Homework 13:

Remember: In addition to this problem, you also have a "Mastering Physics" assignment Due May 8 Due at 10:00 a.m. , Friday, May 8 Write up of the solution to this problem in a coherent fashion.

A one dimensional quantum mechanical system has a wavefunction of the following form: $\psi(x) = \dfrac{A}{1+\dfrac{x^2}{L^2}}$ where A is a normalization constant and L is a constant with dimensions of length.

a. First determine the value of A by ensuring that the wavefunction is normalized.

b. Evaluate

i. $\langle x \rangle = \int_{-\infty}^{\infty} dx\, \psi^*(x)\, x\, \psi(x)$

ii. $\langle x^2 \rangle = \int_{-\infty}^{\infty} dx\, \psi^*(x)\, x^2\, \psi(x)$

iii. $\langle p \rangle = \int_{-\infty}^{\infty} dx\, \psi^*(x) \left(-i\hbar \dfrac{\partial}{\partial x} \psi(x) \right)$

iv. $\langle p^2 \rangle = \int_{-\infty}^{\infty} dx\, \psi^*(x) \left(-\hbar^2 \dfrac{\partial^2}{\partial x^2} \psi(x) \right)$

v. $\Delta x = \sqrt{\langle x^2 \rangle - \langle x \rangle^2}$

vi. $\Delta p = \sqrt{\langle p^2 \rangle - \langle p \rangle^2}$

c. Verify the Heisenberg uncertainty relation for this wave function, namely that $\Delta x\, \Delta p \geq \dfrac{\hbar}{2}$.

Some useful integrals: $\displaystyle\int_{-\infty}^{\infty} dx\, \frac{1}{\left(1+\left(\frac{x}{L}\right)^2\right)^2} = \frac{\mathrm{L}\pi}{2}$, $\displaystyle\int_{-\infty}^{\infty} dx\, \frac{x}{\left(1+\left(\frac{x}{L}\right)^2\right)^2} = 0$,

$\displaystyle\int_{-\infty}^{\infty} dx\, \frac{x^2}{\left(1+\left(\frac{x}{L}\right)^2\right)^2} = \frac{\mathrm{L}^3\pi}{2}$, $\displaystyle\int_{-\infty}^{\infty} dx\, \frac{x}{\left(1+\left(\frac{x}{L}\right)^2\right)^3} = 0$, $\displaystyle\int_{-\infty}^{\infty} dx\, \frac{1}{\left(1+\left(\frac{x}{L}\right)^2\right)^3} = \frac{\mathrm{L}3\pi}{8}$,

$\displaystyle\int_{-\infty}^{\infty} dx\, \frac{x^2}{\left(1+\left(\frac{x}{L}\right)^2\right)^4} = \frac{\mathrm{L}^3\pi}{16}$