MIT simulation of EM waves

G3-01: SHIVE WAVE MACHINE - TRAVELING WAVES

K8-05: ELECTROMAGNETIC PLANE WAVE MODEL

M7-04: MALUS' LAW

K8-42: RADIOWAVES - ENERGY AND DIPOLE PATTERN
Homework set #3
• Due Tuesday by 5PM
• No late homework accepted

Quiz #3
• Sections 34.8-34.10, 35.0-35.5
Last time
E or B? Galilean transformation

Only consider constant velocity between reference frames!

Sharon only observes an electric field from charge

Bill observes an electric field from charged particle
    AND magnetic field produced by the moving charge

Both observe no net force on particle
E or B? Galilean transformation

**Galilean transformation of velocity,**

\[ \vec{v}' = \vec{v} - \vec{V} \quad \text{or} \quad \vec{v} - \vec{v}' + \vec{V} \]

\[ \frac{d\vec{v}'}{dt} = \frac{d\vec{v}}{dt} - \frac{d\vec{V}}{dt} \]

\( \vec{V} \) is a *constant* velocity, so \( d\vec{V}/dt = 0. \)

\[ \vec{a}' = \vec{a} \]

\[ \vec{F}' = \vec{F} \]

Both observers agree on the net force on the particle!
E or B? Galilean transformation

Consider a TEST charge (to measure forces).

Bill (frame S) sets up B-field, observes charge moving at velocity → Force up:

$$\vec{F}_B = q \vec{v} \times \vec{B} = qvB \text{ up}$$

Sharon (frame S’) is moving along with charge so v=0

$$\vec{F}_B = q \vec{v} \times \vec{B} = 0$$

There MUST be a force observed by Sharon since Bill observes one.

There must be an E-field in Sharon’s frame that push’s the charge!
E or B? Galilean transformation

Bill (frame S) sets up B-field, observes charge moving at velocity → Force up:

\[
\vec{F}_B = q \vec{v} \times \vec{B} = q v B \text{ up}
\]

Sharon (frame S') is moving along with charge so \( v = 0 \)

\[
\vec{F}_B' = 0
\]

Must have:

\[
\vec{F}' = \vec{F}
\]

Lorentz Force:

\[
q \vec{E}' + q \frac{\vec{V}}{c} \times \vec{B}' = q \vec{E} + q \frac{\vec{V}}{c} \times \vec{B}
\]

\[
\Rightarrow \vec{E}' = \vec{v} \times \vec{B}
\]
E or B? Two Aspects of same phenomenon

Bill (frame S) Force up:
\[ \vec{F} = q \vec{E} + q \vec{v} \times \vec{B} \]

Charge \( q \) moves through a magnetic field \( \vec{B} \) established by Bill.

Sharon (frame \( S' \)) is moving along with charge so \( v=0 \):

\[ \vec{F}' = q \vec{E}' \]

\[ \vec{F}' = \vec{F} \Rightarrow \vec{E}' = \vec{E} + \vec{v} \times \vec{B} \]
E or B? Two Aspects of same phenomenon

**FIGURE 35.6** A charge in frame S experiences electric and magnetic forces. The charge experiences the same force in frame S’, but it is due only to an electric field.

(a) The electric and magnetic fields in frame S

\[ \vec{F} = q(\vec{E} + \vec{v} \times \vec{B}) \]

(b) The electric field in frame S’, where the charged particle is at rest

\[ \vec{F'} = q\vec{E'} \]

\[ \vec{v'} = 0 \]

\[ \vec{E'} = \vec{E} + \vec{V} \times \vec{B} \]

E field in frame S’ from E and B fields in frame S

How do the B-fields transform from one frame to another?
B-field transformation: Biot Savart Law

\[ E = \frac{1}{4\pi \varepsilon_0} \frac{q}{r^2} \hat{r}, \quad B = 0 \] at \( P \)

In frame \( S \), the static charge creates an electric field but no magnetic field.

\[ \vec{E}' = \vec{E} \quad \text{at} \quad \vec{P} \]

\( \vec{B}' \) is given by the Biot-Savart Law:

\[ \vec{B}' = \frac{\mu_0 q}{4\pi} \frac{\vec{r} \times \hat{r}}{r^2} \quad \text{at} \quad \vec{P} \]

\[ \Rightarrow \vec{B}' = -\mu_0 \varepsilon_0 \vec{\nabla} \times \left( \frac{1}{4\pi \varepsilon_0} \frac{q}{r^2} \hat{r} \right) \]

\[ \Rightarrow \vec{B}' = -\left( \mu_0 \varepsilon_0 \right) \vec{\nabla} \times \vec{E} \]

\[ \frac{1}{\sqrt{\mu_0 \varepsilon_0}} = c = 3 \times 10^8 \text{ m/s} \]

\[ \Rightarrow \text{Biot-Savart Law is derivable from equation 3} \]

\[ \text{E-field of a pt charge transformed into different inertial reference frames yields Biot-Savart Law} \]
E or B? Two Aspects of same phenomenon

The Galilean field transformation equations are

\[
\begin{align*}
\vec{E}' &= \vec{E} + \vec{V} \times \vec{B} \\
\vec{B}' &= \vec{B} - \frac{1}{c^2} \vec{V} \times \vec{E}
\end{align*}
\]

or

\[
\begin{align*}
\vec{E} &= \vec{E}' - \vec{V} \times \vec{B}' \\
\vec{B} &= \vec{B}' + \frac{1}{c^2} \vec{V} \times \vec{E}'
\end{align*}
\]

where \( \vec{V} \) is the velocity of frame S’ relative to frame S and where the fields are measured at the same point in space by experimenters at rest in each reference frame.

NOTE: These equations are only valid if \( \vec{V} \ll c \).
Maxwell’s Equations

\[ \oint_S \vec{E} \cdot d\vec{A} = \frac{Q_{\text{in}}}{\varepsilon_0} \]  
(Gauss's Law)

\[ \oint_c \vec{E} \cdot d\vec{s} = -\frac{d\Phi_B}{dt} \]  
(Faraday's Law)

\[ \oint_S \vec{B} \cdot d\vec{A} = 0 \]  
(Magnetic Gauss's Law)

\[ \oint_c \vec{B} \cdot d\vec{s} = \mu_0 I_{\text{enc}} + \mu_0 \varepsilon_0 \frac{d\Phi_E}{dt} \]  
(Ampere-Maxwell Law)

\[ \vec{F} = q(\vec{E} + \vec{v} \times \vec{B}) \]  
(Lorentz force Law)
Gauss’s Law: \[ \iiint_{S} \vec{E} \cdot d\vec{A} = \frac{q_{\text{in}}}{\varepsilon_{0}} \]

- **Spherical Symmetry**
- **Cylindrical Symmetry**
- **Planar Symmetry**

\[ \oint \vec{E} \cdot d\vec{A} = \Phi_{E} \propto \text{Net \# of } \vec{E} \text{-field lines piercing surface} \]

\[ = \frac{Q_{\text{enc}}}{\varepsilon_{0}} \]
**Magnetic Gauss’s Law**

**Gauss’s Law:**
\[ \oint E \cdot d\mathbf{a} = \Phi_E = \text{Net # of } E\text{-field lines piercing Closed Surface} = 0 \]

**Magnetic Gauss’s Law:**
\[ \oint B \cdot d\mathbf{a} = \Phi_B = \text{Net # of } B\text{-field lines piercing Closed Surface} = 0 \]

Since there are NO magnetic monopoles (only dipoles and conglomerates of dipoles),

**Net number of field lines piercing any closed surface is zero**
Ampere’s Law: \[ \oint \vec{B} \cdot d\vec{s} = \mu_0 I_{enc} \]

- **Long Circular Symmetry**
- **(Infinite) Current Sheet**
- **Solenoid = 2 Current Sheets**
- **Torus/Coax**
Faraday’s Law of Induction

\[ \mathcal{E} = \oint \vec{E} \cdot d\vec{s} = -N \frac{d\Phi_B}{dt} \]

Lenz’s Law:
Induced EMF is in direction that **opposes** the change in flux that caused it
Modification to Ampere’s Law

\[ \oint \mathbf{E} \cdot d\mathbf{a} = \frac{Q_{encl}}{\varepsilon_0} \Rightarrow \mathbf{E} A' = \frac{1}{\varepsilon_0} \frac{Q}{A} A' \]

\[ \Rightarrow \mathbf{E} = \frac{Q}{\varepsilon_0 A} \]

**Displacement Current**

\[ E = \frac{Q}{\varepsilon_0 A} \Rightarrow Q = \varepsilon_0 E A = \varepsilon_0 \Phi_E \]

\[ \frac{dQ}{dt} = \varepsilon_0 \frac{d\Phi_E}{dt} = I_d \]

\[ \oint \mathbf{B} \cdot d\mathbf{s} = \mu_0 (I_{encl} + I_d) \]

\[ = \mu_0 I_{encl} + \mu_0 \varepsilon_0 \frac{d\Phi_E}{dt} \]

Capacitors, EM Waves
Changing B-field induces E-field, Lenz’s law gives direction

$$\oint \vec{E} \cdot d\vec{s} = -\frac{d\Phi_B}{dt}$$

Changing E-field induces B-field, Opposite of Lenz’s law gives B-field direction

$$\oint \vec{B} \cdot d\vec{s} = M_0 \left[ I_{enc} + \varepsilon_0 \frac{d\Phi_E}{dt} \right]$$

A opposite Sign ⇒ opp. “Lenz’s Law”
Maxwell’s Equations

\[ \oiint_{S} \mathbf{E} \cdot d\mathbf{A} = \frac{Q_{\text{in}}}{\varepsilon_{0}} \]  
\text{(Gauss's Law)}

\[ \oint_{c} \mathbf{E} \cdot d\mathbf{s} = -\frac{d\Phi_{B}}{dt} \]  
\text{(Faraday's Law)}

\[ \oiint_{S} \mathbf{B} \cdot d\mathbf{A} = 0 \]  
\text{(Magnetic Gauss's Law)}

\[ \oint_{c} \mathbf{B} \cdot d\mathbf{s} = \mu_{0}I_{\text{enc}} + \mu_{0}\varepsilon_{0} \frac{d\Phi_{E}}{dt} \]  
\text{(Ampere-Maxwell Law)}

\[ \mathbf{F} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B}) \]  
\text{(Lorentz force Law)}
Which Leads To…
EM Waves
Electromagnetic Radiation: Plane Waves

Quickly Review of Traveling Waves
Traveling Sine Wave

Now consider \( f(x) = y = y_0 \sin(kx) \):

Amplitude \((y_0)\)

Wavelength \((\lambda) = \frac{2\pi}{\text{wavenumber} \,(k)}\)

What is \( g(x,t) = f(x+vt) \)? Travels to left at velocity \( \nu \)

\[ y = y_0 \sin(k(x+vt)) = y_0 \sin(kx+kvt) \]
Traveling Sine Wave

\[ y = y_0 \sin(kx + kv t) \]

At \( x = 0 \), just a function of time: \( y = y_0 \sin(kvt) \equiv y_0 \sin(\omega t) \)

Period \((T) = \frac{1}{\text{frequency } (f)} = \frac{2\pi}{\text{angular frequency } (\omega)}\)
Traveling Sine Wave

- Wavelength: $\lambda$
- Frequency: $f$
- Wave Number: $k = \frac{2\pi}{\lambda}$
- Angular Frequency: $\omega = 2\pi f$
- Period: $T = \frac{1}{f} = \frac{2\pi}{\omega}$
- Speed of Propagation: $v = \frac{\omega}{k} = \lambda f$
- Direction of Propagation: $+x$

$$y = y_0 \sin(kx - \omega t)$$
Electromagnetic Radiation: Plane Waves

Traveling E & B Waves

- Wavelength: $\lambda$
- Frequency: $f$
- Wave Number: $k = \frac{2\pi}{\lambda}$
- Angular Frequency: $\omega = 2\pi f$
- Period: $T = \frac{1}{f} = \frac{2\pi}{\omega}$
- Speed of Propagation: $v = \frac{\omega}{k} = \lambda f$
- Direction of Propagation: $+x$

Move along with wave $\Rightarrow kx - \omega t = \text{constant}$,

$\Rightarrow x = \frac{\omega}{k}t + \text{constant}$

$\Rightarrow \frac{dx}{dt} = \frac{\omega}{k}$
Properties of EM Waves

Travel (through vacuum) with speed of light
\[ v = c = \frac{1}{\sqrt{\mu_0 \varepsilon_0}} = 3 \times 10^8 \text{ m/s} \]

At every point in the wave and any instant of time, \( \mathbf{E} \) and \( \mathbf{B} \) are in phase with one another, with
\[ \frac{E}{B} = \frac{E_0}{B_0} = c \]

\( \mathbf{E} \) and \( \mathbf{B} \) fields perpendicular to one another, and to the direction of propagation (they are transverse):
Direction of propagation = Direction of \( \mathbf{E} \times \mathbf{B} \)
Electromagnetic Radiation: Plane Waves

How Do Maxwell’s Equations Lead to EM Waves? Derive Wave Equation
Wave Equation

Start with Ampere-Maxwell Eq: \( \oint \mathbf{B} \cdot d\mathbf{s} = \mu_0 \varepsilon_0 \frac{d}{dt} \int \mathbf{E} \cdot d\mathbf{A} \)
Wave Equation

Start with Ampere-Maxwell Eq:
\[ \oint_c \mathbf{B} \cdot d\mathbf{s} = \mu_0 \varepsilon_0 \frac{d}{dt} \int \mathbf{E} \cdot d\mathbf{A} \]

Apply it to red rectangle:
\[ \oint_c \mathbf{B} \cdot d\mathbf{s} = B_z(x, t)l - B_z(x + dx, t)l \]
\[ \mu_0 \varepsilon_0 \frac{d}{dt} \int \mathbf{E} \cdot d\mathbf{A} = \mu_0 \varepsilon_0 \left( l \, dx \frac{\partial E_y}{\partial t} \right) \]
\[ \frac{B_z(x + dx, t) - B_z(x, t)}{dx} = \mu_0 \varepsilon_0 \frac{\partial E_y}{\partial t} \]

So in the limit that \( dx \) is very small:
\[ \frac{\partial B_z}{\partial x} = \mu_0 \varepsilon_0 \frac{\partial E_y}{\partial t} \]
Wave Equation

Now go to Faraday’s Law

\[ \oint_C \mathbf{E} \cdot d\mathbf{s} = -\frac{d}{dt} \int \mathbf{B} \cdot d\mathbf{A} \]
Wave Equation

Faraday’s Law:
\[ \oint_c \mathbf{E} \cdot d\mathbf{s} = -\frac{d}{dt} \int \mathbf{B} \cdot d\mathbf{A} \]

Apply it to red rectangle:
\[ \oint_c \mathbf{E} \cdot d\mathbf{s} = E_y(x + dx, t)l - E_y(x, t)l \]
\[ -\frac{d}{dt} \int \mathbf{B} \cdot d\mathbf{A} = -ldx \frac{\partial B_z}{\partial t} \]
\[ \frac{E_y(x + dx, t) - E_y(x, t)}{dx} = -\frac{\partial B_z}{\partial t} \]
So in the limit that \( dx \) is very small:
\[ \frac{\partial E_y}{\partial x} = -\frac{\partial B_z}{\partial t} \]
1D Wave Equation for $E$

\[
\frac{\partial E_y}{\partial x} = - \frac{\partial B_z}{\partial t} \quad - \frac{\partial B_z}{\partial x} = \mu_0 \varepsilon_0 \frac{\partial E_y}{\partial t}
\]

Take $x$-derivative of 1st and use the 2nd equation

\[
\frac{\partial}{\partial x} \left( \frac{\partial E_y}{\partial x} \right) = \frac{\partial^2 E_y}{\partial x^2} = \frac{\partial}{\partial x} \left( - \frac{\partial B_z}{\partial t} \right) = - \frac{\partial}{\partial t} \left( \frac{\partial B_z}{\partial x} \right) = \mu_0 \varepsilon_0 \frac{\partial^2 E_y}{\partial t^2}
\]

\[
\frac{\partial^2 E_y}{\partial x^2} = \mu_0 \varepsilon_0 \frac{\partial^2 E_y}{\partial t^2}
\]
1D Wave Equation for $E$

\[
\frac{\partial^2 E_y}{\partial x^2} = \mu_0 \varepsilon_0 \frac{\partial^2 E_y}{\partial t^2}
\]

This is an equation for a wave. Let: $E_y = f(x - vt)$

\[
\begin{align*}
\frac{\partial^2 E_y}{\partial x^2} &= f''(x - vt) \\
\frac{\partial^2 E_y}{\partial t^2} &= v^2 f''(x - vt)
\end{align*}
\]

\[
v^2 = \frac{1}{\mu_0 \varepsilon_0}
\]

\[
V = \frac{1}{\sqrt{\mu_0 \varepsilon_0}} = 3 \cdot 10^8 \text{ m/s}; \text{ Light = EM wave}
\]
1D Wave Equation for B

\[ \frac{\partial B_z}{\partial t} = -\frac{\partial E_y}{\partial x} \]
\[ \frac{\partial B_z}{\partial x} = -\mu_0 \varepsilon_0 \frac{\partial E_y}{\partial t} \]

Take x-derivative of 1st and use the 2nd equation

\[ \frac{\partial}{\partial t} \left( \frac{\partial B_z}{\partial t} \right) = \frac{\partial^2 B_z}{\partial t^2} = \frac{\partial}{\partial t} \left( -\frac{\partial E_y}{\partial x} \right) = -\frac{\partial}{\partial x} \left( \frac{\partial E_y}{\partial t} \right) = \frac{1}{\mu_0 \varepsilon_0} \frac{\partial^2 B_z}{\partial x^2} \]

\[ \frac{\partial^2 B_z}{\partial x^2} = \mu_0 \varepsilon_0 \frac{\partial^2 B_z}{\partial t^2} \]
Electromagnetic Radiation

Both E & B travel like waves:
\[
\frac{\partial^2 E_y}{\partial x^2} = \mu_0 \varepsilon_0 \frac{\partial^2 E_y}{\partial t^2} \quad \frac{\partial^2 B_z}{\partial x^2} = \mu_0 \varepsilon_0 \frac{\partial^2 B_z}{\partial t^2}
\]

But there are strict relations between them:
\[
\frac{\partial B_z}{\partial t} = -\frac{\partial E_y}{\partial x} \quad \frac{\partial B_z}{\partial x} = -\mu_0 \varepsilon_0 \frac{\partial E_y}{\partial t}
\]

Here, \( E_y \) and \( B_z \) are “the same,” traveling along x axis
Amplitudes of E & B

Let $E_y = E_0 f(x - vt); \ B_z = B_0 f(x - vt)$

$$\frac{\partial B_z}{\partial t} = -\frac{\partial E_y}{\partial x} \Rightarrow -vB_0 f'(x - vt) = -E_0 f'(x - vt)$$

$\Rightarrow vB_0 = E_0$

$E_y$ and $B_z$ are “the same,” just different amplitudes.
Energy and momentum of EM radiation

\[ U = \text{Energy density of EM fields} \]
\[ = \varepsilon_0 E^2 + \mu_0 B^2 \]

For EM waves:
\[ B = \frac{E}{c} + \frac{1}{c^2} = \varepsilon_0 \mu_0 \]

\[ u = \varepsilon_0 E^2 \text{ or } \frac{B^2}{\mu_0} \]

\[ u_{\text{avg}} = \varepsilon_0 (E^2)_{\text{avg}} = \frac{1}{2} \varepsilon_0 E_0^2 = \varepsilon_0 \frac{1}{2} c E_0 B_0 \]

Energy flows at speed \( c \):
\[ E_t = U_{\text{avg}} A_{\perp} \Delta x = U_{\text{avg}} A_{\perp} c \Delta t \]
Energy and momentum of EM radiation

Energy flows at speed $c$:

\[ E_t = \nu_{avg} A_\perp \Delta x = \nu_{avg} A_\perp c \Delta t \]

Energy moving through $A_\perp$ in time $\Delta t = E_t \rightarrow \text{Power}

Intensity, $I = \frac{E_t}{\Delta t} = \nu_{avg} \cdot c$ \hspace{1cm} (2)

\[ \text{Def}: \hspace{1cm} \vec{S}_{avg} = \frac{1}{2M_0} \vec{E}_0 \times \vec{B}_0, \text{ or } \vec{S} = \frac{1}{M_0} \vec{E} \times \vec{B} \]

Magnitude gives intensity, instantaneous flow of energy direction
Energy Flow

Poynting vector: \[ \vec{S} = \frac{\vec{E} \times \vec{B}}{\mu_0} \]

- (Dis)charging C, L
- Resistor (always in)
- EM Radiation
Radiation Pressure:

Can derive from Maxwell’s Equations. Complicated!

Quantum mechanics $\Rightarrow$ momentum of photon, $p = \frac{E}{c}$

\[ \text{Force} = \frac{\Delta p}{\Delta t} = \frac{1}{c} \frac{\Delta E}{\Delta t} = \text{Power} \]

\[ \Rightarrow \text{Pressure, } P = \frac{\text{Force}}{A_1} = \frac{1}{c} \frac{\text{Power}}{A_1} \]

$P = \frac{I}{c}$ for perfect absorber

$P = \frac{2I}{c}$ for perfect reflector ($\Delta p$ factor of 2 larger)
Polarization and Malus’s law

**FIGURE 35.27** The plane of polarization is the plane in which the electric field vector oscillates.

(a) Vertical polarization

(b) Horizontal polarization

**FIGURE 35.28** A polarizing filter.

The polymers are parallel to each other.

The electric field of unpolarized light oscillates randomly in all directions.

Only the component of $\vec{E}$ perpendicular to the polymer molecules is transmitted.
Polarization and Malus’s law

\[ \vec{E}_{\text{transmitted}} = E_{\parallel} \hat{j} = E_0 \cos \theta \hat{j} \]

\[ I \propto E^2 \Rightarrow \frac{I_{\text{trans}}}{I_0} = \left( \frac{E_{\text{trans}}}{E_0} \right)^2 \]

\[ I_{\text{transmitted}} = I_0 \cos^2 \theta \quad \text{(incident light polarized)} \]
Producing and Receiving EM waves

Generating Electric Dipole Radiation Applet

Which way is the charge moving?
Producing and Receiving EM waves

Generating Electric Dipole Radiation Applet

Takes time \( \Delta t = \frac{d}{c} \) for anything to point P to know the charge has moved from the point marked by .

\( \vec{E} \) tangent to E-field lines

Charge was here

Which way is charge moving? To Left!
Producing and Receiving EM waves

At large distances, E becomes ‘flat’ \( \rightarrow \) Plane waves
Producing and Receiving EM waves

Quarter-Wavelength Antenna

Accelerated charges are the source of EM waves. Most common example: Electric Dipole Radiation.
Producing and Receiving EM waves

*Figure 13.8.3* Electric and magnetic field lines produced by an electric-dipole antenna. (in the “far field”)
Producing and Receiving EM waves

No radiation along axis of dipole:

Biot-Savart law states there is no B-field along y if there is current parallel to \( \mathbf{r} \)-hat.

**Figure 34.11** Angular dependence of the intensity of radiation produced by an oscillating electric dipole. The distance from the origin to a point on the edge of the gray shape is proportional to the intensity of radiation.

\[ I \propto \frac{\sin^2 \theta}{r^2} \]
General derivation of EM wave equation

\[ \nabla = x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y} + z \frac{\partial}{\partial z} \]

Stoke's Thm: \[ \oint (\nabla \times \mathbf{V}) \cdot d\mathbf{a} = \int \mathbf{V} \cdot d\mathbf{s} \]

Divergence Thm: \[ \iiint \nabla \cdot \mathbf{V} \, dV = \iiint \mathbf{V} \cdot d\mathbf{a} \]

\[ \nabla \times (\nabla \times \mathbf{V}) = \nabla (\nabla \cdot \mathbf{V}) - \nabla^2 \mathbf{V} \]

where \[ \nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \]

\[ (\nabla^2 = \nabla \cdot \nabla) \]

Consider Gauss's Law(s) in free space (q=0, I=0)

\[ \oint \mathbf{E} \cdot d\mathbf{a} = \oiint (\nabla \cdot \mathbf{E}) \, dV = \oiint \frac{\rho}{\varepsilon_0} \, dV \]

\[ \Rightarrow \nabla \cdot \mathbf{E} = \frac{\rho}{\varepsilon_0} \]

free space, \[ \Rightarrow \nabla \cdot \mathbf{E} = 0 \]

Also:

\[ \oint \mathbf{B} \cdot d\mathbf{a} = \oiint \nabla \cdot \mathbf{B} \, dV = 0 \]

\[ \Rightarrow \nabla \cdot \mathbf{B} = 0 \]
General derivation of EM wave equation

Faraday's Law:
\[ \oint E \cdot d\vec{s} = \frac{d\Phi_B}{dt} \]

Stokes' Thm \Rightarrow \oint \vec{E} \cdot d\vec{s} = \oint (\nabla \times \vec{E}) \cdot d\vec{a} \]
\[ + \frac{d\Phi_B}{dt} = -\frac{d}{dt} \oint \vec{B} \cdot d\vec{a} = \oint \frac{d\vec{B}}{dt} \cdot d\vec{a} \]
\[ \therefore \quad \nabla \times \vec{E} = -\frac{d\vec{B}}{dt} \]

Ampere-Maxwell Law:
\[ \oint \vec{B} \cdot d\vec{s} = \mu_0 I_{\text{enc}} + \varepsilon_0 \mu_0 \frac{d\Phi_E}{dt} \]

Stokes' Thm \Rightarrow \oint \vec{B} \cdot d\vec{s} = \oint (\nabla \times \vec{B}) \cdot d\vec{a} \]
\[ = \oint \mu_0 \vec{J} \cdot d\vec{a} \]
\[ = \oint \mu_0 \frac{d\vec{B}}{dt} \cdot d\vec{a} \]
\[ = \varepsilon_0 \mu_0 \oint \frac{d\vec{E}}{dt} \cdot d\vec{a} \]
\[ \therefore \quad \nabla \times \vec{B} = \mu_0 \vec{J} + \varepsilon_0 \mu_0 \frac{d\vec{E}}{dt} \]
General derivation of EM wave equation

Integral Form

Gauss's Law
\[ \oint E \cdot dl = \frac{Q_{in}}{\varepsilon_0} \]

Differential Form
\[ \nabla \cdot E = \frac{\rho}{\varepsilon_0} \]

Gauss's Law of Magnetism
\[ \oint B \cdot dl = 0 \]
\[ \nabla \cdot B = 0 \]

Faraday's Law
\[ \oint E \cdot dl = -\frac{\partial \Phi_B}{\partial t} \]
\[ \nabla \times E = -\frac{\partial \Phi_B}{\partial t} \]

Ampere-Maxwell Law
\[ \oint B \cdot dl = \mu_0 I_{net} + \varepsilon_0 \mu_0 \frac{\partial \Phi_E}{\partial t} \]
\[ \nabla \times B = \mu_0 \mathbf{J} + \varepsilon_0 \mu_0 \frac{d E}{d t} \]

Free space: \( \rho = 0, \mathbf{J} = 0 \) \( \Rightarrow \nabla \cdot E = 0, \nabla \cdot B = 0 \)
\[ \nabla \times E = -\frac{\partial \Phi_B}{\partial t}, \nabla \times B = \varepsilon_0 \mu_0 \frac{d E}{d t} \]
General derivation of EM wave equation

\[ \vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \Rightarrow \vec{\nabla} \times (\vec{\nabla} \times \vec{E}) = -\frac{\partial}{\partial t} (\vec{\nabla} \times \vec{B}) \]

\[ \frac{\vec{\nabla} (\vec{\nabla} \cdot \vec{E}) - \nabla^2 \vec{E}}{\partial t} = -\frac{\partial}{\partial t} (\mu_0 \varepsilon_0 \frac{\partial \vec{E}}{\partial t}) \]

\[ \nabla^2 \vec{E} = \mu_0 \varepsilon_0 \frac{\partial^2 \vec{E}}{\partial t^2} \]

\[ \vec{\nabla} \times \vec{B} = \mu_0 \varepsilon_0 \frac{\partial \vec{E}}{\partial t} \Rightarrow \vec{\nabla} \times (\vec{\nabla} \times \vec{B}) = \mu_0 \varepsilon_0 \frac{\partial}{\partial t} (\vec{\nabla} \times \vec{E}) \]

\[ \frac{\vec{\nabla} (\vec{\nabla} \cdot \vec{B}) - \nabla^2 \vec{B}}{\partial t} = -\frac{\partial \vec{B}}{\partial t} \]

\[ \nabla^2 \vec{B} = \mu_0 \varepsilon_0 \frac{\partial^2 \vec{B}}{\partial t^2} \]

Each component of \( \vec{E} + \vec{B} \) obey wave equation: Answer

\[ \nabla^2 \vec{E} = \hat{x} \frac{\partial^2 \vec{E}_x}{\partial x^2} + \hat{y} \frac{\partial^2 \vec{E}_y}{\partial y^2} + \hat{z} \frac{\partial^2 \vec{E}_z}{\partial z^2} \]

\[ \frac{\partial^2 \vec{E}}{\partial t^2} = \hat{x} \frac{\partial^2 \vec{E}_x}{\partial t^2} + \hat{y} \frac{\partial^2 \vec{E}_y}{\partial t^2} + \hat{z} \frac{\partial^2 \vec{E}_z}{\partial t^2} \]

\[ \frac{\partial^2 \vec{E}_x}{\partial x^2} = \mu_0 \varepsilon_0 \frac{\partial^2 \vec{E}_x}{\partial t^2} \text{ etc.} \]

\[ \nabla^2 \vec{B} = \mu_0 \varepsilon_0 \frac{\partial^2 \vec{B}}{\partial t^2} \text{ etc.} \]
General derivation of EM wave equation

Plane wave solutions:

Assume \( \vec{E} \) only pts. in one direction everywhere \( \vec{E} = \hat{y} E_y(x,t) \)

| \( |\vec{E}| \) only varies with time & position \( x \)
| \( \nabla \times \vec{B} = \mu_0 \frac{d\vec{E}}{dt} \) \( \Rightarrow \vec{B} = \hat{z} B_z(x,t) \)

Big Assumptions! However, superposition of plane wave solutions that have various amplitudes, wavelengths, frequencies, & direction of travel will reproduce any possible solution

(Exactly like any time domain waveform can be thought of as a superposition of Sin's & Cos's of various Amplitudes & frequencies; Principle of Fourier Analysis.)

\[ \ddot{E}_y = \frac{\partial^2 E_y}{\partial x^2} = \mu_0 \varepsilon_0 \frac{\partial^2 E_y}{\partial t^2} \]

Looks like a wave equation:

\[ \frac{\partial^2 E_y}{\partial x^2} = \frac{1}{V^2} \frac{\partial^2 E_y}{\partial t^2} \]

\( V = \frac{1}{\sqrt{\varepsilon_0 \mu_0}} = 3 \times 10^8 \text{ m/s} \)

Same as light!!!