Physics 263: Electromagnetism and Modern Physics

Sections 0101- 0105 **Final Exam** *Prof. J.J. Kelly*

Instructions:

- This closed book, closed notes exam contains 8 problems to be completed in 2 hours. You may use a basic scientific calculator, but no other aids are permitted. The final page provides a brief compilation of *possibly useful information*, including relevant physical constants.
- Work each problem in the space provided. If additional space is needed, use the back of the *previous* page and indicate that you have done so.
- Please write your name on each page, including this one. Do not use red ink.
- **Explain your reasoning and show your work**. Partial credit will be awarded when the relevant physical principles are applied even if mistakes are made in execution of the steps. However, correct guesses without any explanation may be penalized.
- **Algebraic answers must have consistent dimensions and numerical answers must have consistent units**.
- **Honor pledge:** please copy and sign "I pledge on my honor that I have not given or received any unauthorized assistance on this examination."

Pledge:

Signature: Solutions **Student ID number:** ______________

Printed name: ______________________ **Section:** ______________

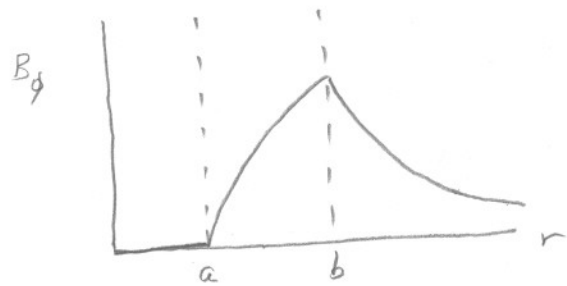
Problem	Score	Problem	Score
1		5	
2		6	
3		7	
4		8	
total			

1. A long cylindrical shell with inner radius *a* and outer radius *b* carries total current *I* along its length.

a) Find expressions for the magnetic field in the regions (i) $r < a$, (ii) $a < r < b$, and (iii) $r > b$. Sketch the function $B(r)$. *(10 pts)*

Apply Ampere's law to find azimuthal component, B_ϕ.

$$2\pi r\, B_\phi = \mu_0 I\,(\text{enclosed}) \Rightarrow \frac{2\pi r}{\mu_0 I} B_\phi = \begin{cases} 0 & r \le a \\ \dfrac{r^2 - a^2}{b^2 - a^2} & a \le r \le b \\ I & b \le r \end{cases}$$



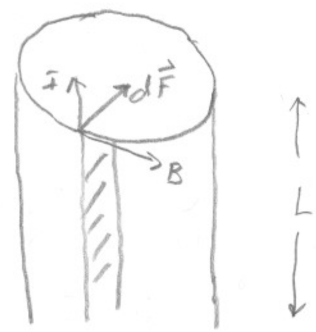
b) Evaluate the pressure assuming that the shell is thin, such that $a \approx b \approx R$. Is the pressure inward or outward? [Hint: consider the force upon a narrow strip of current running the length of the cylinder.] *(5 pts)*

$$2\pi R\, B_\phi = \mu_0 I \Rightarrow B_\phi = \frac{\mu_0 I}{2\pi R}$$

$$dF = \frac{\mu_0 I}{2\pi R} I L \frac{d\theta}{2\pi} \Rightarrow F = \frac{\mu_0 I^2 L}{2\pi R}$$

$A = 2\pi R L$ = surface area

∴ inward pressure $P = \mu_0 \left(\frac{I}{2\pi R}\right)^2$



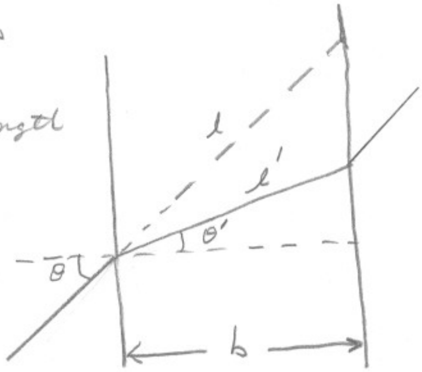
c) Compute the pressure for $R = 0.2\ cm$ and $I = 10\ A$. *(5 pts)*

$$P = 0.796\ \frac{N}{m^2}$$

2. Interference and diffraction

   a) A beam of monochromatic light of wavelength λ_0 in vacuum impinges upon two slits of width *a* and separation *d* and forms a diffraction pattern upon a distant screen. A thin sheet of transparent material, such as glass or plastic, with thickness *b* and refractive index *n* is then inserted behind the top slit only. Derive an expression for the angular shift of the resulting diffraction pattern. Does the pattern shift up or down? Explain your reasoning carefully with the aid of a clear diagram. *(10 pts)*

The diagram at right compares the phases accumulated on the dashed path of length l for $n \to 1$ with the actual solid path of length l' for $n > 1$, such that



$$\Delta\phi_1 = \frac{2\pi}{\lambda_0}(nl' - l), \quad l = b/\cos\theta, \quad l' = b/\cos\theta', \quad n\sin\theta' = \sin\theta$$

Using $\cos\theta' = \sqrt{1 - n^{-2}\sin^2\theta} = \frac{1}{n}\sqrt{n^2 - \sin^2\theta}$,

we find $\Delta\phi_1 = \frac{2\pi}{\lambda_0} b \left(\frac{n^2}{\sqrt{n^2 - \sin^2\theta}} - \frac{1}{\cos\theta} \right)$

The net phase difference between two beams is

$$\Delta\phi = \Delta\phi_1 - \frac{2\pi}{\lambda_0} d\sin\theta$$

The phase delay in the material tends to compensate the longer geometrical path for the bottom slit, so the pattern shifts up. For small θ,

$$\Delta\phi \approx \frac{2\pi}{\lambda_0} b\left(n - \frac{\theta^2}{2n} - 1 - \frac{\theta^2}{2}\right) - \frac{2\pi}{\lambda_0} d\theta_0$$

$$\Delta\phi = 0 \implies \frac{n+1}{2n}\theta_0^2 + \frac{d}{b}\theta_0 + 1 - n = 0 \implies \theta_0 \approx \frac{n}{n+1}\left[\sqrt{\left(\frac{d}{b}\right)^2 + 2\frac{n^2-1}{n}} - \frac{d}{b}\right]$$

where the positive root ensures $\theta_0 \to 0$ for $n \to 1$.

   b) Use diffraction theory to estimate the maximum distance at which you can resolve the two headlights of an approaching car. *(10 pts)*

$\Delta\theta = 1.22\frac{\lambda}{D} = \frac{W}{L}$;

$W \sim 2\text{m}$ = separation between lamps

$D \sim 3\text{mm}$ = pupil diameter

$\lambda \sim 550\text{ nm}$

$$\therefore L = \frac{WD}{1.22\lambda} \sim 9\text{ km}$$

3. A coil consisting of N circular turns with area A is connected to an external circuit with resistance R. A uniform magnetic field B is directed perpendicular to the coil. The coil is now turned over, thereby reversing the magnetic flux through it.

a) Derive an expression for the total charge Q that flows through the external circuit as the coil is flipped. Measurement of this charge can be used to determine the magnetic field strength. *(10 pts)*

$$\mathcal{E} = -\frac{d\Phi_B}{dt} = R\frac{dQ}{dt} \Rightarrow Q = R^{-1}\Delta\Phi_B \text{ in magnitude}$$

$$\Delta\Phi_B = 2NAB \text{ under reversal}$$

$$\therefore \quad Q = \frac{2NAB}{R}$$

b) Compute Q for $B=1\ T$, $N=100$, $A=5\ cm^2$, and $R=0.2\ \Omega$. *(10 pts)*

$$Q = 0.5\ C$$

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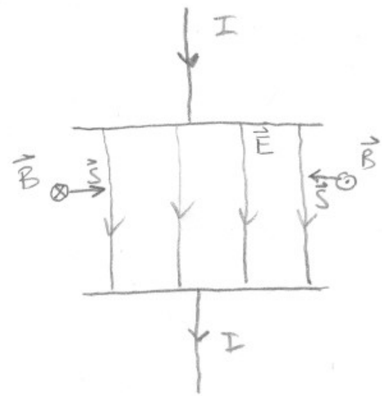
4. A parallel-plate capacitor consists of two circular plates with radius R separated by distance d. A steady current I charges the capacitor. Use idealized electric and magnetic fields, neglecting fringing effects near the edges.

a) Find an expression for the Poynting vector at the edge of the capacitor. From what direction does energy enter the capacitor? *(10 pts)*

assume uniform E $\Rightarrow$ $E = \frac{Q}{A\varepsilon_0} = \frac{It}{\pi R^2 \varepsilon_0}$

Ampere-Maxwell $\Rightarrow$ $2\pi R B = \mu_0 \varepsilon_0 \dot{\Phi}_E = \mu_0 I$

$\vec{S} = \frac{\vec{E} \otimes \vec{B}}{\mu_0}$ is radially inward for charging capacitor with $\vec{E}, \vec{B}$ directions shown



$S = \frac{EB}{\mu_0} = \frac{Q}{A\varepsilon_0} \frac{I}{2\pi R} = \frac{I^2 t}{2\pi^2 R^3 \varepsilon_0}$

b) Find an expression for the rate at which the energy stored in the capacitor increases. Compare this result with the electromagnetic energy flux into the capacitor. *(10 pts)*

energy stored in electric field $U_E = \frac{\varepsilon_0}{2} E^2 V = \frac{\varepsilon_0}{2} E^2 \pi R^2 d$

$\dot{U}_E = \varepsilon_0 E \dot{E} \pi R^2 d = \frac{I^2 t}{(\pi R^2)^2 \varepsilon_0} \pi R^2 d = \frac{I^2 t d}{\pi R^2 \varepsilon_0}$

energy flux through cylindrical surface: $\dot{U} = S \, 2\pi R d$

$\dot{U} = \frac{I^2 t d}{\pi R^2 \varepsilon_0} = \dot{U}_E$

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5. Phase shift for a series *RLC* circuit.

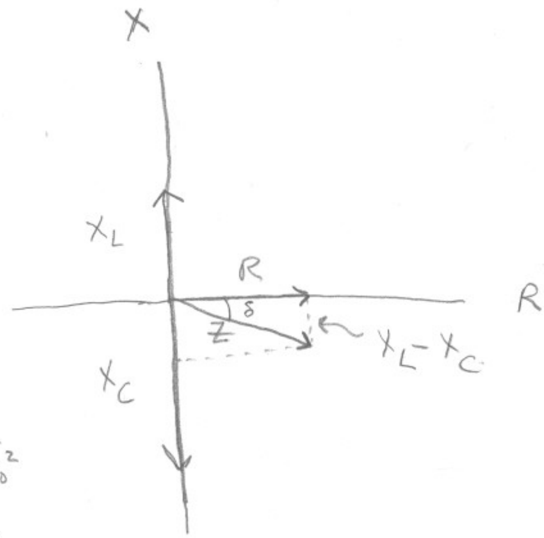
a) Use a phasor diagram to write an expression for the phase shift δ between the current and voltage in a series *RLC* circuit as a function of the driving frequency ω. *(10 pts)*

$X_L = \omega L \qquad X_C = \frac{1}{\omega C}$

$\tan\delta = \frac{X_L - X_C}{R} = \frac{\omega^2 - \omega_0^2}{\omega\Gamma}$

with $\omega_0^2 = \frac{1}{LC}$

$\Gamma = \frac{R}{L}$



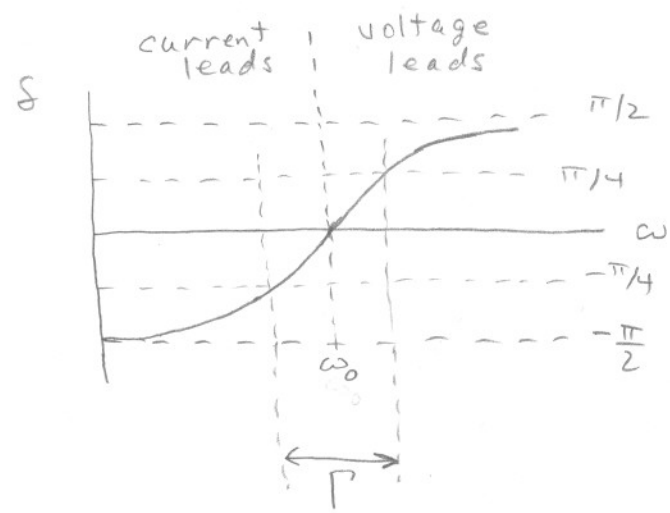
note $\tan\delta_\pm = \pm 1 \Rightarrow \omega_\pm^2 - \omega_0^2 = \pm\,\omega_\pm\Gamma$

$\omega_\pm = \frac{\pm\Gamma + \sqrt{\Gamma^2 + 4\omega_0^2}}{2}$

$\therefore\ \omega_+ - \omega_- = \Gamma$ is width of resonance

$\omega = \omega_0 \Rightarrow \tan\delta = 0$ at resonance

b) Sketch the function $\delta(\omega)$ and determine the resonant frequency in terms of *R*, *L*, and *C*. Identify the high and low frequency limits and determine for each whether the current leads or lags the voltage. Explain. *(10 pts)*



The current phasor is parallel to R, voltage to Z.

$\therefore$ voltage leads for $\omega > \omega_0 \Rightarrow \delta > 0$ when inductance dominates
voltage lags for $\omega < \omega_0 \Rightarrow \delta < 0$ when capacitance dominates

6. A charged kaon is an unstable particle that has a rest mass of $mc^2 = 493.6\,\text{MeV}$ and a proper lifetime of 1.24×10^{-8} s. Suppose that kaons with kinetic energy of 800 MeV are created in a nuclear physics experiment.

a) What is the momentum of the kaons in MeV/c? *(5 pts)*

$$(mc^2)^2 = (K + mc^2)^2 - (pc)^2 \Rightarrow pc = [K(K+2mc^2)]^{1/2}$$

$$\therefore\ p = 1196\ \text{MeV/c}$$

b) What is the average distance that these particles travel in the laboratory before decaying? *(5 pts)*

$$E = K + mc^2 = \gamma mc^2 \Rightarrow \gamma = 1 + \frac{K}{mc^2} = 2.62$$

$$\gamma = \left[1 - \left(\frac{v}{c}\right)^2\right]^{-1/2} \Rightarrow \frac{v}{c} = (1-\gamma^{-2})^{1/2} = 0.924$$

time dilation $\Rightarrow t = \gamma\tau = 2.62 \times 1.24\times10^{-8}\,\text{s} = 3.25\times10^{-8}\,\text{s}$

$$\therefore\ d = vt = 9.0\,\text{m}$$

OR: $d = \gamma v \tau = \frac{p}{m}\tau$

c) The dominant decay mode produces a muon with mass 105.7 MeV/c^2 and a massless neutrino. Compute the neutrino energy and the muon kinetic energy in the kaon rest frame. *(10 pts)*

$$m_K c^2 = E_\mu + E_\nu$$

$$p_\mu = p_\nu = E_\nu / c$$

$$(m_\mu c^2)^2 = E_\mu^2 - (p_\mu c)^2 = (m_K c^2 - p_\nu c)^2 - (p_\nu c)^2$$

$$p_\nu c = \frac{m_K^2 - m_\mu^2}{2m_K} c^2 = 235\ \text{MeV}$$

$$\therefore\ E_\nu = 235\,\text{MeV}$$

$$K_\mu = 152\ \text{MeV}$$

7. Suppose that a particle of mass m is found in a harmonic oscillator potential $V(x) = \frac{1}{2}kx^2$ where k is a positive constant.

a) Show that with an appropriate choice for b the wave function $\psi(x) = \exp(-x^2/2b^2)$ is a solution to Schrödinger's equation and express its energy in terms of the classical oscillation frequency ω. How do you know that this is the ground state? *(10 pts)*

$$\psi = e^{-x^2/2b^2} \Rightarrow \psi' = -\frac{x}{b^2}\psi$$

$$\psi'' = -\frac{1}{b^2}\psi - \frac{x}{b^2}\psi' = -\frac{1}{b^2}\left(1 - \frac{x^2}{b^2}\right)\psi$$

$$-\frac{\hbar^2}{2m}\psi'' + V\psi = E\psi \Rightarrow -\frac{\hbar^2}{2m}\frac{1}{b^2}\left(\frac{x^2}{b^2} - 1\right)\psi + \frac{1}{2}kx^2 = E$$

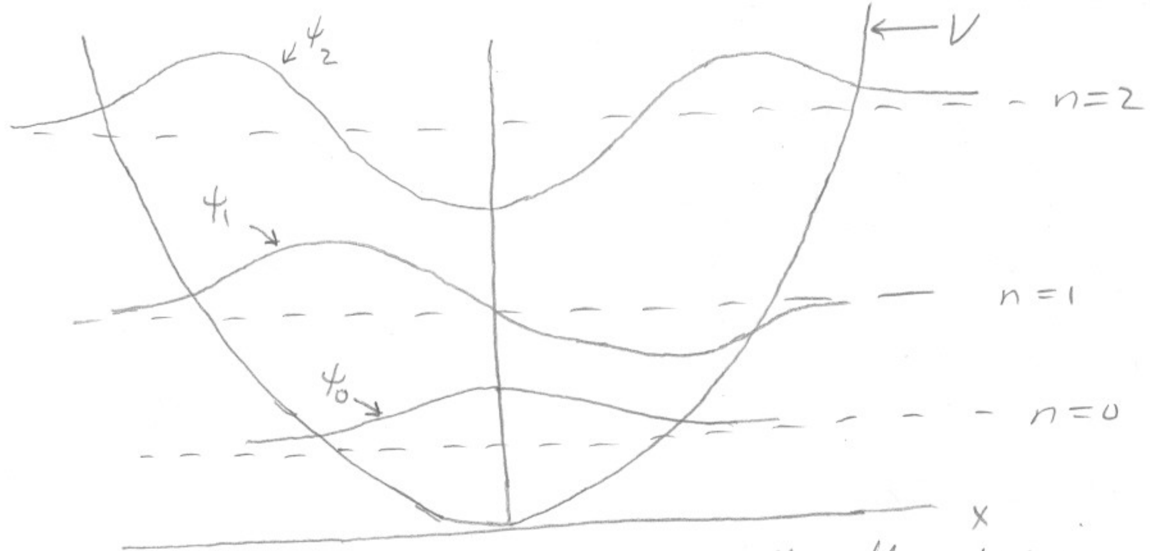
match coefficients for powers of x $\Rightarrow \begin{cases} \frac{\hbar^2}{2m} b^{-4} = \frac{1}{2}k \\ E = \frac{\hbar^2}{2m} b^{-2} \end{cases}$

$$\therefore \quad b = \left(\frac{\hbar^2}{mk}\right)^{1/4}$$

$E = \frac{1}{2}\hbar\omega$ with $\omega = \sqrt{k/m}$ being the classical oscillation frequency

Having no nodes, this is the ground state.

b) Sketch the wave functions for the lowest three states of the harmonic oscillator and explain their basic features. *(10 pts)*



The wavefunctions are oscillatory in the classically allowed region with E>V and decay rapidly in the classically forbidden regions with E<V. The smoothest wavefunction, with no nodes, is the ground state. The energy increases with oscillation frequency, i.e. number of nodes.

8. Describe two phenomena that classical physics failed to describe correctly near the end of the nineteenth century and the new ideas that explained those phenomena and contributed to the development of the early quantum theory. (*10 pts each*)

Possibly useful information

$$\oint d\vec{a}\cdot\vec{E}=\frac{Q}{\varepsilon_0}$$

$$\oint d\vec{a}\cdot\vec{B}=0$$

$$\oint d\vec{s}\cdot\vec{E}=-\frac{d\Phi_B}{dt}$$

$$\oint d\vec{s}\cdot\vec{B}=\mu_0\left(I+\varepsilon_0\frac{d\Phi_E}{dt}\right)$$

$$\vec{D}=\varepsilon\vec{E}=\varepsilon_0\vec{E}+\vec{P}$$

$$\vec{B}=\mu\vec{H}=\mu_0(\vec{H}+\vec{M})$$

static: $\vec{E}=\frac{1}{4\pi\varepsilon_0}\int dq\frac{\hat{r}}{r^2}$

static: $\vec{B}=\frac{\mu_0}{4\pi}\int\frac{I\,d\vec{s}\times\hat{r}}{r^2}$

$$u=\frac{\varepsilon_0E^2}{2}+\frac{B^2}{2\mu_0}$$

$$\vec{S}=\vec{E}\times\vec{B}/\mu_0$$

$$\frac{1}{f}=(n-1)(\frac{1}{R_1}-\frac{1}{R_2})$$

$$\frac{1}{s_o}+\frac{1}{s_i}=\frac{1}{f}$$

single slit: $I(\theta)=I_0\left(\frac{\sin\alpha}{\alpha}\right)^2$, $\alpha=\frac{\pi a}{\lambda}\sin\theta$

grating: $I(\theta)=I_1\left(\frac{\sin N\beta}{\sin\beta}\right)^2$, $\beta=\frac{\pi d}{\lambda}\sin\theta$

$$x'=\gamma(x-ut)$$

$$t'=\gamma(t-\frac{ux}{c^2})$$

$$\gamma=(1-(\tfrac{u}{c})^2)^{-1/2}$$

$$E=\gamma mc^2$$

$$\vec{p}=\gamma m\vec{v}$$

$$(mc^2)^2=E^2-(pc)^2$$

$$-\frac{\hbar^2}{2m}\psi''(x)+V(x)\psi(x)=E\psi(x)$$

$$c^2 = \frac{1}{\varepsilon_0 \mu_0}$$

$$c = 2.9979 \times 10^8 \ \frac{m}{s}$$

$$\varepsilon_0 = 8.8542 \times 10^{-12} \ \frac{F}{m}$$

$$\mu_0 = 4\pi \times 10^{-7} \ \frac{N}{A^2}$$

$$e = 1.6022 \times 10^{-19} \ C$$

$$N_A = 6.022 \times 10^{23} \ mole^{-1}$$

$$m_e = 9.109 \times 10^{-31} \ kg = 0.511 \, MeV / c^2$$

$$m_p = 1.672 \times 10^{-27} \ kg = 938.27 \, MeV / c^2$$

$$\mu_B = 9.27 \times 10^{-24} \ \frac{J}{T}$$

$$\mu_N = 5.05 \times 10^{-27} \ \frac{J}{T}$$

$$h = 6.626 \times 10^{-34} \ J \cdot s$$

$$\hbar = 1.055 \times 10^{-34} \ J \cdot s$$

$$\hbar c = 197.3 \text{ eV nm}$$

$$\frac{e^2}{4\pi\varepsilon_0 \hbar c} = \frac{1}{137}$$

Dimensions and SI Units for Basic Electromagnetic Quantities

Quantity	**Unit**	**Abbreviation**	**Conversions**	**Dimensions**
Charge	Coulomb	*C*		Q
Electric potential	Volt	*V*	*J/C*	$ML^2T^{-2}Q^{-1}$
Electric field	Volt/meter	*V/m*	*N/C*	$MLT^{-2}Q^{-1}$
Capacitance	Farad	*F*	*C/V*	$M^{-1}L^{-2}T^2Q^2$
Current	Ampere	*A*	*C/s*	QT^{-1}
Magnetic field	Tesla	*T*	*N /(A m)*	$MT^{-1}Q^{-1}$
Magnetic flux	Weber	*Wb*	*J/A*	$ML^2T^{-1}Q^{-1}$
Inductance	Henry	*H*	J/A^2	ML^2Q^{-2}
Resistance	ohm	Ω	*V/A*	$ML^2T^{-1}Q^{-2}$