Differential Form of Gauss's Law

The integral form of Gauss's law for electric fields and magnetic fields is given by

$$\oint \mathbf{E} \cdot d\mathbf{A} = \frac{Q_{enclosed}}{\varepsilon_o} \tag{1}$$

$$\oint \mathbf{B} \cdot d\mathbf{A} = 0, \tag{2}$$

where $\oint d\mathbf{A}$ means to take the integral around a surface bounding a volume. The quantity $d\mathbf{A}$ is vector representing a small area element of the surface; $d\mathbf{A}$ points away from the volume and is normal to the area element. These equations measure the flux of electric and magnetic fields lines through the surface. A more useful quantity, however, is the strength and direction of the flux lines at a given point. This is given by the divergence of the E and B fields. To write down equations that will allow us to quantify these strengths, we will change the integral equations in Eqs. 1 and 2 into differential equations. To that end, we first give a formal definition of the divergence, $\nabla \cdot \mathbf{F}$, of a vector:

Definition 1 *The divergence of a vector is the limit of its surface integral per unit volume as the volume enclosed by the surface goes to zero.*

Mathematically we write,

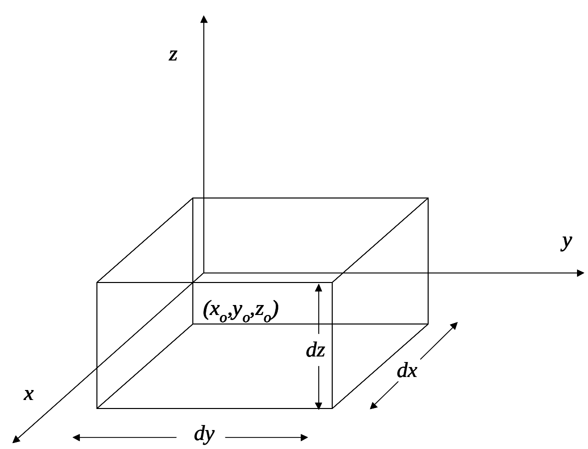
$$\nabla \cdot \mathbf{F} = \lim_{V \longrightarrow 0} \frac{1}{V} \oint \mathbf{F} \cdot d\mathbf{A}. \tag{3}$$

where

$$\nabla = \widehat{x}\frac{\partial}{\partial x} + \widehat{y}\frac{\partial}{\partial y} + \widehat{z}\frac{\partial}{\partial z}. \tag{4}$$

The divergence is a scalar quantity defined at a point on the surface of integration.

To find an explicit form for $\nabla \cdot \mathbf{F}$ we will work in rectangular coordinates where the area and volume elements are given by $dxdy$ and $dxdydz$ respectively. Consider a rectangular volume with one corner at (x_o, y_o, z_o). The left-hand size of Gauss's law applied to vector $\mathbf{F}$, $\oint \mathbf{F} \cdot d\mathbf{A}$, then can be written as:



$$-\int F_x(x_o, y, z)\, dydz + \int F_x(x_o + dx, y, z)\, dydz$$
$$-\int F_y(x, y_o, z)\, dxdz + \int F_y(x, y_o + dy, z)\, dxdz$$
$$-\int F_z(x, y, z_o)\, dxdy + \int F_z(x, y, z_o + dz)\, dxdy. \tag{5}$$

Here, we have assumed that the components of $\mathbf{F}$ point along the positive x-, y- and z-axes. Thus, we pick up a minus sign for the faces closest to the origin. In the limit as we shrink the volume, the integrals can be replaced by

$$
\begin{aligned}
&\left[F_x\left(x_o+dx,y,z\right)-F_x\left(x_o,y,z\right)\right]dydz \\
&\qquad\qquad\left[F_y\left(x,y_o+dy,z\right)-F_y\left(x,y_o,z\right)\right]dxdz \\
&\qquad\qquad\qquad\qquad\left[F_z\left(x,y,z_o+dz\right)-F_z\left(x,y,z_o\right)\right]dxdy. \quad (6)
\end{aligned}
$$

To evaluate these expressions consider the terms that multiplies $dydz$. We note that we can make a Taylor expansion about x_o for $F_x\left(x_o+dx,y,z\right)$ and write it as

$$F_x\left(x_o,y,z\right)+dx\frac{\partial}{\partial x}F_x\left(x_o,y,z\right)+\cdots \tag{7}$$

Then we write

$$F_x\left(x_o+dx,y,z\right)-F_x\left(x_o,y,z\right)=dx\frac{\partial}{\partial x}F_x\left(x_o,y,z\right). \tag{8}$$

It is also possible to recognize that

$$F_x\left(x_o+dx,y,z\right)-F_x\left(x_o,y,z\right)=dF_x\left(x_o,y,z\right). \tag{9}$$

Using the chain rule we have

$$dF_x\left(x_o,y,z\right)=dx\frac{\partial}{\partial x}F_x\left(x_o,y,z\right). \tag{10}$$

Doing the same for the terms that multiply $dxdz$ and $dxdy$ we can write expression 5 in the limit of a shrinking volume as

$$dxdydz\left[\frac{\partial}{\partial x}F_x\left(x_o,y,z\right)+\frac{\partial}{\partial y}F_y\left(x,y_o,z\right)+\frac{\partial}{\partial z}F_z\left(x,y,z_o\right)\right]. \tag{11}$$

Finally, putting it all together we write the divergence as

$$\nabla\cdot\mathbf{F}=\lim_{V\longrightarrow 0}\frac{1}{V}\oint\mathbf{F}\cdot d\mathbf{A}=\frac{\partial}{\partial x}F_x\left(x,y,z\right)+\frac{\partial}{\partial y}F_y\left(x,y,z\right)+\frac{\partial}{\partial z}F_z\left(x,y,z\right). \tag{12}$$

We have dropped the subscripts because the divergence applies to all points on the surface. Gauss's law for the E and B field then become

$$\nabla\cdot\mathbf{E}=\frac{\rho}{\varepsilon_o} \tag{13}$$

$$\nabla\cdot\mathbf{B}=0, \tag{14}$$

where ρ is the charged enclosed per unit volume.