, the net charge (typically zero) must be conserved in the circuit.

If you send a current I into one terminal then you build + charge on that plate. But if the circuit started at net zero charge then that + charge must have come from somewhere. Tracing backwards through the circuit, the + charge must have come from the opposite plate which is now building an equal amount of - charge. Therefore, although current does not physically travel across the plates, as far as mathematical calculations are concerned the current effectively flows between the two terminals in the sense that it goes in one terminal and comes out the other. [We will go much more in depth into these ideas when we cover current and flux.]

Using conservation of energy and charge we can derive the following relations for the equivalent capacitor given an assembly of capacitors

$$\begin{array}{r|l} C_{\text{eq}} = \sum_i C_i & \textbf{Capacitors in parallel} \\ \frac{1}{C_{\text{eq}}} = \sum_i \frac{1}{C_i} & \textbf{Capacitors in series} \end{array}$$

given that the capacitors that all begin uncharged or equivalent to that. [Example of a complication to this with switches]

Capacitors store energy: It takes work to charge a capacitor. If we have charge Q and therefore $\Delta V(Q) = \frac{Q}{C}$, then the work required to store dQ more charge is $dQ\,\Delta V(Q)$. But if Q increases to $Q + dQ$, then $\Delta V(Q)$ also increases and it takes more work to build up more charge.

Starting uncharged, to build up to charge Q the work required is $\int_0^Q dQ\,\Delta V(Q)$ and therefore

$$\begin{array}{r|l} U_C = \frac{1}{2}\frac{1}{C}Q^2 & \textbf{Energy stored in a capacitor} \end{array}$$

Compare this to the spring force and potential energy stored in a stretched or compressed spring

$$F_k = k\,x \tag{III.1}$$

$$U_k = \frac{1}{2}k\,x^2 \tag{III.2}$$

Dielectric materials: Consider inserting a dielectric material of freely orientable electric charge or dipoles between the two plates of a capacitor. Placing charge Q_0 on the plates would have created a field between the plates E_0. But E_0 moves the dielectric charge around which then has an induced field E_{ind} counter to E_0 and therefore the total field strength is reduced to $E_{\rm tot} = E_0 - E_{\rm ind}$. The bigger E_0 the bigger $E_{\rm ind}$. Assuming a simple linear relationship, $E_{\rm ind} = \alpha\, E_0$ where $0 < \alpha < 1$ we have

$$Q = C_0\,\Delta V \qquad \text{(III.3)}$$

$$Q = C_0(1-\alpha)\Delta V_0 \qquad \text{(III.4)}$$

Fixing Q, inserting the dielectric reduced ΔV by the factor $0 < (1-\alpha) < 1$. But fixing V, inserting the dielectric reduced the charge stored on the capacitor by the factor $1 < \kappa = \frac{1}{1-\alpha}$ or the Dielectric constant. By the definition of a capacitor we have a new capacitance $\kappa\, C_0$.

B. Current and Resistance

1. *Chapter 31 (Knight)*

An n-dimensional current $\mathbf{J}_n$ transports charge Q through n-dimensions and can be expressed

$$\mathbf{J}_n = \rho_n\,\mathbf{v} \qquad \text{(III.5)}$$

where is the n-dimensional charge density and $\mathbf{v}$ is the transport velocity.

We can setup an $(n-1)$ dimensional boundary B_{n-1} and measure how much charge is transported across the boundary. This is the flux integral

Flux	Current	
$\frac{dQ}{dt} = \int_{B_2} \mathbf{J}_3 \cdot \mathbf{dA}$	$\mathbf{J}_3 = \frac{dQ}{dV}\mathbf{v}$	3-dimensions
$\frac{dQ}{dt} = \int_{B_1} \mathbf{J}_2 \cdot \boldsymbol{d\ell}$	$\mathbf{J}_2 = \frac{dQ}{dA}\mathbf{v}$	2-dimensions
$\frac{dQ}{dt} = \mathbf{J}_1 \cdot \hat{\mathbf{n}}$	$\mathbf{J}_1 = \frac{dQ}{d\ell}\mathbf{v}$	1-dimension

In your book's notation: $\rho_3 = \frac{dQ}{dV} = \rho$, $\rho_2 = \frac{dQ}{dA} = \eta, \rho_3 = \frac{dQ}{d\ell} = \lambda$. In 1-dimension, the flux integral is a zero dimensional integral (no integration) and the only effect is to account for the direction of the current. J_1 is the linear current I.

Conservation of charge can be stated that the closed flux integral must be zero: all current flowing into a closed volume (of whatever dimension) must equal the output else there is some charge buildup or loss within the volume.

Ohm's Law as Linear Media: Let's model our resistor as an isotropic material with uniform voltage gradient (and electric field) placed across it. The material is neither a perfect conductor nor perfect insulator. Material charge may be made flow, but it takes some work to do so. Given the input field $\mathbf{E}$, we must determine the induced current $\mathbf{J}(\mathbf{E})$. We will work in 3-dimensions.

For small field strength, a physically reasonable and *analytic* response must take the form

$$\mathbf{J}(\mathbf{E}) \;=\; \sigma_1\,\mathbf{E} + \sigma_3(\mathbf{E}\cdot\mathbf{E})\mathbf{E} + \sigma_5(\mathbf{E}\cdot\mathbf{E})^2\mathbf{E} + \cdots \tag{III.6}$$

where σ are the conductivity coefficients. Therefore $I \propto \Delta V$ is a pretty good approximation (good to second order). A linear resistor is defined

$$\Delta V = R\,I \quad \Big| \quad \textbf{Ohm's Law}$$

Using conservation of energy and charge we can derive the following relations for the equivalent resistor given an assembly of resistors

$$\begin{array}{r|l} R_{\text{eq}} = \sum_i R_i & \textbf{Resistors in series} \\ \frac{1}{R_{\text{eq}}} = \sum_i \frac{1}{R_i} & \textbf{Resistors in parallel} \end{array}$$

Resistivity, a material property: Consider a standard resistor R of length ℓ and cross-sectional area A lined up along the x-axis. If we line up n_x resistors in series along the x-axis then we get a $n_x R$ resistor of length $n_x\ell$. If we stack n_y resistors in parallel along the y-axis then we get a $\frac{1}{n_y}R$ resistor of cross-sectional $n_y A$. If we stack n_z resistors in parallel along the z-axis then we get a $\frac{1}{n_z}R$ resistor of cross-sectional $n_z A$. If we do all three then we get a $\frac{n_x}{n_y n_z}R$ resistor with length $n_x\ell$ and cross-sectional $n_y n_z A$. If our standard resistors are very small then we can take a continuum limit and say

$$R = \rho\frac{\ell}{A} \quad \Big| \quad \textbf{Resistivity}$$

where ρ here is the resistivity, not the charge density (it's actually $\frac{1}{\sigma_1}$, the inverse of how easily current is induced in the resistor). ρ is the material property and the remainder is determined by the geometry of the resistor.

Resistors dissipate energy:

$$P_R = R\,I^2 \quad \Big| \quad \textbf{Power dissipated by a Resistor}$$

Compare $P = I\,\Delta V$ to $P = \mathbf{v}\cdot\mathbf{F}$ for the power generated (or dissipated) by a force. This energy is lost in the form of heat.

Ohm's Law as Near Equilibrium Dissipation: There is also a motivation/derivation of Ohm's law by considering the resistor as a thermal environment which can exchange energy with the current flowing through it. This is too difficult to go into but it yields a fluctuation-dissipation relation: the more resistance you have, the more thermal noise you get in the signal. Specifically $\langle V_{\mathrm{noise}}^2 \rangle \propto RT$ where T is the temperature of the resistor. The noise is random (on average zero) and $\langle \cdot \rangle$ denotes the noise average.

C. Inductors

1. *Chapter 34 (Knight)*

The solenoid gave us

$$\Delta V = L\,\tfrac{dI}{dt} \;\Big|\; \textbf{Inductor Voltage}$$

where L happened to be $\mu_0 N \frac{N}{\ell} A$.

We can work out that inductors add just like resistors. We can also work out that capacitors would have added just like resistors if we had used $\frac{1}{C}$ instead of C.

Inductors store energy: It takes work to get current flowing through an inductor. If we have current I and therefore $\Delta V(I) = L\,I$, then the work required to run dI more current is $dI\,\Delta V(I)$. But if I increases to $I + dI$, then $\Delta V(I)$ also increases and it takes more work to run more current.

Starting from rest, to build up to current I the work required is $\int_0^I dI\,\Delta V(I)$ and therefore

$$U_L = \tfrac{1}{2} L\,I^2 \;\Big|\; \textbf{Energy stored in an inductor}$$

Compare this to Newton's second law

$$F = m\,\frac{dv}{dt} \tag{III.7}$$

$$K = \frac{1}{2} m\,v^2 \tag{III.8}$$

In this sense, induction is like inertia for electricity.

Mutual Inductance: Two solenoids which share the same ferromagnetic core will obey the relation

$$\tfrac{V_1}{V_2} = \tfrac{N_1}{N_2} \;\Big|\; \textbf{Transformer}$$

and remember that in each case $V \propto \frac{dI}{dt}$ like an inductor.

DC is a constant voltage, and can't be transformed by induction. AC is an oscillating voltage $V(t) = V_0 \cos(\omega t)$, which will produce an oscillating current, and so the transformer puts out an oscillating voltage just at a different amplitude and phase.

Why is this important? Consider the delivery of power to your house.

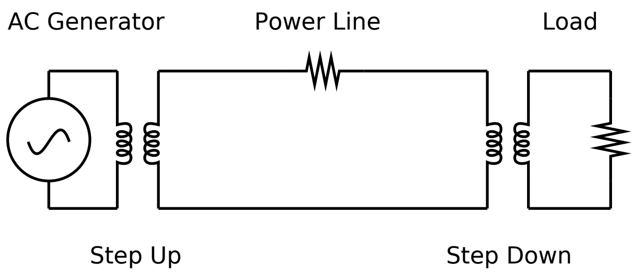

FIG. 2: Power line transmission

Let's consider the limit in which the line resistance is very small $R_{\text{line}} = r$. In this case

$$I_{\text{line}} \sim I_0 \tag{III.9}$$

$$V_{\text{line}} \sim r\, I_0 \tag{III.10}$$

where I_0 is what the current would be with no resistance in the line.

What is the power lost by the line resistance?

$$P_{\text{line}} = V_{\text{line}} I_{\text{line}} \tag{III.11}$$

$$\sim r\, I_0^2 \tag{III.12}$$

Obviously we must lower the line current.

Fortunately the power transmitted through the transformers and to the load does not care about the current but the combination $P_{\text{tran}} = V_{\text{tran}}\, I_{\text{line}}$. So we can reduce the current by some factor, say n, as long as we increase the voltage by that same factor. Therefore we step the voltage way up for the line and then step it back down for our house (so that we don't destroy ourselves).

IV. CIRCUITS

A. DC Circuits

1. *Chapter 32, 34, 14 (Knight)*

Now we can more carefully examine how capacitors charge and discharge.

$$\begin{array}{rl|l} R\,\frac{d}{dt}Q + \frac{1}{C}Q = & V_0 & \textbf{RC Circuit} \\ \langle Q \rangle = & CV_0 & \text{Equilibrium Charge} \\ \gamma = & \frac{1}{RC} & \text{Characteristic Decay Rate} \end{array}$$

We can also revisit the problem with the battery, switch, and two capacitors.

We can more careully examine how current accelerates from rest.

$$\begin{array}{rl|l} L\,\frac{d}{dt}I + R\,I = & V_0 & \textbf{RL Circuit} \\ \langle I \rangle = & \frac{V_0}{R} & \text{Equilibrium Current} \\ \gamma = & \frac{R}{L} & \text{Characteristic Decay Rate} \end{array}$$

The following circuit conserves energy.

$$\begin{array}{rl|l} L\,\frac{d^2}{dt^2}Q + \frac{1}{C}Q = & V_0 & \textbf{LC Circuit} \\ \langle Q \rangle = & CV_0 & \text{Equilibrium Charge} \\ \omega = & \frac{1}{\sqrt{LC}} & \text{Characteristic Frequency} \end{array}$$

It is just the harmonic oscillator. This is our first second order equation, it requires two boundary conditions, e.g. initial charge on capacitor and initial current. Note that we can express solutions in terms of amplitudes, amplitude and phase, or amplitude and initial time.

$$Q(t) = A_1\,\cos(\omega t) + A_2\,\sin(\omega t) \tag{IV.1}$$

$$Q(t) = A_3\,\cos(\omega t + \phi_3) \tag{IV.2}$$

$$Q(t) = A_4\,\sin(\omega[t - t_0]) \tag{IV.3}$$

We can do this because of the double angle formulas.

This following is not in Knight, except in Chp. 14 for mechanical.

$$
\begin{array}{r|l}
L\,\frac{d^2}{dt^2}Q + R\,\frac{d}{dt}Q + \frac{1}{C}Q = V_0 & \textbf{RLC Circuit} \\
\langle Q \rangle = C V_0 & \text{Equilibrium Charge} \\
m\,\frac{d^2}{dt^2}x + b\,\frac{d}{dt}x + k\,x = F_0 & \textbf{Damped Harmonic Oscillator} \\
\langle x \rangle = \frac{F_0}{k} & \text{Equilibrium Position}
\end{array}
$$

What is the timescale here? It's a linear ODE with constant coefficients, we can assume a (homogeneous) solution of the form $Q_f(t) = A_f\, e^{f\,t}$. You must solve the characteristic equation

$$
\begin{array}{r|l}
L\,f^2 + R\,f + \frac{1}{C} = 0 & \textbf{Characteristic Equation}
\end{array}
$$

The real part will be a negative decay rate (reciprocal of decay time, $\gamma = \frac{1}{\tau}$) and the imaginary part will be the oscillation frequency. The discriminant $R^2 - 4\frac{L}{C}$ is very important.

Under-damped $f = -\gamma \pm \imath\,\omega$ Damped oscillations

Critically damped $f = -\gamma$ Double root, pure decay but not purely exponential decay. There is a second secular solution $Q_\mathrm{s}(t) = A_\mathrm{s}\, t\, e^{f\,t}$.

Over-damped $f = -\gamma_0 \pm \delta\gamma$ Purely damped evolution with two decay rates

I also want to briefly mention circuits with multiple loops. After applying conservation of charge and energy we get

$$
\mathbf{L}\,\frac{d^2}{dt^2}\mathbf{Q} + \mathbf{R}\,\frac{d}{dt}\mathbf{Q} + \mathbf{C}^{-1}\,\mathbf{Q} \;=\; \mathbf{V}_0 \tag{IV.4}
$$

$$
\langle \mathbf{Q} \rangle \;=\; \mathbf{C}\mathbf{V}_0 \tag{IV.5}
$$

Circuits with one kind of component were algebraic, but now differential. This is not something that I expect you to be able to *solve* in class.

Use the fact that $\frac{d}{dt}\mathbf{Q} = \mathbf{I}$ to write the block-matrix equation

$$
\frac{d}{dt}\begin{bmatrix} \Delta\mathbf{Q} \\ \Delta\mathbf{I} \end{bmatrix} = \begin{bmatrix} \mathbf{0} & \mathbf{1} \\ -\mathbf{L}^{-1}\mathbf{C}^{-1} & -\mathbf{L}^{-1}\mathbf{R} \end{bmatrix} \begin{bmatrix} \Delta\mathbf{Q} \\ \Delta\mathbf{I} \end{bmatrix} \tag{IV.6}
$$

One can solve this with matrix diagonalization.

B. AC Circuits

1. *Chapter 36 (Knight)*

Consider any circuit with linear components and and AC voltage. The differential equation is linear, so the solution comes in two parts, the homogeneous part of the solution which evolves at RLC timescales and the driven part of the solution which is pushed by the voltage. The homogeneous part of the solution fixes the initial conditions and with any resistance decays away in time. We will focus on the driven part of the solution which is the steady state solution and exists indefinitely.

Let's say we have an AC voltage $V(t) = V_0 \cos(\omega t)$ with a resistor, we then get

$$I(t) = \frac{V_0}{R}\cos(\omega t) \tag{IV.7}$$

the resistor resists current just the same.

For an AC voltage with an inductor we get

$$I'(t) = \frac{V_0}{L}\cos(\omega t) \tag{IV.8}$$

$$I(t) = \frac{V_0}{\omega L}\sin(\omega t) \tag{IV.9}$$

$Z_L = \omega L$ is like the resistance but the current and voltage are out of phase.

For an AC voltage with an capacitor we get

$$Q(t) = C\,V_0\cos(\omega t) \tag{IV.10}$$

$$I(t) = -\omega C\,V_0\sin(\omega t) \tag{IV.11}$$

$Z_C = \frac{1}{\omega C}$ is like the resistance but the current and voltage are out of phase.

Here's the trick that will keep track of the phases.

$$V(t) = \mathrm{Re}\big[V_0\,e^{+\imath\omega t}\big] \tag{IV.12}$$

$$\tilde{V}(t) = V_0\,e^{+\imath\omega t} \tag{IV.13}$$

Assume a solution of the form $\tilde{I}(t) = \tilde{I}_0\,e^{+\imath\omega t}$ For each individual component we would then get

$$\tilde{I}(t) = \frac{V_0}{R}e^{+\imath\omega t} \tag{IV.14}$$

$$\tilde{I}(t) = \frac{V_0}{\imath\omega L}e^{+\imath\omega t} \tag{IV.15}$$

$$\tilde{I}(t) = \imath\omega C\,V_0\,e^{+\imath\omega t} \tag{IV.16}$$

The real parts of these equations are precisely equal to the above. We therefore have the complex impedances R, $\imath\omega L$, and $\frac{1}{\imath\omega C}$ denoted $\tilde{Z}$. The magnitude is like the resistance but the $\imath$ keeps track of the phase information.

Why is this trouble necessary? Consider an AC RLC circuit, solving for the driven solution would be extremely tedious with sines and cosines being differentiated twice and collected together. Here we can assume a solution of the form $\tilde{I}(t) = \tilde{I}_0\, e^{+\imath\omega t}$ and say

$$\imath\omega L\, \tilde{I}(t) + R\, \tilde{I}(t) + \frac{1}{\imath\omega C}\tilde{I}(t) \;=\; V_0\, e^{+\imath\omega t} \tag{IV.17}$$

$$\tilde{I}(t) \;=\; \frac{V_0}{\imath\omega L + R + \frac{1}{\imath\omega C}} e^{+\imath\omega t} \tag{IV.18}$$

The amplitude of the current is then

$$|\tilde{I}| \;=\; \frac{V_0}{\sqrt{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2}} \tag{IV.19}$$

The the phase difference is

$$\tan(\phi) \;=\; -\frac{\omega L + \frac{1}{\omega C}}{R} \tag{IV.20}$$

The really incredible thing about this method is that you can do multiple loops with complex algebra instead of differential equations. There is one thing to note: this works because of linearity. Consider the power dissipated in the circuit

$$P(t) \;=\; V(t)\, I(t) \tag{IV.21}$$

This is not a linear relation, so we can't get fancy by taking the real part of a complex variable. We have to take the real parts, then stick them into this relation.

$$P(t) \;=\; V_0 \cos(\omega t)\, I_0 \cos(\omega t + \phi) \tag{IV.22}$$

$$\langle P \rangle \;=\; \frac{1}{2} I_0\, V_0\, \cos(\phi) \tag{IV.23}$$

And you can see that the *average* power dissipated through a resistor is $\frac{1}{2} R\, I_0$ and 0 for an inductor or capacitor.

Important stuff to cover.

- Resonance
- Multiple loop circuits
- High and low pass filters
- Diodes and rectifiers

V. OPTICS

A. Electromagnetic Waves

1. *Chapter 20, 35 (Knight)*

Finalizing Maxwell's Equations Up to now we have

Integral	Differential	
$\oint \mathbf{E}\cdot d\mathbf{A} = \frac{1}{\epsilon_0}q$	$\nabla\cdot\mathbf{E} = \frac{1}{\epsilon_0}\rho$	**Gauss' Law**
$\oint \mathbf{E}\cdot d\boldsymbol{\ell} = -\frac{\partial}{\partial t}\int \mathbf{B}\cdot d\mathbf{A}$	$\nabla\times\mathbf{E} = -\frac{\partial}{\partial t}\mathbf{B}$	**Faraday's Law**
$\oint \mathbf{B}\cdot d\mathbf{A} = 0$	$\nabla\cdot\mathbf{B} = 0$	No magnetic charge
$\oint \mathbf{B}\cdot d\boldsymbol{\ell} = \mu_0 I$	$\nabla\times\mathbf{B} = \mu_0\mathbf{J}$	**Ampere's Law**

One can see, by assuming a kind of symmetry in the field, that there is a changing electric flux term missing from Ampere's law as compared to Faraday's law.

We can determine the proper term by considering a wire of current feeding into a parallel plate capacitor. Around the wire, Ampere's law determines there to be a magnetic field. But if we stretch the area enclosed by the loop over the closest plate, the enclosed current is now zero for the same magnetic field around the same loop! So we are missing something. What is there which is equal to $\mu_0 I$? Well I causes a build up of $\frac{d}{dt}Q$ on the plate. There is the time derivative we need, but how does Q relate to the electric flux? From the definition of the capacitor we have $Q = CV$ or $Q = Cd\,E$. But we want a flux so let's use $C = \frac{\epsilon_0 A}{d}$. Combining all of this together we have the relation

$$\mu_0 I \;=\; \epsilon_0\mu_0\frac{d}{dt}EA \tag{V.1}$$

and therefore the complete equation must be

Integral	Differential	
$\oint \mathbf{B}\cdot d\boldsymbol{\ell} = \mu_0 I + \epsilon_0\mu_0\frac{\partial}{\partial t}\int \mathbf{E}\cdot d\mathbf{A}$	$\nabla\times\mathbf{B} = \mu_0\mathbf{J} + \epsilon_0\mu_0\frac{\partial}{\partial t}\mathbf{E}$	**Ampere-Maxwell Law**

With the Lorentz force law we now have a complete theory of electromagnetism.

Energy in the Field We know that the capacitor stores the potential energy $U(Q) = \frac{Q^2}{2C}$ but we can also think of this in terms of the field by substituting Q(E) to get $\frac{U}{Ad} = \frac{1}{2}\epsilon_0 E^2$ or and generally

$\frac{dU}{dV} = \frac{\epsilon_0}{2}\mathbf{E}^2$	**Energy in the Electric Field**

where V here is volume and $\frac{dU}{dV}$ is the local energy density due to the electric field.

We can do the same thing for the solenoid and magnetic field using $B = \mu_0 \frac{N}{\ell} I$ and $L = \mu_0 \frac{N^2 A}{\ell}$ to get $\frac{U}{A\ell} = \frac{1}{2\mu_0} B^2$ or and generally

$$\frac{dU}{dV} = \frac{1}{2\mu_0}\mathbf{B}^2 \quad \Bigg| \quad \textbf{Energy in the Magnetic Field}$$

Both of these relations are completely general.

Electromagnetic Waves Notice that Maxwell's equations for electrodynamics are linear with source (or driving) terms. Therefore we can decompose solutions in to homogeneous and generated (driven) parts. Let's inspect the (homogeneous) source-less, free field solutions.

$$\nabla \cdot \mathbf{E} = 0 \tag{V.2}$$

$$\nabla \cdot \mathbf{B} = 0 \tag{V.3}$$

$$\nabla \times \mathbf{E} = -\frac{\partial}{\partial t}\mathbf{B} \tag{V.4}$$

$$\nabla \times \mathbf{B} = \epsilon_0 \mu_0 \frac{\partial}{\partial t}\mathbf{E} \tag{V.5}$$

Next we will use the mathematical identity

$$\nabla \times (\nabla \times \mathbf{A}) = \nabla(\nabla \cdot \mathbf{A}) - \nabla^2 \mathbf{A} \tag{V.6}$$

and come to the equations

$$\nabla^2 \mathbf{E} = \epsilon_0 \mu_0 \frac{\partial^2}{\partial t^2}\mathbf{E} \tag{V.7}$$

$$\nabla^2 \mathbf{B} = \epsilon_0 \mu_0 \frac{\partial^2}{\partial t^2}\mathbf{B} \tag{V.8}$$

which are wave equations in 3-dimensions with wave speed $v = c = \frac{1}{\sqrt{\epsilon_0 \mu_0}}$. But note that this is only 2 equations so there are $4 - 2 = 2$ more equations constraining our solutions.

1-dimensional wave solutions were all of the form $f(x \mp vt)$ or $g(kx \mp \omega t)$ where $\frac{\omega}{k} = v$. There are more kinds of waves in higher dimensions but perhaps the most analogous are plane waves: $f(\hat{\mathbf{k}} \cdot \mathbf{x} - vt)$ or $g(\mathbf{k} \cdot \mathbf{x} - \omega t)$ where $\frac{\omega}{k} = v$ and the wave travels in the $+\hat{\mathbf{k}}$ direction. Let's try to find plane wave solutions

$$\mathbf{E} = f_E(\hat{\mathbf{k}}_E \cdot \mathbf{x} - ct)\,\hat{\mathbf{E}}_0 \tag{V.9}$$

$$\mathbf{B} = f_B(\hat{\mathbf{k}}_B \cdot \mathbf{x} - ct)\,\hat{\mathbf{B}}_0 \tag{V.10}$$

these satisfy the wave equations, but what about all of maxwell's equations? Let's plug them in and simplify

$$\hat{\mathbf{k}}_E \cdot \hat{\mathbf{E}}_0 = 0 \tag{V.11}$$

$$\hat{\mathbf{k}}_B \cdot \hat{\mathbf{B}}_0 = 0 \tag{V.12}$$

$$f'_E(\hat{\mathbf{k}}_E \cdot \mathbf{x} - ct)\,\hat{\mathbf{k}}_E \times \hat{\mathbf{E}}_0 = +c\,f'_B(\hat{\mathbf{k}}_B \cdot \mathbf{x} - ct)\,\hat{\mathbf{B}}_0 \tag{V.13}$$

$$f'_B(\hat{\mathbf{k}}_B \cdot \mathbf{x} - ct)\,\hat{\mathbf{k}}_B \times \hat{\mathbf{B}}_0 = -\frac{1}{c}f'_E(\hat{\mathbf{k}}_E \cdot \mathbf{x} - ct)\,\hat{\mathbf{E}}_0 \tag{V.14}$$

The divergence-less equations imply that the waves are both **transverse** This means that the field points in a direction perpendicular to the direction the wave travels.

From the curl equations we can dot the first with with the electric field direction and the second with the magnetic field direction to get

$$0 = +c\,f'_B(\hat{\mathbf{k}}_B \cdot \mathbf{x} - ct)\,\hat{\mathbf{E}}_0\,\hat{\mathbf{B}}_0 \tag{V.15}$$

$$0 = -\frac{1}{c}f'_E(\hat{\mathbf{k}}_E \cdot \mathbf{x} - ct)\,\hat{\mathbf{E}}_0\,\hat{\mathbf{B}}_0 \tag{V.16}$$

so the electric and magnetic fields must point in different directions.

To inspect the curl equations more thoroughly let's use the mathematical identity "bac cab"

$$\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = \mathbf{B}\,(\mathbf{A} \cdot \mathbf{C}) - \mathbf{C}\,(\mathbf{A} \cdot \mathbf{B}) \tag{V.17}$$

to get

$$+f'_E(\hat{\mathbf{k}}_E \cdot \mathbf{x} - ct)\,\hat{\mathbf{k}}_E = +c\,f'_B(\hat{\mathbf{k}}_B \cdot \mathbf{x} - ct)\,\hat{\mathbf{E}}_0 \times \hat{\mathbf{B}}_0 \tag{V.18}$$

$$-f'_B(\hat{\mathbf{k}}_B \cdot \mathbf{x} - ct)\,\hat{\mathbf{k}}_B = -\frac{1}{c}f'_E(\hat{\mathbf{k}}_E \cdot \mathbf{x} - ct)\,\hat{\mathbf{E}}_0 \times \hat{\mathbf{B}}_0 \tag{V.19}$$

From outer vectors we can see that the fields must propagate in the same direction. Then we can see that the wave shapes must be the same except for a factor of c. Our fully determined plane wave solutions are then

$$\mathbf{E} = f(\hat{\mathbf{k}} \cdot \mathbf{x} - ct)\,\hat{\mathbf{E}}_0 \tag{V.20}$$

$$\mathbf{B} = \frac{1}{c}f(\hat{\mathbf{k}} \cdot \mathbf{x} - ct)\,\hat{\mathbf{B}}_0 \tag{V.21}$$

$$\hat{\mathbf{k}} = \hat{\mathbf{E}}_0 \times \hat{\mathbf{B}}_0 \tag{V.22}$$

They are transverse waves which propagate at right angles and have proportional amplitudes.

Recalling previous formulas, if the energy density in the electric field is $\frac{\epsilon_0}{2}\mathbf{E}^2$ then the energy density in the magnetic field is $\frac{1}{2\mu_0}\mathbf{B}^2 = \frac{1}{2\mu_0 c^2}\mathbf{E}^2$... the same. There is equal energy distributed between the two fields. Now let's look at the vector

$$\mathbf{S} = \frac{1}{\mu_0}\mathbf{E}\times\mathbf{B} \quad \Big| \quad \textbf{Poynting Vector}$$

This vector points in the direction that the wave propagates at and note that it's magnitude is

$$S = \frac{EB}{\mu} \tag{V.23}$$

$$= c\frac{B^2}{\mu} \tag{V.24}$$

which is equal to the total field energy density times c. Therefore $\mathbf{S}$ is the **energy current**. If we do a flux integral, then we get the energy per unit time (power) passing through a surface area.

$P = \oint \mathbf{S}\cdot d\mathbf{A}$	**Instantaneous Power**
$\langle P\rangle = \frac{1}{2}\oint \mathbf{S}_0\cdot d\mathbf{A}$	**Average Power** for sinusoidal waves

where the average $\langle \mathbf{S}\rangle = \frac{1}{2}\mathbf{S}_0$ for a sinusoidal wave is also called the intensity I.

Also, the E&M wave also carries momentum proportional to its energy

$E = p\,c$	**Photon Kinetic Energy**
$E = \frac{p^2}{2m}$	**Regular Kinetic Energy**

You can consider this the kinetic energy of a massless particle.

Now as the wave transfers momentum, it must deliver a force.

$$F = \frac{dp}{dt} \tag{V.25}$$

$$= \frac{P}{c} \tag{V.26}$$

And a force over area is pressure.

$$\frac{F}{A} = \frac{\frac{P}{A}}{c} \tag{V.27}$$

$$= \frac{S}{c} \tag{V.28}$$

so the Poynting vector delivers pressure. Remember that the above equation is instantaneous and not a time average.

Polarization: A sinusoidal wave carrying a certain amount of energy in a certain direction is not unique. You may rotate the E&B fields 180° before you are out of phase again, 360° before you are back in phase. This angle is called the polarization angle.

Polaroid filters block out one polarization of the field and let the other pass

$$E_0 \rightarrow \cos(\theta)\, E_0 \tag{V.29}$$

$$I_0 \rightarrow \cos^2(\theta)\, I_0 \tag{V.30}$$

Natural light is randomly polarized in a uniform distribution (unpolarized is terrible language). So the average intensity you get out is the average over $\cos^2(\theta)\, I_0$ or $\frac{1}{2}\, I_0$.

B. Geometric Optics

1. *Chapter 23 (Knight)*

Reflection: If reflection is a reversible process then we must have

$$\theta_r = \theta_i \quad \Big| \quad \textbf{Law of Reflection}$$

- Reflection as a collision
- Flat Mirrors
- Corner Reflectors
- Spherical Mirrors (23.56)

Index of Refraction: We have discussed that for linear materials the electromagnetic wave speed is $v = \frac{1}{\sqrt{\epsilon\mu}}$, which is strictly less than the speed of light in vacuum. For such a material the index of refraction is defined $n = \frac{c}{v}$ where c is the speed of light in vacuum so this value is dimensionless and strictly greater than one (baring some very exotic material (which don't give faster light but does something else)).

Refraction: Why does light travel in a straight line? Perhaps to get from point 1 to point 2 the light is minimizing distance or transit time.

It is a basic fact of the universe that light refracts between the interface of different media. You can stick your hand in water and see how its image is bent at the interface. It is very likely that light has different wave speeds in the different media. But what is their relation to the refraction?

Consider light passing between the two points (x_1, y_1) and (x_2, y_2) in two the different media n_1 and n_2. Let the interface point be (x_i, y_i).

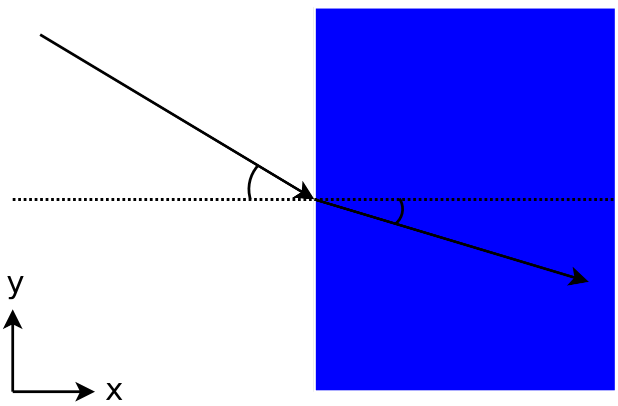

FIG. 3: Refraction

x_i is fixed on our diagram, but y_i will be dependent up on the two indices of refraction.

Light can't be minimizing distance traveled here, otherwise we would have a single straight line. Let's see what minimizing the transit time will give us. We must minimize the total transit time $t = t_1 + t_2$

$$t_1 = \frac{\ell_1}{v_1} \tag{V.31}$$

$$t_2 = \frac{\ell_2}{v_2} \tag{V.32}$$

where

$$\ell_1 = \sqrt{(x_i - x_1)^2 + (y_i - y_1)^2} \tag{V.33}$$

$$\ell_2 = \sqrt{(x_i - x_2)^2 + (y_i - y_2)^2} \tag{V.34}$$

y_i must be such to minimize $t(y_i)$. Calculus then tells us (after some simplification)

$$\frac{dt}{y_i} = \frac{1}{v_1}\frac{y_i - y_1}{\ell_1} + \frac{1}{v_2}\frac{y_i - y_2}{\ell_2} \tag{V.35}$$

We can then relate the hight/distance ratios to trigonometry and the wave speeds to refraction index to get

$$n_1 \sin(\theta_1) = n_2 \sin(\theta_2) \quad \Big| \quad \textbf{Snell's Law}$$

- Refraction through a window
- Refraction through a prism
- Total internal reflection
- Fiber optics

Optics

- Flat interface (23.27)
- Spherical interface (23.44)
- Thin Lens

C. Wave Interference

1. *Chapter 21, 22 (Knight)*

The key theme for interference is that while we can add fields, we cannot add intensities. When two fields combines they can interfere. If the intensity decreases, they are interfering destructively. If the intensity increases, they are interfering constructively.

Thin Flims: There are two facts we need to consider for thin films

1. When light reflects off of a *higher* index medium, it reflects back 180° out of phase. (Mechanical analogy with rope)
2. When light travels into a medium, the wave length changes but not the frequency.

The frequency can't change for any continuous wave, otherwise it would become discontinuous at the interface.

$$n = \frac{c}{v} \tag{V.36}$$

$$= \frac{\lambda_0 f_0}{\lambda_n f_n} \tag{V.37}$$

$$\lambda_n = \frac{\lambda_0}{n} \tag{V.38}$$

Everything then works out directly. Do not memorize the resulting equations, they will differ for every problem.

I will also cover single slit and double slit interference, but this takes more time to derive the resulting equations - you will not have to be able to do it.

VI. RELATIVITY

A. Galilean Relativity

1. Chapter 3, 37 (Knight)

Consider the three points 1,2,3.

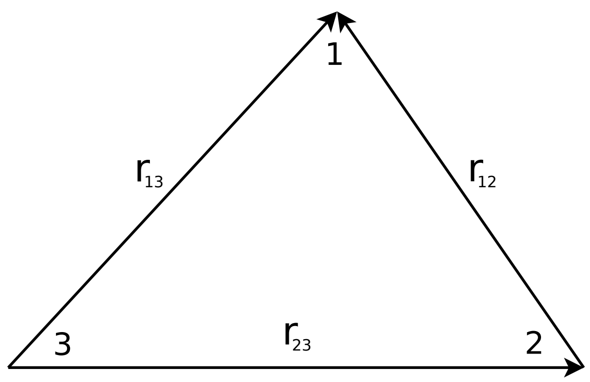

FIG. 4: Galilean vector addition

Galilean relativity dictates that we can relate these vectors in the manner

$$\mathbf{r}_{12} = \mathbf{r}_{13} - \mathbf{r}_{23} \tag{VI.1}$$

$$\mathbf{r}_{12} = -\mathbf{r}_{21} \tag{VI.2}$$

We will refer to this as a **frame shift**.

Differentiating the first relation gives us our Galilean velocity addition rule

$$\mathbf{v}_{12} = \mathbf{v}_{13} - \mathbf{v}_{23} \tag{VI.3}$$

If we think of $\mathbf{v}_{13}$ as the velocity of particle 1 in frame 3, then $\mathbf{v}_{12}$ is also the velocity of particle 1 but in frame 2. They two velocities are related by the velocity of frame 2 relative to frame 3. We will refer to this as a **frame boost** - it is simply any non-constant frame shift.

What is special about Galilean relativity is that it preserves the separation vector. I.e. as any particle i has transformed position

$$\mathbf{r}_{i1} = \mathbf{r}_{i2} - \mathbf{r}_{12} \tag{VI.4}$$

then the separation vector $\mathbf{r}_{ij}$ and therefore r_{ij} is unaltered no matter what frame you measured it in.

We generally don't care how our frame is oriented. We really just want to make sure that all separation distances r_{ij} are preserved. Mathematically we say that we want to preserve the metric

$$(\Delta r)^2 = (\Delta x)^2 + (\Delta y)^2 + (\Delta z)^2 \quad \Big| \quad \textbf{Euclidean metric}$$

Two other kinds of transformations will do this. One is **inverting** the axes, say $x \to -x$. The other is **rotations**

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \quad \Bigg| \quad \text{2-dimensional rotation}$$

One can easily prove that this preserves our notion of distance.

Newton's Second Law and Inertial Frames All of science is guided by the principle of parsimony. In the context of physics this gives rise to the desire that physical laws be the same in all valid frames. A valid frame for us will the the inertial frame.

Newton's second law dictates that dynamics of particles are governed by the differential equation

$$m_i \frac{d^2}{dt^2}\mathbf{r}_i \; = \; \sum_j \mathbf{F}_{ij} \tag{VI.5}$$

and up until magnetics all of our conservative forces took the form

$$m_i \frac{d^2}{dt^2}\mathbf{r}_i \; = \; \sum_j F_{ij}(r_{ij})\,\hat{\mathbf{r}}_{ij} \tag{VI.6}$$

We will consider inertial frames to be those in which the above law is applicable. The right hand side is compatible with any shift, inversion, or rotation, though note that the force would get inverted or rotated but its magnitude would remain invariant. But the left hand side is only compatible with **constant** shifts, boosts, rotations, and inversions. Thus our family of inertial frames are related by such transformations. Any other transformation would introduce fictitious forces. One can always identify a fictitious force as it comes from the left hand side and is always proportional to mass. Examples of fictitious forces that you may have encountered are centrifugal and Coriolis forces. There is actually one more...

2. Chapter 35 (Knight)

Galilean relativity and inertial frames was all well and good until we ventured into magnetics. In the rest frame of a point charge we only see an electric field, but if we boost we see a moving charge and thus a magnetic field. Thus electric and magnetic fields transform into each other. They are relative. If we take the Lorentz force law to be true, and consider a charge at rest, then we can get half of the boost transformations. If we further consider how the Coulomb field must transform into a Biot-Savart field we can then complete the relations.

$$\mathbf{E}' \; = \; \mathbf{E} - \mathbf{v} \times \mathbf{B} \tag{VI.7}$$

$$\mathbf{B}' \; = \; \mathbf{B} + \frac{1}{c^2}\mathbf{v} \times \mathbf{B} \tag{VI.8}$$

given the Galilean boost

$$t' \; = \; t \tag{VI.9}$$

$$\mathbf{x}' \; = \; \mathbf{x} - t\mathbf{v} \tag{VI.10}$$

This might seem acceptable but recall back to the exam problem with co-moving point charges. You calculated that their forces can cancel at $v = c$. But that contradicts the rest frame solution where they always attract!

Something is definitely wrong here. It turns out that out old Lorentz force law and the above transformations are only good for speeds much less than the speed of light. It turns out that Maxwell's equations do not respect Galilean relativity as Newton's second law does.

B. Special Relativity

1. *Chapter 37 (Knight)*

Maxwell's equations and Newton's 2nd law (where inertial frames are determined by Galilean relativity) (given the Lorentz force) law cannot both be correct. They are fundamentally inconsistent with each other. Before any of this was well understood, Poincare and Lorentz had already worked out the mathematics of what kind of boosts will work with Maxwell's equations. What they did not have was a proper understanding what this mathematics meant.

[A Lorentz boost in the x-direction.]

$$t' = \gamma\left(t - \frac{v}{c^2}x\right) \tag{VI.11}$$

$$x' = \gamma(x - v\,t) \tag{VI.12}$$

$$y' = y \tag{VI.13}$$

$$z' = z \tag{VI.14}$$

where

$$\gamma = \frac{1}{\sqrt{1-\beta^2}} \tag{VI.15}$$

$$\beta = \frac{v}{c} \tag{VI.16}$$

$$\alpha = \tanh^{-1}(\beta) \tag{VI.17}$$

So the Lorentz boost and Galilean boost look the same for speeds much less than the speed of light.

A little bit cleaner and we have

$$\begin{bmatrix} ct' \\ x' \\ y' \\ z' \end{bmatrix} = \begin{bmatrix} \cosh(\alpha) & -\sinh(\alpha) & 0 & 0 \\ -\sinh(\alpha) & \cosh(\alpha) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} ct \\ x \\ y \\ z \end{bmatrix} \tag{VI.18}$$

A Galilean boost looked like a translation, but a Lorentz boost looks like a hyperbolic rotation with respect to α the **rapidity**.

Keeping the above in mind, Maxwell's equations predict electromagnetic waves which travel at the speed of light, but with respect to what? Sound waves travel at the speed of sound with respect to a medium (say air). The proposed medium for E&M waves was called the **aether**. The aether would have it's own frame relative to which the speed of light would be c.

Utilizing the speed at which the Earth travels around the Sun (roughly 0.01% the speed of light), Michelson and Morely attempted to locate the aether frame relative to our solar system. They failed to find the aether. This is perhaps the most famous null experiment ever. All kinds of crazy ideas were proposed to explain this phenomena of a wave with no apparent reference frame. Einstein's theory was the simplest and eventually won out.

Before we started with two basic assumptions: (1) the laws of nature are the same in all inertial frames and (2) 3-dimensional spatial vector lengths are invariant quantities. **Special Relativity** sides with the Michelson-Morely experiment and drops to invariance of 3-dimensional lengths for the invariance of the speed of light itself. What is the metric which leaves the speed of light invariant? Let Δx denote the distance between two points on a light ray and Δt the time it took the light ray to travel that distance.

$$c = \left| \frac{\Delta x}{\Delta t} \right| \tag{VI.19}$$

$$c^2 = \frac{(\Delta x)^2}{(\Delta t)^2} \tag{VI.20}$$

$$(c\Delta t)^2 = (\Delta x)^2 \tag{VI.21}$$

$$0 = (c\,\Delta t)^2 - (\Delta x)^2 \tag{VI.22}$$

this quantity must be zero (for light) in any reference frame. For non-light paths it could be nonzero. The general metric is then.

$$(\Delta s)^2 = (c\,\Delta t)^2 - (\Delta x)^2 - (\Delta y)^2 - (\Delta z)^2 \quad \Big| \quad \textbf{Lorentzian metric (space-time separation)}$$

If all we agree upon this notion of 4-dimensional "length" then we will all agree upon the speed of light. The minus sign makes it hyperbolic as opposed to Euclidean. [Just as rotations were compatible with the Euclidean metric, it is easy to show that Lorentz boosts are compatible with this metric. Ordinary rotations and (time-independent) translations are still compatible as well.]

The **proper frame** is the co-moving frame for space-time events in which we simply have $(\Delta s)^2 = (c\Delta\tau)^2$ where τ is the **proper time**. So we may also express the metric

$$(c\,\Delta\tau)^2 = (c\,\Delta t)^2 - (\Delta x)^2 - (\Delta y)^2 - (\Delta z)^2 \quad \Big| \quad \textbf{Lorentzian metric (proper time)}$$

But what have we done in giving up the invariance of distances? All of the following can be motivated from the metric or Lorentz boosts.

Simultaneity Is now strictly local.

Time Dilation Consider $\Delta\tau$ in the proper frame where one is at rest as compared to a relatively moving frame. This is easy to derive from the metric.

Length Contraction Given time dilation, it is then easy to derive length contraction from the difference in proper times to travel set distances.

Velocity Addition Two boosts, rapidity adds but velocity does not add so simply.

We also need to reconsider Newton's second law and everything accompanying. Galilean relativistic theories are constructed with 3-dimensional vectors like (x, y, z). Special relativistic theories are constructed with 4-dimensional vectors like $(c\,t, x, y, z)$. We need to figure out all of the relevant 4-dimensional vectors. This will unify older 3-dimensional vectors with 1-dimensional quantities and perhaps modify things with factors of γ.

Let's compare the space-time separation of the proper frame to a moving frame

$$(c\,d\tau)^2 = (c\,dt)^2 - (dx)^2 - (dy)^2 - (dz)^2 \tag{VI.23}$$

$$c^2 = c^2\left(\frac{dt}{d\tau}\right)^2 - \left(\frac{dx}{d\tau}\right)^2 - \left(\frac{dy}{d\tau}\right)^2 - \left(\frac{dz}{d\tau}\right)^2 \tag{VI.24}$$

$$(mc)^2 = (mc)^2\left(\frac{dt}{d\tau}\right)^2 - \left(m\frac{dx}{d\tau}\right)^2 - \left(m\frac{dy}{d\tau}\right)^2 - \left(m\frac{dz}{d\tau}\right)^2 \tag{VI.25}$$

$$(mc)^2 = (mc)^2\left(\frac{dt}{d\tau}\right)^2 - \left(m\frac{d}{d\tau}\mathbf{r}\right)^2 \tag{VI.26}$$

This is a constant and must be invariant in any reference frame. The second term looks like our old $\mathbf{p}^2$ except that we have proper time differentiation. The full equation describes the length^2 (using our metric - a hyperbolic pythagorean theorem) of some 4-dimensional vector (4-momentum). Let us write this 4-momentum in terms of older quantities and γ.

$$\mathbf{p}_4 = \begin{bmatrix} mc\frac{dt}{d\tau} \\ m\frac{d}{d\tau}\mathbf{r} \end{bmatrix} \tag{VI.27}$$

$$= \begin{bmatrix} \gamma mc \\ \gamma \mathbf{p}_3 \end{bmatrix} \tag{VI.28}$$

This is our 4-momentum. It looks like our old 3-momentum in the last 3 spatial dimensions, but what is this new temporal term? Let us Taylor expand it suggestively

$$\gamma mc^2 = mc^2 + \frac{1}{2}mv^2 + \mathcal{O}\left(\frac{v}{c}\right)^4 \tag{VI.29}$$

In the non-relativistic limit it is a constant plus the kinetic energy. This is a constant rest mass energy. Constant energies were never noticed before, only energy gradients. Special relativity has unified energy and momentum (and mass) and we have the 4-vector length relation

$$(mc)^2 = \left(\frac{E}{c}\right)^2 - \mathbf{p}^2 \tag{VI.30}$$

which unifies the previous relations $E = \frac{1}{2}mv^2$ and $E = pc$ for the kinetic energy of massive and massless particles.

Massless particles have a 4-momentum, only no 4-velocity.

$$\mathbf{p}_4 = m\mathbf{v}_4 \tag{VI.31}$$

$$\mathbf{v}_4 = \gamma \begin{bmatrix} c \\ \mathbf{v}_3 \end{bmatrix} \tag{VI.32}$$

Their mass is zero and 4-velocity is infinite, but the product of the two (momentum) is finite.

Newton's law can be naturally reconstructed as

$$\frac{d^2}{d\tau^2}\mathbf{p}_4 = \frac{d^2}{d\tau^2}m\begin{bmatrix} ct \\ \mathbf{r} \end{bmatrix} \tag{VI.33}$$

$$= \begin{bmatrix} ? \\ \gamma\mathbf{F}_3 \end{bmatrix} \tag{VI.34}$$

The temporal part of the 4-force must be power. Special relativity has unified force and power.

Now you might ask, what about E&M fields, how can special relativity help two 3-vectors (the magnetic field is actually on a pseudo-vector) transform into each other. The answer is that what we thought were two 3-vectors was actually the single anti-symmetric matrix

$$\boldsymbol{\mathcal{F}} = \begin{bmatrix} 0 & +\frac{E_x}{c} & +\frac{E_y}{c} & +\frac{E_z}{c} \\ -\frac{E_x}{c} & 0 & +B_z & -B_y \\ -\frac{E_y}{c} & -B_z & 0 & +B_x \\ -\frac{E_z}{c} & +B_y & -B_x & 0 \end{bmatrix} \tag{VI.35}$$

and transforms quite readily via Lorentz boosting.

The Lorentz force law becomes

$$\mathbf{F}_4 = q\mathbf{v}_4 \cdot \boldsymbol{\mathcal{F}} \tag{VI.36}$$

$$\mathbf{v}_4 = \gamma\left(c, \mathbf{v}_3\right) \tag{VI.37}$$

where the 3-dimensional dot and cross products are embedded into the anti-symmetric structure of $\boldsymbol{\mathcal{F}}$.

The continuity equation becomes

$$\boldsymbol{\nabla}_4 \cdot \mathbf{J}_4 = 0 \tag{VI.38}$$

$$\boldsymbol{\nabla}_4 = \left(\frac{\partial}{\partial\, ct}, \boldsymbol{\nabla}_3\right) \tag{VI.39}$$

$$\mathbf{J}_4 = \rho\mathbf{v}_4 \tag{VI.40}$$

unifying density and current.

And Maxwell's equations reduce from 4 to 2.

$$\nabla_4 \cdot \boldsymbol{\mathcal{F}} = \mu_0 \mathbf{J}_4 \tag{VI.41}$$

$$\nabla_4 \cdot \boldsymbol{\mathcal{F}}^\dagger = 0 \tag{VI.42}$$

where $\boldsymbol{\mathcal{F}}^\dagger$ is the dual matrix (swaps electric and magnetic fields). The existence of any magnetic charge would enter in to the second equation. Relativity has not illuminated anything about its nonexistence.

One last object which will be referenced in General Relativity - the stress matrix $\mathbf{T}$ (the generalization of anisotropic pressures; you feed it a direction vector and it gives you a stress force) becomes a part of the stress-energy-momentum tensor. Neglecting c, γ, ... it looks like

$$\mathbf{T}_4 = \begin{bmatrix} E & \mathbf{p}_3^{\mathrm{T}} \\ \mathbf{p}_3 & \mathbf{T}_3 \end{bmatrix} \tag{VI.43}$$

C. General Relativity

There are several issues that remain to be resolved with respect to gravity.

- The theory of Newtonian Gravity must be made (special) relativistic. Strictly speaking, this could be done by considering an analog to the magnetic field, and this will give some improvement to Newtonian gravity.
- Newtonian gravity fails to correctly account for the precise orbit of Mercury (planet closest to the sun). Some thought an extra planet Vulcan to be causing this discrepancy.
- Gravity takes the form of a fictitious force. Can we explain gravity without using forces? What would be the correct inertial frame?

The beginning of our resolution lies in Einstein's **equivalence principle**. There are really two.

1. The inertial mass on the left hand side of Newton's second law is exactly the same as the gravitational mass (charge) on the right hand side. No experiment can measure any discrepancy.

2. Being in an accelerating frame (standing in an accelerating rocket ship) is no different than standing on the earth and feeling its gravitational tug.

Already, even without mathematics we have made physical predictions. Specifically that the path of light must be bent by gravity as light shares our frame. This prediction was first confirmed by measuring the gravitational deflection of starlight around the Sun during a lunar eclipse.

Now on to the theory of General Relativity. The key idea is to consider the existence of local inertial frames free of gravity (local special relativity) and then to patch them together, forming a space-time manifold. It is the coordinate transformations from one frame to the next which will induce the appearance of gravity. The coordinate transformations are not mere abstractions as they are necessary in getting from one location to the next in the manifold. Mathematically speaking, gravity becomes synonymous with curvature in the space-time manifold. And so the simplest theory of gravity would then directly relate curvature to the source of gravity: mass, energy, stuff.

The mathematics of manifold curvature had already been worked out by Riemann, but this was very exotic mathematics at the time and it took a while for Einstein to digest it and discover the correct theory. The simplest theory of gravity (which is relativistic, equates gravitational and inertial mass, ...) is simply a proportionality relation between a specific combination of curvature tensors and the stress-energy tensor mentioned at the end of the previous section. Einstein's theory is the only such theory which reduces to the correct Newtonian limit, though one can consider the existence of terms higher order in the curvature.

I have made Einstein's theory of gravity (the equation) seem a bit simple. It is actually a second order, nonlinear, partial differential equation which requires (for dynamics) a sensible model of matter in the stress energy tensor.

For this class we will only consider the most general, spherically symmetric, vacuum solution.

$$d\tau^2 = \left(1 - \frac{2M}{r}\right) dt^2 - \left(1 - \frac{2M}{r}\right)^{-1} dr^2 - r^2 d\Omega^2 \tag{VI.44}$$

with gravitational units $c = G = 1$, in spherical coordinates... specifically the spherical coordinates of an observer at $r = \infty$ where this metric looks like special relativity. This metric can be applied to the exterior of spherically symmetric planets, stars, etc, but only one of them.

Let's look at some predictions. Let's compare clocks with the far away observer and a local observer at fixed radius.

$$d\tau^2 = \left(1 - \frac{2M}{r}\right) dt^2 \tag{VI.45}$$

$$dt = \frac{1}{\sqrt{1 - \frac{2M}{r}}} d\tau \tag{VI.46}$$

This is like a gravitational time dilation formula.

Let's compare radial distances as measured by the far away observer and a local observer.

$$d\sigma^2 = \left(1 - \frac{2M}{r}\right)^{-1} dr^2 \tag{VI.47}$$

$$dr = \sqrt{1 - \frac{2M}{r}} d\sigma \tag{VI.48}$$

This is like a gravitational length contraction formula.

But what if we want to compare two things at finite radii? Then we can do the following

$$\frac{d\tau_1}{d\tau_2} = \frac{d\tau_1}{dt}\frac{dt}{d\tau_2} \tag{VI.49}$$

$$= \sqrt{\frac{1 - \frac{2M}{r_1}}{1 - \frac{2M}{r_2}}} \tag{VI.50}$$

Now what if we are interested in reference frames in circular orbit?

$$d\tau^2 = \left(1 - \frac{2M}{r}\right) dt^2 - r^2 d\Omega^2 \tag{VI.51}$$

$$\left(\frac{d\tau}{dt}\right)^2 = \left(1 - \frac{2M}{r}\right) - r^2 \left(\frac{d\Omega}{dt}\right)^2 \tag{VI.52}$$

$$\frac{d\tau}{dt} = \sqrt{1 - \frac{2M}{r} - v^2} \tag{VI.53}$$

This formula combines special relativistic and gravitational effects and is used in accurate GPS.

VII. QUANTUM MECHANICS

A. Old Quantum Mechanics

1. *Chapter 39 (Knight)*

From your book and labs I will assume you know

- The energy of a photon $E = h\,f$.
- The de Broglie wavelength $\lambda\,p = h$.

Here in the notes I'm going to cover something not in your book but very accessible. It was known that **Plank's constant** is the constant which appears in any quantized quantity, so given it's units of action it was natural to think of things to quantize. This is **Bohr-Sommerfeld quantization**: one takes some (classical) periodic solution in phase space $[x(t), p(t)]$ (position and momentum) and then perform the closed integral

$$n\,h = \oint p \cdot dx \tag{VII.1}$$

and if there are multiple degrees of freedom, there will be multiple quantum numbers n.

The integral is typically converted to a time integral via

$$n\,h = \oint p \cdot \frac{dx}{dt} dt \tag{VII.2}$$

We will consider a few simple problems.

Particle in a Box:

For a particle in a box, reflecting off the walls (L apart) we calculate

$$n\,h = 2pL \tag{VII.3}$$

$$p = \frac{nh}{2L} \tag{VII.4}$$

and then we can compute the energy levels via $E = \frac{p^2}{2m}$. It turns out that this is pretty accurate.

Harmonic Oscillator:

Let's take a solution

$$x(t) = A\,\sin(\omega t) \tag{VII.5}$$

$$x'(t) = \omega A\,\cos(\omega t) \tag{VII.6}$$

$$p(t) = m\omega A\,\cos(\omega t) \tag{VII.7}$$

and so the quantization condition is

$$n\,h \;=\; \oint m\omega^2 A^2\;\cos^2(\omega t)dt \tag{VII.8}$$

$$=\; \frac{1}{2}m\omega^2 A^2\,\frac{2\pi}{\omega} \tag{VII.9}$$

but the energy at any point in time is

$$E \;=\; \frac{1}{2m}p(t)^2 + \frac{m\omega^2}{2}x(t)^2 \tag{VII.10}$$

$$=\; \frac{m\omega^2}{2}A^2\cos^2(\omega t) + \frac{m\omega^2}{2}A^2\sin^2(\omega t) \tag{VII.11}$$

$$=\; \frac{1}{2}m\omega^2 A^2 \tag{VII.12}$$

and so we have the energy quantization

$$E \;=\; n\frac{h}{2\pi}\omega \tag{VII.13}$$

$$E \;=\; n\hbar\omega \tag{VII.14}$$

It turns out that this relation is good for relative energies (differences).

Hydrogen-like Atom:

For our final example we will consider the circular orbit of a Hydrogen-like atom (one light charge in orbit and one massive, opposite charge at the focus). For the circular orbit we have

$$\frac{mv^2}{r} \;=\; k_e\frac{qQ}{r^2} \tag{VII.15}$$

$$\frac{p^2}{m} \;=\; k_e\frac{qQ}{r} \tag{VII.16}$$

The quantization condition gives us

$$n\,h \;=\; p\,2\pi r \tag{VII.17}$$

$$n^2\hbar^2 \;=\; p^2 r^2 \tag{VII.18}$$

$$\frac{p^2}{2m} \;=\; \frac{n^2\hbar^2}{2mr^2} \tag{VII.19}$$

which combined with the previous relation gives us

$$k_e\frac{qQ}{r} \;=\; \frac{n^2\hbar^2}{mr^2} \tag{VII.20}$$

$$\frac{mk_e^2q^2Q^2}{n^2\hbar^2} \;=\; k_e\frac{qQ}{r} \tag{VII.21}$$

$$\tag{VII.22}$$

Therefore the energy must be

$$E = \frac{p^2}{2m} - k_e \frac{qQ}{r} \tag{VII.23}$$

$$E = -\frac{m k_e^2 q^2 Q^2}{2\hbar^2} \frac{1}{n^2} \tag{VII.24}$$

and this is precisely the Hydrogen spectrum.

What is wrong with this old quantum mechanics?

1. This kind of quantization is fairly ad-hoc. There is no reason given why only some special family of classical solutions may exist other than hand-wavy arguments referencing "adiabatic invariants".

2. This kind of quantization is incomplete. There are many other quantum phenomena (such as superposition) which have no place in this theory.

3. Perhaps the most damaging is that it can't really explain any atom more complicated than Hydrogen. Other atoms, like neutral Helium, are classically unstable. But phenomenologically we do observe spectral lines for them, they are just more complicated.

B. Hamiltonian Mechanics

There are three valid methods of working out classical mechanics problems

Newton The second-order dynamics of each trajectory x are generated from forces.

Hamliton The first-order dynamics of phase space (x, p) are generated via the Hamiltonian.

Langrange Proper solutions are those which minimize the action.

Unfortunately quantum mechanics can only be related to the second two formalisms, which you have probably never seen. These formalisms are not completely equivalent. With Newton's second law one can introduce phenomenological forces such as damping, $\mathbf{F} = -b\mathbf{v}$. With the Hamiltonian and Lagrangian formalism one must work from conservative forces (potentials). Non-conservative forces such as damping must be modeled with a dissipative environment.

Here I will give a short synopsis of what Hamiltonian mechanics looks like. For our purposes the Hamiltonian is equal to the energy (not generally true), but specifically represented with phase space coordinates.

$$H(x,p) = \frac{p^2}{2m} + U(x) \tag{VII.25}$$

The first order dynamics of trajectories in phase space can then be represented

$$\frac{d}{dt}x = +\frac{\partial H}{\partial p} \tag{VII.26}$$

$$\frac{d}{dt}p = -\frac{\partial H}{\partial x} \tag{VII.27}$$

and the dynamics of any function of phase space coordinates can be represented as being generated by the Hamiltonian

$$\frac{d}{dt}A(t,x,p) = \frac{\partial A}{\partial t} + \{A,H\} \tag{VII.28}$$

$$\{A,B\} \equiv \frac{\partial A}{\partial x}\frac{\partial B}{\partial p} - \frac{\partial B}{\partial x}\frac{\partial A}{\partial p} \tag{VII.29}$$

where $\{A,B\}$ here is our classical commutator. In the Hamiltonian formalism, classical mechanics has a very deep mathematical structure. In going from classical mechanics to quantum mechanics many of these landmarks will stay the same, but the mathematics which connects them will be transplanted.

It is because of this very beautiful mathematical structure that we are allowed to luxury of **statistical mechanics**. Say we let our particle have some probability of being at a particular location in phase space $\rho(x,p)$. According the Hamiltonian formalism, the equation of motion for this phase space distribution function is

$$\frac{d}{dt}\rho = \{\rho,H\} \tag{VII.30}$$

If ρ begins normalized, $1 = \int dx\,dp\,\rho(x,p)$, then it will remain normalized as it evolves in time. Hamiltonian evolution preserves probabilities. Moreover, canonical coordinate transformations also preserve probabilities.

One of the more important distributions is the **stationary distribution**. This distribution does not change in time and can be thought of as a collection of orbits in phase space. For the harmonic oscillator, the stationary states will be the orbit circles in phase space.

Quantum mechanics will look most similar to this Hamiltonian formalism when considering distributions instead of individual trajectories. But in comparing these two things (what is truly comparable), quantum mechanics is actually simpler!

C. Quantum Mechanics

1. *Chapter 40, 41 (Knight)*

Strictly speaking there is no derivation of the Schrödinger equation. We will motivate it from the phenomenological de Broglie wavelength and an abstract notion of Hamiltonian dynamics. De Broglie says that a particle will exhibit wave interference on our detection screen with wavelength given by

$$\lambda\, p \;=\; h \tag{VII.31}$$

So let us consider what the wave function of a free particle might look like. If we simply use complex numbers instead of sinusoidal functions we could say the wave function is

$$\psi(x,t) \;=\; A\, e^{\imath kx - \imath\omega t} \tag{VII.32}$$

and the "intensity" (probability) for the screen detector would be given by the modulus $|\psi|^2$ after considering all interference effects.

From the de Broglie wavelength, we should have that $k = \frac{2\pi}{\lambda} = \frac{p}{\hbar}$ and therefore we have

$$\psi(x,t) \;=\; A\, e^{\imath \frac{px}{\hbar} - \imath\omega t} \tag{VII.33}$$

We also have another relation similar to the de Broglie wavelength, the energy in a quanta (for photons at least)

$$E \;=\; h\, f \tag{VII.34}$$

and applying this relation we get

$$\psi(x,t) \;=\; A\, e^{\imath \frac{px}{\hbar} - \imath \frac{Et}{\hbar}} \tag{VII.35}$$

Now we must consider that the momentum should relate to the energy via

$$E \;=\; \frac{p^2}{2m} \tag{VII.36}$$

at least for the free particle. What equation of motion would force this constraint? Note that differential operations are needed to extract this information

$$+\,\imath\hbar \frac{\partial}{\partial t}\psi(x,t) \;=\; E\,\psi(x,t) \tag{VII.37}$$

$$-\imath\hbar \frac{\partial}{\partial x}\psi(x,t) \;=\; p\,\psi(x,t) \tag{VII.38}$$

and that the correct equation of motion would be

$$+\imath\hbar\frac{\partial}{\partial t}\psi(x,t) \;=\; -\frac{1}{2m}\hbar^2\frac{\partial^2}{\partial x^2}\psi(x,t) \tag{VII.39}$$

Let us identify our differential operators with what information they extract from the wavefunction

$$\hat{E} \;=\; +\imath\hbar\frac{\partial}{\partial t} \tag{VII.40}$$

$$\hat{p} \;=\; -\imath\hbar\frac{\partial}{\partial x} \tag{VII.41}$$

and so now our equation looks like

$$\hat{E}\,\psi(x,t) \;=\; \frac{1}{2m}\hat{p}^2\,\psi(x,t) \tag{VII.42}$$

and to generalize to a particle which is not free we should have the Hamiltonian dynamics

$$\hat{E}\,\psi \;=\; \hat{H}\,\psi \tag{VII.43}$$

$$\hat{H} \;\equiv\; \frac{1}{2m}\hat{p}^2 + U(\hat{x}) \tag{VII.44}$$

This is the **Schrödinger equation**. For $\psi(x,t)$ as we have used, we are said to be in the position representation and the position operator $\hat{x}$ simply acts as the position value x.

This equation correctly produces our energy spectra and interference phenomena. This is a matter wave theory where the modulus $|\psi|^2$ yields detector probability. The discrete energy spectra comes about because this is a partial differential equation with boundary conditions. [Mathematically this is analogous to the discrete harmonics found in a flute or guitar string.]

The wavefunction would be best related to the marginal distribution of the classical phase space distribution

$$|\psi(x)|^2 \;\approx\; \int dp\,\rho(x,p) \tag{VII.45}$$

But there are severe differences behind the scenes. There is no interference in the classical distribution. And there is in generally no quantum distribution in phase space (there is a pseudo-distribution however). Quantum mechanically we can only formulate a distribution in position space or momentum space, but not both. This is because they are conjugate variables, i.e. $\hat{p} = -\imath\hbar\frac{\partial}{\partial x}$. A similar condition exists between time and frequency (here also energy).

This is true of any wave. Sound waves also have this relation between time and frequency. You cannot perfectly describe any sound wave by both time and frequency. You might think you can as that is exactly what sheet music describes, but try as you might, you cannot accurately play a very low frequency note for a very short amount of time. In fact, you cannot accurately play any note for an arbitrarily short amount of time. One has the mathematically "uncertainty relation" $\Delta\omega\,\Delta t \geq \frac{1}{2}$ which will extend to any wavefunction. For quantum mechanics we have the Heisenberg uncertainty relation

$$\Delta x\,\Delta p \;\geq\; \frac{\hbar}{2} \tag{VII.46}$$

though you should hesitate to think of this as an uncertainty of something which is exists in some kind of certain way. [Also note that your book has really bad typo's in this chapter. It's an $\hbar$ here, not h.]

Just as sound files can be represented in the time domain or frequency domain (what an MP3 would use), the wave function not only has a position $\psi(x)$ representation but a momentum representation, $\mathcal{F}\{\psi\}(p)$, related by a Fourier transformation. As we still have $\psi \approx e^{\imath\frac{px}{\hbar}}$ that makes $\hat{x} = -\imath\hbar\frac{\partial}{\partial p}$. This representation is not as useful because the Hamiltonian looks like

$$\hat{H}\,\mathcal{F}\{\psi\}(p) \;=\; \left\{\frac{1}{2m}p^2 + U\left(-\imath\hbar\frac{\partial}{\partial p}\right)\right\}\mathcal{F}\{\psi\}(p) \tag{VII.47}$$

and this is not easier to solve, except for free particles.

Table of Classical and Quantum Analogs

Quantum	**Classical**
Probability of being at location x.	
$\lvert\psi(x)\rvert^2$	$\int dp\, \rho(x,p)$
modulus of position wavefunction	marginal position distribution
Probability of having momentum p.	
$\lvert\mathcal{F}\{\psi\}(p)\rvert^2$	$\int dx\, \rho(x,p)$
modulus of momentum wavefunction	marginal momentum distribution
(Fourier transform of position wavefunction)	
Expectation value $\langle A \rangle$	
$\int dx\, \psi^* \, \hat{A}\, \psi$	$\int\int dp\, dx\, A(x,p)\, \rho(x,p)$
$\int dp\, \mathcal{F}\{\psi\}^* \, \hat{A}\, \mathcal{F}\{\psi\}$	$\int\int dx\, dp\, A(x,p)\, \rho(x,p)$
Stationary state with energy E	
$\hat{H}\, \psi_E = E\, \psi_E$	$\rho(x,p) \propto \begin{cases} 1 \ : \ H(x,p) = E \\ 0 \ : \ H(x,p) \neq E \end{cases}$
wavefunction solution ψ_E	the entire orbit for that energy
(time-independent Schrödinger Equation)	(not normalized here)
Commutators	
$[\hat{A}, \hat{B}] = \hat{A}\, \hat{B} - \hat{B}\, \hat{A}$	$\{A, B\} = \frac{\partial A}{\partial x}\frac{\partial B}{\partial p} - \frac{\partial B}{\partial x}\frac{\partial A}{\partial p}$
$[\hat{x}, \hat{p}] = \imath\hbar$	$\{x, p\} = 1$
Hamiltonian Dynamics	
$\frac{d}{dt}\hat{\rho} = \imath\hbar[\hat{\rho}, \hat{H}]$	$\frac{d}{dt}\rho = \{\rho, H\}$

D. The Schrödinger Equation

1. *Chapter 41 (Knight)*

Now that we have the Schrödinger equation we can proceed to solving it. Physically we are most interested in the the energy spectrum and the corresponding energy states. The energy states are normalized solutions to the time-independent Schrödinger equation

$$\hat{H}\, \psi_E(x) \ = \ E\, \psi_E(x) \tag{VII.48}$$

and they are also required in determining any dynamical solutions to the time-dependent Schrödinger equation

$$\hat{H}\,\psi(x,t) \;=\; \hat{E}\,\psi(x,t) \tag{VII.49}$$

This is because any arbitrary initial state can be expanded in terms of the energy wavefunctions (Sturm-Liouville theory; this is kind of like a Fourier expansion).

$$\psi(x,0) \;=\; \sum_E c_E\,\psi_E(x) \tag{VII.50}$$

$$c_E \;=\; \int dx\,\psi_E(x)^*\,\psi(x,0) \tag{VII.51}$$

Each individual energy wavefunction then evolves in a fairly trivial manner in accord with the time-dependent Schrödinger equation.

$$\psi(x,0) \;=\; \sum_E c_E\,\psi_E(x)\,e^{-\imath\frac{Et}{\hbar}} \tag{VII.52}$$

Particle in a Box

This is perhaps the simplest problem to work out. Let us recall the classical solution which we previously quantized in phase space.

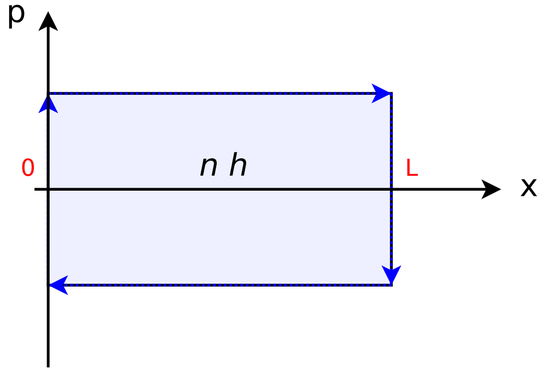

FIG. 5: Particle in a box: stationary orbit with phase space area shaded

Let us consider the classical marginal distributions. The position distribution is uniform within the box with no probability of being outside of the box. The momentum distribution would be a 50-50 chance of $\pm p$ or equivalently $\pm\sqrt{2mE}$. Old quantum mechanics gave us $2pL = nh$ and therefore $p_n = \frac{h}{2L}n$ and $E_n = \frac{h^2}{8mL^2}n^2$.

Let us solve the corresponding Schrödinger equation and see what we get. First we need the potential for our Hamiltonian operator. For the rigid box to trap (classical) particles perfectly, the potential must be an infinite square well

$$U(x) \;=\; \begin{cases} 0 \;:\; 0 < x < L \\ \infty \;:\; \text{else} \end{cases} \tag{VII.53}$$

Our Schrödinger equation then looks like

$$\begin{cases} -\frac{\hbar}{2m}\frac{d^2}{dx^2}\psi_E(x) \;=\; E\,\psi_E(x) \;:\; 0 < x < L \\ -\frac{\hbar}{2m}\frac{d^2}{dx^2}\psi_E(x) \;=\; \infty\,\psi_E(x) \;:\; \text{else} \end{cases} \tag{VII.54}$$

Obviously even the quantum particle must vanish outside of the box. Inside the box we have the equation

$$\psi_E''(x) + \frac{2mE}{\hbar^2}\,\psi_E(x) \;=\; 0 \tag{VII.55}$$

and we've seen this equation a few times now. The solutions are of the two equivalent forms

$$\psi_E(x) \;=\; A\cos(kx) + B\sin(kx) \tag{VII.56}$$

$$\psi_E(x) \;=\; A_+ e^{+\imath kx} + A_- e^{-\imath kx} \tag{VII.57}$$

with $k^2 = \frac{2mE}{\hbar^2}$. Note that the second set of solutions is a linear combination of one wavefunction traveling to the right with momentum $p = +\hbar k$ and another wavefunction traveling to the left with momentum $p = -\hbar k$.

Next we enforce the boundary conditions and normalization. What are our boundary conditions? We must match the inner solution with the outer (vanishing) solution as continuous as possible. This means that the inner solution must vanish at the boundary of the box.

$$\psi(0) \;=\; 0 \tag{VII.58}$$

$$\psi(L) \;=\; 0 \tag{VII.59}$$

There is only one nontrivial solution which can do this

$$\psi_E(x) = B\sin(kx) \tag{VII.60}$$

$$\psi_E(x) = \frac{B}{2\imath}\left(e^{+\imath kx} - e^{-\imath kx}\right) \tag{VII.61}$$

$$\tag{VII.62}$$

where we can only have $k = \frac{\pi}{L}n$ with n any integer. Therefore the energy spectrum is $E = \frac{\pi^2\hbar^2}{2mL^2}n^2 = \frac{h^2}{8mL^2}n^2$ which is the same result from old quantum mechanics. The momentum distribution is also the same; we have $p_n = \pm\frac{\pi\hbar}{L}n = \pm\frac{h}{2L}n$.

There are two differences however. The first difference is that the position distribution is not uniform, it is instead like $\sin^2(kx)$. The second is that there is no zero energy solution. If we apply $n = 0$ then we cannot normalize the wave function; we get nothing.

Speaking of normalization, we must fix B via

$$1 = \int_0^L |B_n|^2 \sin^2\left(n\pi\frac{x}{L}\right) \tag{VII.63}$$

$$B_n = \sqrt{\frac{2}{L}} \tag{VII.64}$$

and our energy solutions are

$$\psi_E(x) = \sqrt{\frac{2}{L}}\sin\left(n\pi\frac{x}{L}\right) \tag{VII.65}$$

They are standing waves with half of their momentum traveling to the right and half to the left. For very large quantum numbers $|\psi_n(x)|^2$ can effectively be treated as a uniform distribution in position related expectation values as the $\sin^2(kx)$ will oscillate too rapidly for any integral to resolve such detail.