# Error Propagation

Suppose that we make $N$ observations of a quantity $x$ that is subject to random fluctuations or measurement errors. Our best estimate of the true value for this quantity is then $\overline{x} \pm \sigma_x$ where

$$\overline{x} = \frac{1}{N} \sum_{i=1}^{N} x_i \, , \qquad {\sigma_x}^2 = \frac{1}{N-1} \sum_{i=1}^{N} (x_i - \overline{x})^2$$

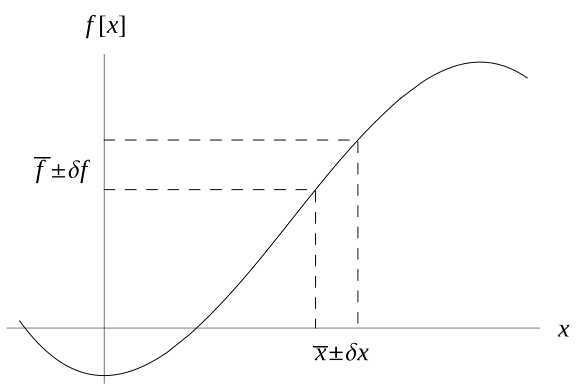
are the sample mean and variance. Next, suppose that we compute a derived quantity $f[x]$. Our best estimate for this quantity is $\overline{f} = f[\overline{x}]$, but what is the uncertainty in this quantity? This question falls under the heading of *error propagation*. Let us assume that $\sigma_x$ is small enough to use a linear approximation to $f[x]$ near $\overline{x}$, such that

$$f - \overline{f} \approx \frac{\partial f}{\partial x} (x - \overline{x})$$

where the derivative is evaluated at $\overline{x}$. Thus, if we expect the true value of $x$ to lie in the range $\overline{x} \pm \delta x$, the true value for $f[x]$ should lie in the corresponding range $\overline{f} \pm \delta f$ where

$$\delta f = \left| \frac{\partial f}{\partial x} \right| \delta x$$

Under the present circumstances we would interpret the uncertainty $\delta x$ as the standard deviation $\sigma_x$, but often we must estimate this quantity by other means. As illustrated in the figure below, the steeper is $f[x]$ near $\overline{x}$ the larger is $\delta f$ for given $\delta x$; this follows simply from the definition of derivative as rate of change. Note that we use the absolute value because uncertainties are expressed as positive numbers giving the width of an interval.




Now suppose that we measure two variables $x$ and $y$ and wish to compute $f = f[x, y]$. How accurately do we know $f$ if the uncertainties in the measured quantities are $\delta x$ and $\delta y$? A crude estimate might simply add two contributions analogous to the one-dimensional result above. However, this is likely to be an overestimate because sometimes the fluctuations in the two variables will add and sometimes they will subtract. A more realistic estimate requires statistical analysis. The simplest situation occurs when $x$ and $y$ are normally distributed random variables. Our best estimates of the true values for these parameters are then their mean values, $\overline{x}$ and $\overline{y}$, with uncertainties given by their standard deviations,

$\sigma_x$ and $\sigma_y$. Furthermore, we again assume that the uncertainties are small enough to approximate variations in $f[x, y]$ as linear with respect to variation of these variables, such that

$$f - \overline{f} \approx \frac{\partial f}{\partial x}(x - \overline{x}) + \frac{\partial f}{\partial y}(y - \overline{y})$$

where the partial derivatives are evaluated at $(\overline{x}, \overline{y})$. If we perform many measurements, the variance of $f$ becomes

$$\sigma_f{}^2 = \frac{1}{N-1}\sum_{i=1}^{N}(f_i - \overline{f})^2 = \frac{1}{N-1}\sum_{i=1}^{N}\left(\frac{\partial f}{\partial x}(x_i - \overline{x}) + \frac{\partial f}{\partial y}(y_i - \overline{y})\right)^2$$

Expanding this expression

$$\sigma_f{}^2 = \frac{1}{N-1}\left(\left(\frac{\partial f}{\partial x}\right)^2\sum_{i=1}^{N}(x_i - \overline{x})^2 + \left(\frac{\partial f}{\partial y}\right)^2\sum_{i=1}^{N}(y_i - \overline{y})^2 + 2\,\frac{\partial f}{\partial x}\,\frac{\partial f}{\partial y}\sum_{i=1}^{N}(x_i - \overline{x})(y_i - \overline{y})\right)$$

and identifying its terms, we find

$$\sigma_f{}^2 = \left(\frac{\partial f}{\partial x}\right)^2\sigma_x{}^2 + \left(\frac{\partial f}{\partial y}\right)^2\sigma_y{}^2 + 2\,\frac{\partial f}{\partial x}\,\frac{\partial f}{\partial y}\,\sigma_{x,y}$$

where

$$\sigma_x{}^2 = \frac{1}{N-1}\sum_{i=1}^{N}(x_i - \overline{x})^2\,, \qquad \sigma_y{}^2 = \frac{1}{N-1}\sum_{i=1}^{N}(y_i - \overline{y})^2$$

are the sample variances for each variable and

$$\sigma_{x,y} = \frac{1}{N-1}\sum_{i=1}^{N}(x_i - \overline{x})(y_i - \overline{y})$$

is their *covariance*.

This analysis can readily be generalized to an arbitrary number of variables. Let $x = \{x_\mu, \mu = 1, m\}$ represent the set of variables, such that

$$\sigma_f{}^2 = \sum_{\mu,\nu=1}^{m}\frac{\partial f}{\partial x_\mu}\,\sigma_{\mu,\nu}\,\frac{\partial f}{\partial x_\nu}$$

where

$$\sigma_{\mu,\nu} = \frac{1}{N-1}\sum_{i=1}^{N}(x_{\mu,i} - \overline{x}_\mu)(x_{\nu,i} - \overline{x}_\nu)$$

is the covariance matrix and $x_{\mu,i}$ is the $i^{\text{th}}$ measurement of $x_\mu$. Note that the diagonal elements of the covariance matrix, $\sigma_{\mu,\mu} = \sigma_\mu{}^2$, are simply variances for each variable.

The covariance measures the tendency for fluctuations of one variable to be related to fluctuations of another. A closely related quantity is the *correlation*

$$C_{x,y} = \frac{\sigma_{x,y}}{\sigma_x \sigma_y} \Longrightarrow -1 \le C_{x,y} \le 1$$

which is normalized to the range $-1 \le C_{x,y} \le 1$. If a positive deviation in $x$ (such that $x_i - \overline{x} > 0$) is more likely to be accompanied by a positive deviation in $y$, then $C_{x,y}$ will be positive, whereas $C_{x,y}$ would be negative if a positive deviation in one variable is likely to be accompanied by a negative deviation in the other. If the deviations in one variable are equally likely to be accompanied by deviations of either sign in the other variable , the sum of products of fluctuations will tend to average to zero and $C_{x,y}$ will be small. Thus, when $C_{x,y}$ is negligibly small, the variables $x$ and $y$ are described as *statistically independent* or as *uncorrelated*.

It is often impractical to repeat measurements many times. We must then estimate the uncertainties in various quantities by other means. For example, if we are using a ruler, the uncertainty in length will be about half the smallest division. In the absence of contrary information, we usually assume that random fluctuations in different quantities are independent and omit the covariance. The error propagation formula then reduces to

$$(\delta f)^2 = \left(\frac{\partial f}{\partial x}\,\delta x\right)^2 + \left(\frac{\partial f}{\partial y}\,\delta y\right)^2 + \cdots$$

where we use the notation $\delta x$ to represent an uncertainty instead of $\sigma_x$ because we use an estimated uncertainty instead of an observed variance. This formula can be extended to an arbitrary number of statistically independent variables whose contributions to the net uncertainty are said to *add in quadrature* because fluctuations sometimes add and sometimes subtract. Each term is a partial uncertainty determined by the uncertainty in one variable and the rate of change with respect to that variable. Notice that if the partial uncertainties vary significantly in size, only the largest contributions matter because squaring before adding strongly emphasizes the larger terms. For example, suppose that $\delta f$ consists of six contributions where one term is five units and the other five terms are each one unit, such that

$$(\delta f)^2 = (5)^2 + (1)^2 + (1)^2 + (1)^2 + (1)^2 + (1)^2 \Longrightarrow \delta f = 5.5$$

The total contribution of the five smaller terms is only one tenth the contribution of the single largest term. Thus, the net uncertainty

$$\delta f \le \left|\frac{\partial f}{\partial x}\,\delta x\right| + \left|\frac{\partial f}{\partial y}\,\delta y\right| + \cdots$$

is less than the linear sum of partial uncertainties. When designing an experiment, identify the dominant partial uncertainties and attempt to minimize them; the smaller terms do not require as much attention if their contribution to the quadrature sum is negligible.

A couple of simple examples are listed below. Here $a$, $b$, $m$, $n$, $\lambda$ are considered exact numbers while $x$, $y$ are experimental quantities measured with finite precision.

$$f = a\,x + b\,y \Longrightarrow (\delta f)^2 = (a\,\delta x)^2 + (b\,\delta y)^2$$

$$f = a\,x^m\,y^n \Longrightarrow \left(\frac{\delta f}{f}\right)^2 = \left(m\,\frac{\delta x}{x}\right)^2 + \left(n\,\frac{\delta y}{y}\right)^2$$

$$f = a\,\mathrm{Exp}[-\lambda\, x] \implies \delta f = |\lambda\, f|\,\delta x$$

### ■ example: measuring *g* with a pendulum

The period, $T$, of a simple pendulum is related to its length, $L$, by

$$T = 2\pi\sqrt{\frac{L}{g}}$$

Therefore, if we measure $L$ and $T$, we can deduce the gravitational acceleration, $g$, using

$$g = 4\pi^2 \frac{L}{T^2} \implies \frac{\delta g}{g} = \sqrt{\left(\frac{\delta L}{L}\right)^2 + \left(\frac{2\,\delta T}{T}\right)^2}$$

Notice that the relative uncertainty in the period carries a greater weight than the precision of the length because it enters the formula for $g$ with a larger power.

### ■ example: mean of *N* measurements

Suppose that we make $N$ observations of the random variable $x$ and compute its mean value

$$\overline{x} = \frac{1}{N}\sum_{i=1}^{N} x_i$$

What is the uncertainty in the mean? If we assume that the uncertainty, $\delta x$, is the same for each observation, then

$$(\delta\,\overline{x})^2 = \sum_{i=1}^{N}\left(\frac{\partial\,\overline{x}}{\partial x_i}\,\delta x_i\right)^2 = \sum_{i=1}^{N}\left(\frac{\delta x_i}{N}\right)^2 = \frac{(\delta x)^2}{N} \implies \delta\,\overline{x} = \frac{\delta x}{\sqrt{N}}$$

Therefore, the uncertainty in the mean is smaller than the uncertainty in a single observation by a factor of $1\big/\sqrt{N}$.

### ■ example: weighted mean

Next consider the weighted mean

$$\overline{x} = \sum_{i=1} \frac{x_i}{\sigma_i^2} \Big/ \sum_i \frac{1}{\sigma_i^2}$$

where the uncertainties in each observation may be different. Notice that the terms with the smallest uncertainty carry the most weight. The uncertainty in the weighted mean can be evaluated using standard error propagation to be

$$(\sigma_{\overline{x}})^2 = \frac{\sum_{i=1}(\sigma_i^{-2}\,\sigma_i)^2}{(\sum_i \sigma_i^{-2})^2} \implies \sigma_{\overline{x}} = \sqrt{\frac{1}{\sum_i \sigma_i^{-2}}}$$

Notice that if uncertainties are uniform, such that $\sigma_i = \delta x$ is the same for each observation, then this result is the same as the preceding result for the unweighted average.