

40 Wave Functions and Uncertainty

Recommended class days: 2

Background Information

Classical particles have a well-defined position at all instants of time and are described by a trajectory $x(t)$. The experimental evidence for wave-particle duality should have convinced students that atomic-level particles do *not* have a well-defined position. Hence they cannot be described by a classical trajectory.

This chapter raises the issue of how to *describe* the behavior of atomic particles. The double-slit experiment and the analogy between electrons and photons lead us to the idea of a wave function. This chapter looks at wave functions from a pictorial and graphical representation. The “law” of wave functions—the Schrödinger equation—is deferred to Chapter 41. It is important to emphasize that the wave function is a new *hypothesis*. Only subsequent experimental tests will reveal if there is any merit to the hypothesis.

The wave function is an abstract concept, although no more abstract than the concept of a field. Not surprisingly, it takes quite some time before students begin to feel comfortable with this idea or can use it to reason about physical situations. The focus of this chapter is to connect the wave function with experimental outcomes and measurable probabilities, thus linking this new idea with physical reality.

The probability ideas of quantum mechanics are especially difficult for many students. It takes a number of examples for them to catch on to measuring the probability of finding a particle in an interval δx at position x . To keep the idea clear, this text writes explicitly $\text{Prob}(\text{in } \delta x \text{ at } x)$. The concept of *probability density* is especially difficult. The analogy of linear mass density along a string of variable diameter is helpful to many students.

Student Learning Objectives

- To introduce the *wave function* as the descriptor of particles in quantum mechanics.
- To provide the wave function with a probabilistic interpretation.
- To understand the wave function through pictorial and graphical exercises.
- To introduce the idea of normalization.
- To recognize the limitations on knowledge imposed by the Heisenberg uncertainty principle.

Pedagogical Approach

Double-slit interference is the experiment that leads most naturally to the idea of a wave function. Both photons and electrons make wave-like interference patterns that are built up particle by particle. For light interference, a straightforward argument links the probability of a photon landing in an interval δx at position x to the amplitude function $A(x)$ of the classical light wave across the

screen. Because electrons make the same pattern, we can postulate that there is some new wave-like function $\psi(x)$ that is associated with the probability of an electron landing in an interval δx at position x . Our goal, in this chapter and the next, is to learn:

- What are the properties of this function $\psi(x)$?
- What *law* governs the behavior of $\psi(x)$?

Figure 40.1 is especially important. Students need to think about and understand how the photon dot pattern, the graph of $A(x)$, and the graph of the light intensity I are related to each other. Once students understand the reasoning that leads to $P(x) \propto |A(x)|^2$, then the wave function $\psi(x)$ can be introduced to characterize the arrival positions for electrons. Point out that $\psi(x)$ is a “wave-like function” in the sense that it is oscillatory, but that no physical quantity is actually waving. Comment that this is a hypothesis, nothing that is mandated by the experiments. Only after the implications of the hypothesis are tested will we know if the hypothesis has merit.

Most of the student effort in this chapter should be focused on matching graphs of $\psi(x)$ and $|\psi(x)|^2$ with dot pictures of electrons arriving at a detector. It is important that students associate the wave function with measurable quantities. Students should practice drawing dot patterns from graphs and, conversely, deducing wave function graphs from dot patterns.

Wave functions must be normalized in order for the probabilistic interpretation to work. Students readily agree that a photon or electron that passes through the slits must land *somewhere* on the screen. Thus there is *unit probability* of being detected somewhere. The difficulty for students is relating this common-sense observation to the mathematical normalization condition. Few students yet have the sophistication to understand what you’re doing if you plunge right into probability densities and integrals. It’s best to approach this as a sum $\sum \text{Prob}(\text{in } \delta x \text{ at } x)$, summed over all intervals—that is, “everywhere.” Once they agree that this expresses the idea of the electron landing *somewhere* on the screen, you can change the sum to $\sum P(x)\delta x$ and then extend the sum to an integral. Both graphical and analytical practice with normalization are important.

No introduction to quantum physics is complete without some mention of the Heisenberg uncertainty principle. I find it best to approach the uncertainty principle from the classical result $\Delta f \Delta t > 1$. By starting with beats and then superimposing more waves, students are led to see that this relationship is a property of “waviness.” The essential point they need to understand is that a wave packet yields no precise answer to the question, “What is the frequency?” Because of the superposition of many frequencies, the frequency of the wave packet is “uncertain” by Δf .

Once students get past this critical point, it is straightforward to ask, “If $\Delta f \Delta t > 1$ is an inherent aspect of waviness, how does it apply to a matter wave?” This leads easily to the Heisenberg uncertainty principle as a statement about the wave nature of matter. Students follow this approach more easily than finding the uncertainty principle through an analysis of single-slit diffraction. Note that whether the right side of the uncertainty principle is h , $h/2$, or $\hbar$ is irrelevant, because we haven’t given a precise specification of how Δx and Δp are to be defined.

Using Class Time

DAY 1: It’s best to spend a good part of day 1 on photons. If students really understand

- that the probability of detecting a photon is associated with $|A(x)|^2$, and
- the concept of *probability density*,

then they've overcome one of the conceptual barriers to understanding $\psi(x)$. In anticipation of $\psi(x)$ as the *wave function*, Part V introduced the terminology *amplitude function* for $A(x)$. Thus it's good to use this term explicitly when referring to $A(x)$.

It's reasonable to start out with a mini-lecture that summarizes the reasoning linking the probability of detecting a photon in δx to the square of the amplitude $|A(x)|^2$. They can read the details in the text, so the main points to make in class are that:

- If N_{tot} photons pass through an apparatus and $N(\text{in } \delta x \text{ at } x)$ arrive in a small interval δx at position x , the *probability* for any one photon to be detected there is

$$\text{Prob}(\text{in } \delta x \text{ at } x) = \frac{N(\text{in } \delta x \text{ at } x)}{N_{\text{tot}}}$$

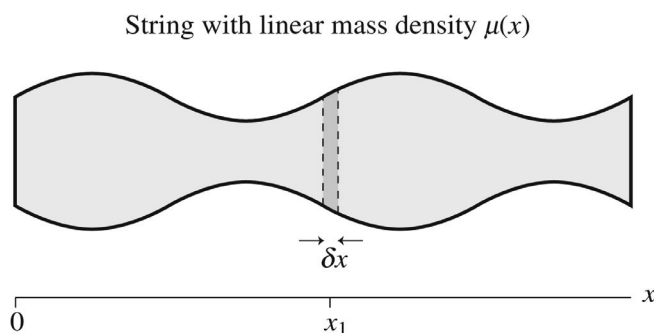
Thus probability can be *measured* by counting photons (or electrons).

- The energy landing in a small interval of width δx at position x is $N(\text{in } \delta x \text{ at } x)hf$.
- The energy is also related to the intensity and thus to $|A(x)|^2 \delta x$.

Combining these three pieces leads to $\text{Prob}(\text{in } \delta x \text{ at } x) \propto |A(x)|^2 \delta x$.

The single-slit experiment forms a useful class exercise. Have students sketch a single-slit diffraction pattern, then have them articulate that this is the *intensity* of the light. Then have them draw a dot pattern right under their graph to show how a few dozen photons would appear. Make sure their density of dots matches the graph, with most in the central maximum. Finally, ask them to graph $A(x)$ beneath the dot pattern. This will be hard for most students because they don't recognize that $A(x)$ oscillates between positive and negative value.

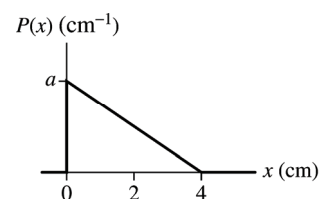
It's useful to introduce probability density by drawing a picture of a string that has a variable diameter. Include an x -axis beneath the string. Ask students how they would determine the mass of a small segment of string of length δx at position x_1 . They should recall from waves on strings that the mass is related to the linear density μ by $\delta m = \mu \delta x$. Now, because the string isn't uniform, the linear density is a function of position: $\mu(x)$. Thus $\text{mass}(\text{in } \delta x \text{ at } x) = \mu(x) \delta x$. This separates the string property $\mu(x)$ from the experimenter-chosen interval δx .



By analogy, we *define* the probability density $P(x)$ such that $\text{Prob}(\text{in } \delta x \text{ at } x) = P(x) \delta x$. By separating out δx , we believe that $P(x)$ is the physically meaningful quantity that describes the photons (or electrons). Have students discuss and articulate the fact that $P(x)$ has units. You can then proceed to introduce the normalization condition, as discussed in the Pedagogical Approach.

As an exercise to put these pieces together, draw the probability density graph shown in the figure and state that this is the probability density of photons in some unspecified experiment. Ask students to:

- Draw the photon dot pattern.
- Determine the value of the constant a .



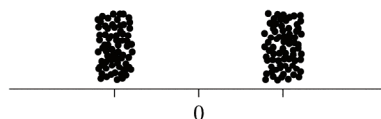
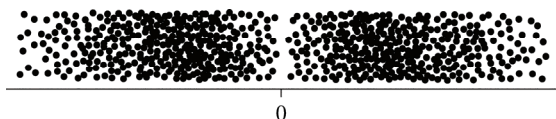
- Determine how many out of 10^6 total photons land in a 0.001 cm interval located at $x = 1$ cm. Then at $x = 3$ cm.
- Determine how many out of 10^6 total photons land in the interval $2 \text{ cm} \leq x \leq 4 \text{ cm}$.
- Sketch a graph of the light wave's amplitude function $A(x)$.

DAY 2: After noting that single-slit and double-slit patterns with electrons look the same as they do with photons—a fact for which they've seen evidence—you can *postulate* the existence of a wave function $\psi(x)$ for electrons that is analogous to $A(x)$ for photons. Note explicitly that:

- This is a postulate, subject to confirmation by experiment.
- The function $\psi(x)$ is a “wave-like function.” There is no physical quantity actually waving or oscillating. This is different than $A(x)$, which represents an oscillating electric field.
- We do not, at this point, have any way to predict what $\psi(x)$ should be for a given experimental arrangement. But if the postulate is valid, there must be a new law of nature that would allow us to calculate $\psi(x)$, just as we can calculate $A(x)$ from the laws of electromagnetism. Promise them they will learn the “law of psi” in the next chapter.

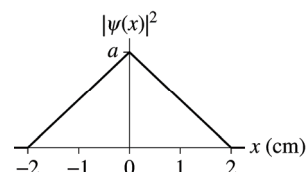
The formalism is the same as for photons, so you can define the probability density, remind them of the normalization condition, and then get right into examples of using and thinking about wave functions. (Note that many of the examples in this chapter do not satisfy the continuity requirements of true wave functions. They are chosen for pedagogical value, not realism.) Be sure to note that $\psi(x)$ has units!

Example 1: For each of the electron dot patterns shown, (a) draw a graph of the probability density $P(x)$ and (b) Draw a graph of the wave function $\psi(x)$. Normalization isn't necessary for these.



Example 2: A graph of $|\psi(x)|^2$ for electrons is shown.

- What is the value of a ?
- Draw a graph of $\psi(x)$.
- Draw a dot pattern showing the arrival of electrons.
- Determine the probability that an electron arrives on the screen with $|x| \leq 1 \text{ cm}$.



Example 3: An electron wave function is

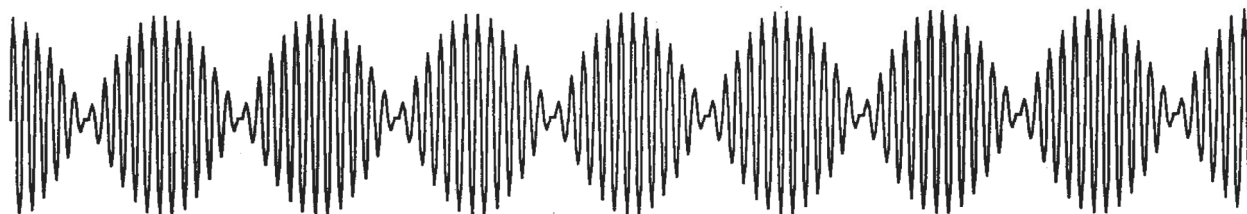
$$\psi(x) = \begin{cases} 0 & x < 0 \\ cx & 0 \leq x \leq 2 \text{ nm} \\ 0 & x > 2 \text{ nm} \end{cases}$$

- Draw graphs of $\psi(x)$ and $|\psi(x)|^2$.
- Determine the normalization constant c .
- Draw a dot pattern showing the arrival of electrons.
- Find the probability that an electron is located at $x > 1 \text{ nm}$.

As noted above, the Heisenberg uncertainty principle is approached from an analysis of the superposition of classical waves. Students already know that the superposition of two frequencies causes beats, with a net wave that looks rather like a wave packet. This is an easy introduction of Δf as the *range* of frequencies that are superimposed to form a wave packet.

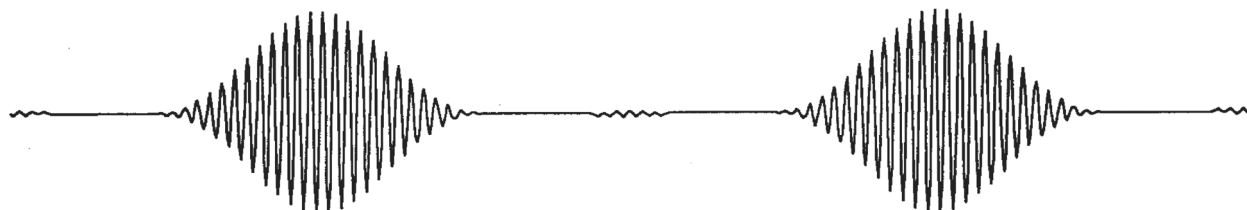
A nice demonstration is to use computer software to superimpose an increasing number of waves of similar frequency. The figure below, which shows the superposition of two and seven waves centered on $\omega = 100$, illustrates vividly how wave packets are the superposition of many waves over a finite range of frequencies Δf .

Superposition of two frequencies: $y = \sin(96t) + \sin(104t)$



Superposition of seven frequencies, same horizontal scale as above.

$$y = 0.4\sin(94t) + 1.0\sin(96t) + 1.7\sin(98t) + 2.0\sin(100t) + 1.7\sin(102t) + 1.0\sin(104t) + 0.4\sin(106t)$$



You can finish day 2 with standard applications of the uncertainty principle to a particle in a box. Because Δv_x is a velocity uncertainty, and because the average velocity of a particle in a box is zero, example problems interpret Δv_x by saying that the range of possible (or likely) velocities is from $-\frac{1}{2}\Delta v_x$ to $+\frac{1}{2}\Delta v_x$. End-of-chapter homework problems expect a similar interpretation.

Sample Reading Quiz Questions

1. What basic experiment is used in this chapter to suggest a common description for both photons and electrons?
2. What is the quantity $\psi(x)$ called?
3. The quantity $|\psi(x)|^2$ is called the
 - a. wave function.
 - b. probability.
 - c. probability density.
 - d. intensity.
 - e. amplitude density.
 - f. Schrödinger function.
4. Requiring the sum of all probabilities to be equal to one is called
 - a. equalization.
 - b. normalization.
 - c. unification.
 - d. quantization.

Sample Exam Questions

Sample exam questions for Chapters 40–41 are at the end of Chapter 41.

