Physics for Scientists and Engineers

Chapter 10
Energy

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Introduction to Energy

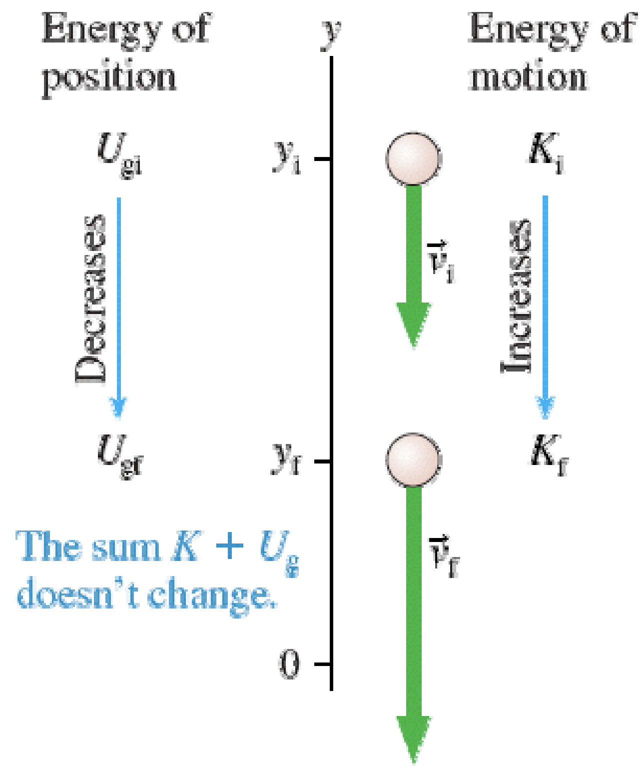
- Energy is one of the most important concepts in science although it is not easily defined
- Every physical process that occurs in the Universe involves energy and energy transfers
- The energy approach to describing motion is particularly useful when the force is *not constant*
- An approach will involve *Conservation of Energy*
  - This could be extended to biological organisms, technological systems and engineering situations

Energy of Falling Object

- Kinematic eq. $v_f^2 = v_i^2 + 2a_y(y_f - y_i)$
  - Gravity is the only force acting
- Multiplying $\frac{1}{2}m$ & putting $a_y = g$,

$$\frac{1}{2}mv_f^2 = \frac{1}{2}mv_i^2 - mg(y_f - y_i)$$

$$\Rightarrow \frac{1}{2}mv_f^2 + mgy_f = \frac{1}{2}mv_i^2 + mgy_i$$



- Define $K = \frac{1}{2}mv^2$: *kinetic energy*

  $U_g = mgy$: *gravitational potential energy*
- Then, $K_f + U_{gf} = K_i + U_{gi}$: *Conservation of Energy*

Kinetic Energy

- Kinetic energy is the energy associated with the *motion* of an object
- Kinetic energy is always given by $K = \frac{1}{2}mv^2$, where v is the speed of the object
  - Kinetic energy is a scalar, *independent of direction* of **v**
  - Kinetic energy cannot be negative
- Units of energy: 1 kg $m^2/s^2 \equiv$ 1 Joule
  - Energy has dimensions of $[M\ L^2\ T^{-2}]$

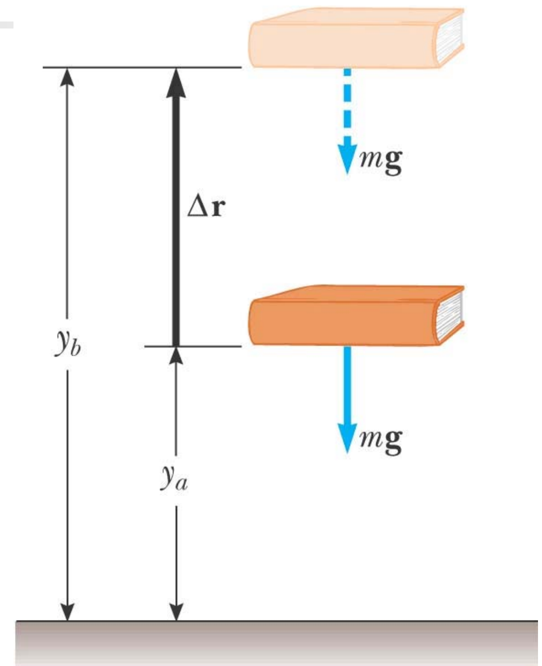
Potential Energy

- Potential energy is the energy associated with the *relative position* of objects that exert forces on each other
  - A potential energy can only be associated with specific types of forces, i.e. *conservative forces*
- There are many forms of potential energy: gravitational, electromagnetic, chemical, nuclear
- Gravitational potential energy is associated with the *distance above* Earth's surface: $U_g = mgy$

Total Mechanical Energy

- The sum of kinetic and potential energies is the mechanical energy of the system: $E = K + \sum U_i$
- When *conservative* forces act in an *isolated* system, kinetic energy gained (or lost) by the system as its members change their relative positions is balanced by an equal loss (or gain) in potential energy
  - This is the *Conservation of Mechanical Energy*
  - If there is friction or air drag, the total mechanical energy is *not* conserved, i.e. constant

Gravitational Potential Energy



- The system consists of Earth and a book
- Initially, the book is at rest ($K = 0$) at $y = y_b$ ($U_g = mgy_b$)
- When the book falls and reaches y_a,

$$K_a + mgy_a = 0 + mgy_b$$

$$\Rightarrow K_a = mg\,(y_b - y_a) > 0$$

- The kinetic energy that was converted from potential energy depends only on the *relative* displacement $y_b - y_a$
  - It doesn't matter where we take $y = 0$

Example 1: Launching a Pebble

Bob uses a slingshot to shoot a 20.0 g pebble straight up with a speed of 25.0 m/s.

How high does a pebble go?

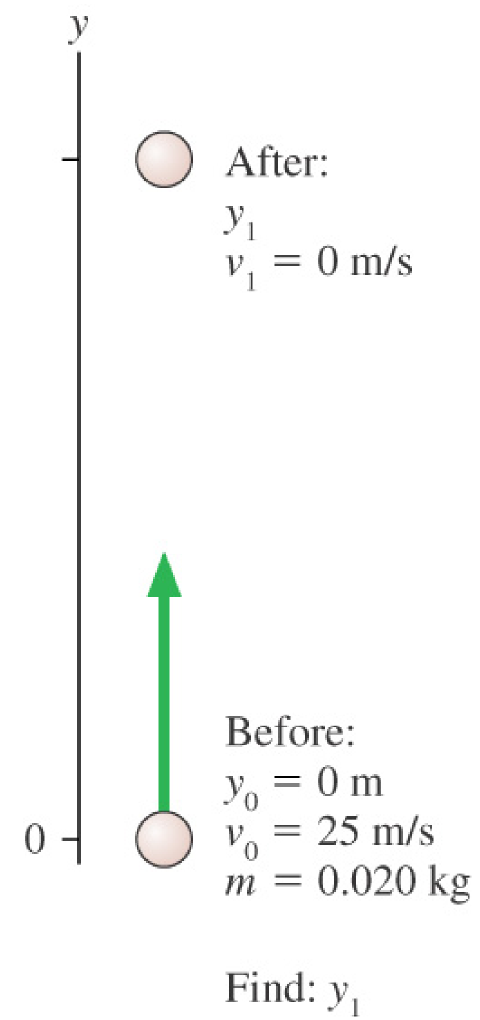
$$\text{Before}: K_b = \frac{1}{2}mv^2 = 6.25\text{ J},\ U_{gb} = 0$$

$$\text{After}: \ K_a = 0,\ U_{ga} = mgy = 0.196\, y_a$$

Using energy conservation, $K_a + U_{ga} = K_b + U_{gb}$

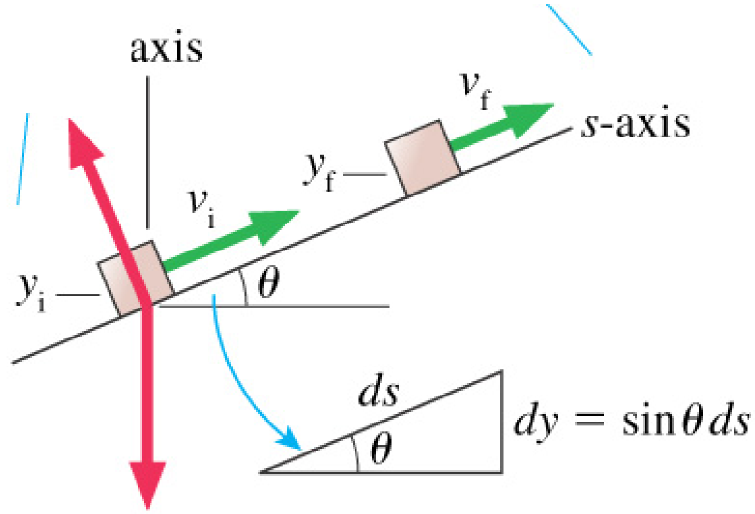
$$0 + 0.196 y_a = 6.25 + 0,\ \text{or}\ y_a = 31.9\text{ m}$$

Kinetic energy was converted into gravitational potential energy by 6.25 J.



Energy Conservation in 2-D

We will now analyze 2-D motion along a frictionless incline.



$$\sum F_s = ma_s = m\frac{dv_s}{dt},$$

$$-mg\sin\theta = m\frac{dv_s}{ds}\frac{ds}{dt} = mv_s\frac{dv_s}{ds}$$

Multiplying both sides by ds, $-mg\sin\theta ds = mv_s dv_s$

But $\sin\theta ds = dy$. Integrating from initial to final positions,

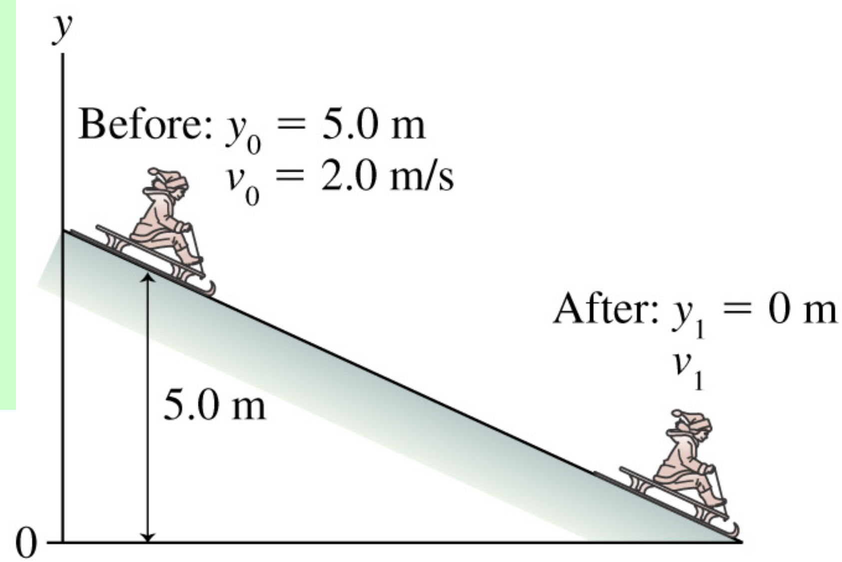
$$\frac{1}{2}mv_f^2 + mgy_f = \frac{1}{2}mv_i^2 + mgy_i$$

This is the same equation as in a 1-D motion.

Example 2: Downhill Sledding

Christine runs forward with her sled at 2.0 m/s. She hops onto the sled at the top of a 5.0 m high, very slippery slope.

What is her speed at the bottom?



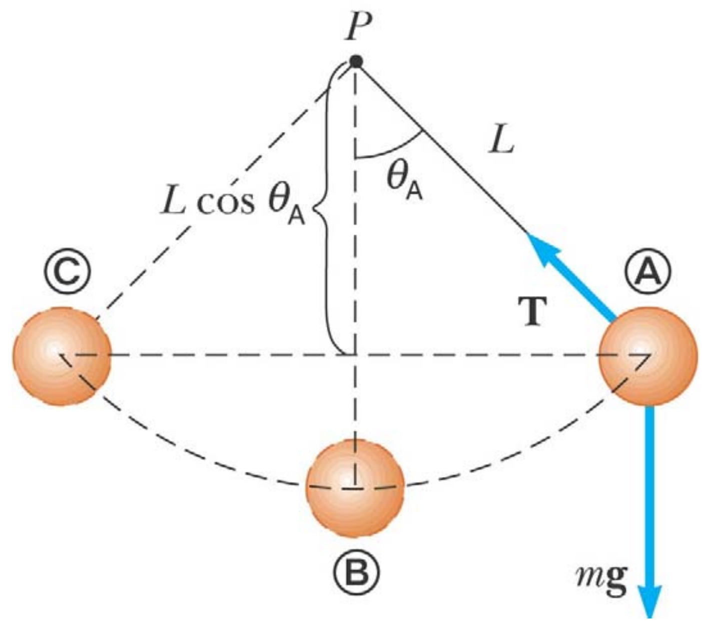
Using the conservation of energy,

$$\frac{1}{2}mv_0^2 + mgy_0 = \frac{1}{2}mv_1^2 + mgy_1$$

$$v_1 = \sqrt{v_0^2 + 2g(y_0 - y_1)} = 10\text{ m/s}$$

Notice that (1) the mass cancelled out and (2) only Δy is needed.
We could have taken the bottom of the hill as y = 10.0 m and the top as y = 15.0 m, and gotten the same answer.

Energy Conservation: Pendulum

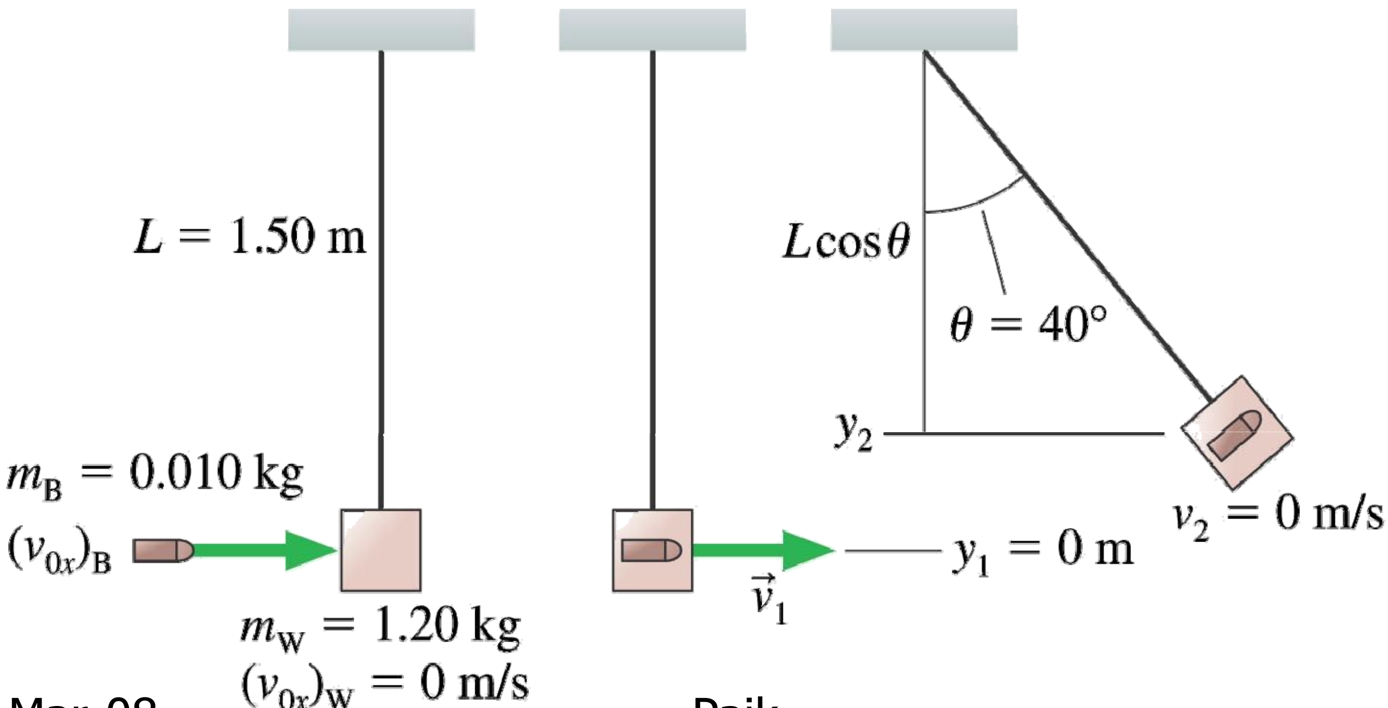


- As the pendulum swings, there is a continuous exchange between potential and kinetic energies
- At A, all the energy is potential
- At B, all of the potential energy at A is transformed into kinetic energy
    - Let zero potential energy be at B
- At C, the kinetic energy has been transformed back into potential energy

Example 3: Ballistic Pendulum

A 10 g bullet is fired into a 1200 g wood block hanging from a 150 cm long string. The bullet embeds itself into the block, and the block then swings out to an angle of 40°.

What was the speed of the bullet?



Example 3, cont

This problem has two parts: (a) the bullet and the block make a perfectly inelastic collision, in which *momentum* is conserved, and (b) the total mass as a system undergoes a pendulum motion, in which *energy* is conserved.

(a) Momentum conservation: $(m_w + m_b)v_1 = m_w v_{w0} + m_b v_{b0}$

Since $v_{w0} = 0,\ v_{b0} = \dfrac{m_w + m_b}{m_b} v_1$

(b) Energy conservation: $\frac{1}{2} m_{tot} v_2^2 + m_{tot} g y_2 = \frac{1}{2} m_{tot} v_1^2 + m_{tot} g y_1$

Since $v_2 = 0$ and we can choose $y_1 = 0,\ v_1 = \sqrt{2 g y_2} = \sqrt{2gL(1 - \cos\theta)}$

Therefore, $v_{b0} = \dfrac{m_w + m_b}{m_b} \sqrt{2gL(1 - \cos\theta)}$

$$= \frac{1.210\ \text{kg}}{0.010\ \text{kg}} \sqrt{2(9.80\ \text{m/s}^2)(1.50\ \text{m})(1 - 0.766)} = 320\ \text{m/s}$$

Example 4: Roller Coaster

A roller-coaster car is released from rest at the top of the first rise and then moves freely with negligible friction. The roller coaster has a circular loop of radius *R* in a vertical plane.

(a) Suppose first that the car barely makes it around the loop: at the top of the loop the riders are upside down and feel weightless, i.e. $n = 0$. Find the height of the release point above the bottom of the loop, in terms of R so that $n = 0$.

Example 4, cont

(a) At the top of the loop, the car and riders are in free fall. From Newton's 2nd law,

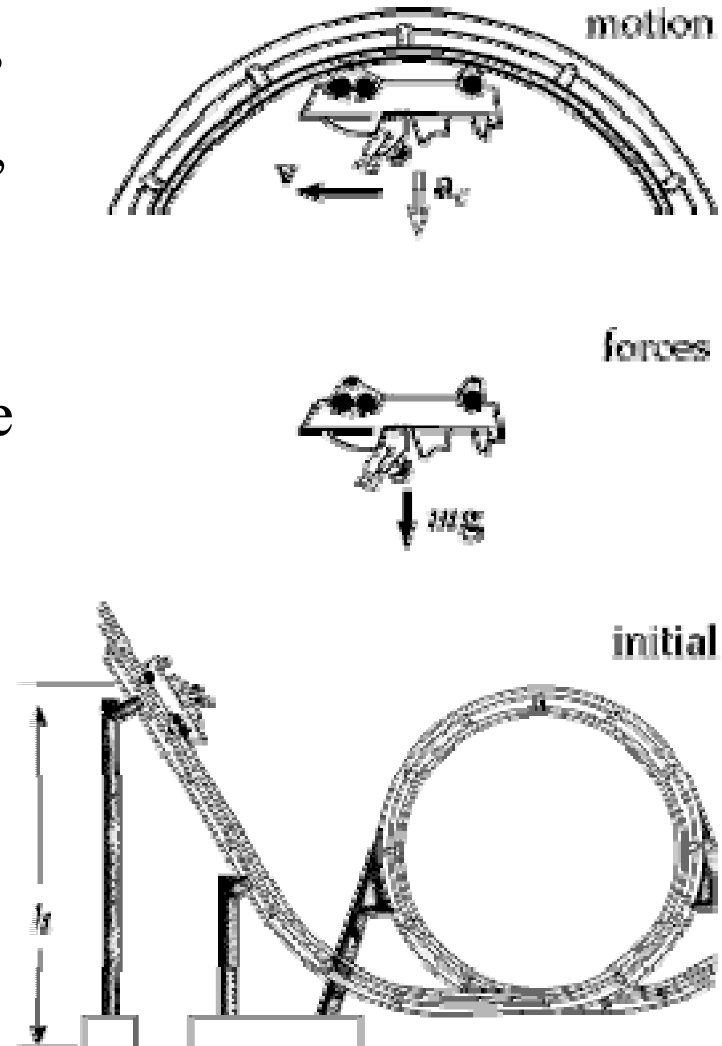
$$-mg = -m\frac{v^2}{R} \;\Rightarrow\; v = \sqrt{gR}$$

Energy is conserved between the release and the top of the loop:

$$K_i + U_{gi} = K_f + U_{gf}$$

$$0 + mgh = \frac{1}{2}mv^2 + mg(2R)$$

$$gh = \frac{1}{2}gR + 2gR \;\Rightarrow\; h = 2.5R$$



Example 4, cont

(b) Show that the normal force on the car at the bottom of the loop exceeds the normal force at the top of the loop by six times the weight of the car.

(b) Energy conservation : (bottom) $mgh = \frac{1}{2}mv_b^2$ (top) $mgh = \frac{1}{2}mv_t^2 + mg(2R)$

$$\Rightarrow \quad v_b^2 = 2gh \qquad v_t^2 = 2gh - 4gR$$

Newton's 2nd law : (bottom) $n_b - mg = +m\frac{v_b^2}{R}$ (top) $-n_t - mg = -m\frac{v_t^2}{R}$

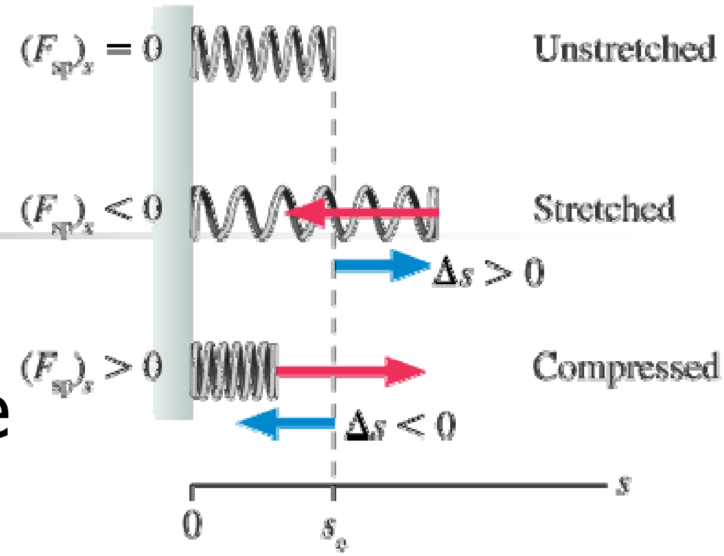
Substituting v_b^2 and v_t^2, $n_b = mg\left(1 + 2h/R\right)$ $n_t = mg\left(-5 + 2h/R\right)$

$$\Rightarrow \quad n_b - n_t = 6mg$$

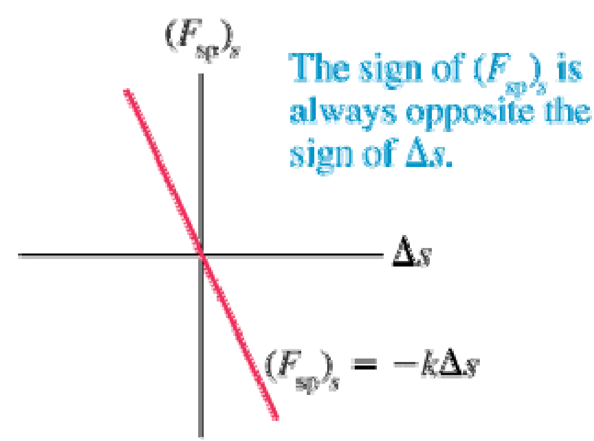
The normal force on each rider follows the same rule. Such a large normal force is very uncomfortable for the riders. Roller coasters are therefore built *helical* so that some of the gravitational energy goes into axial motion.

Hooke's Law

- Force exerted by a spring is always directed opposite to the displacement from equilibrium



- Hooke's Law: $F_s = -k\Delta s$
  - Δs is the displacement *from the equilibrium position*
  - k is the spring constant and measures the stiffness of the spring
  - F is called the *restoring force*



- If it is released, it will oscillate back and forth

Measuring the Force Constant

$$\sum F_y = F_s - mg = ma_y$$

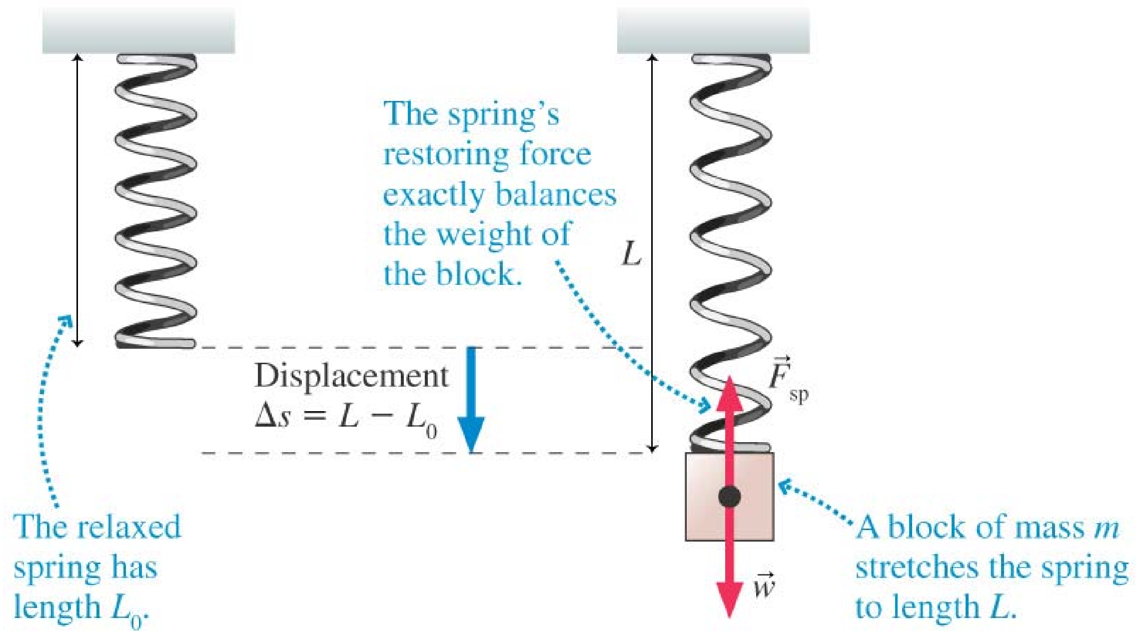
At equilibrium,

$$a_y = 0, \; -k\Delta s - mg = 0$$

Therefore,

$$k = \frac{mg}{\Delta s} \; (\text{N/m})$$

Measure m and Δs to determine k.



Energy of Object on a Spring

$$\sum F_s = ma_s, \; -k(s-s_e) = m\frac{dv_s}{dt}$$

Use the chain rule,

$$-k(s-s_e) = m\frac{dv_s}{ds}\frac{ds}{dt} = mv_s\frac{dv_s}{ds}$$

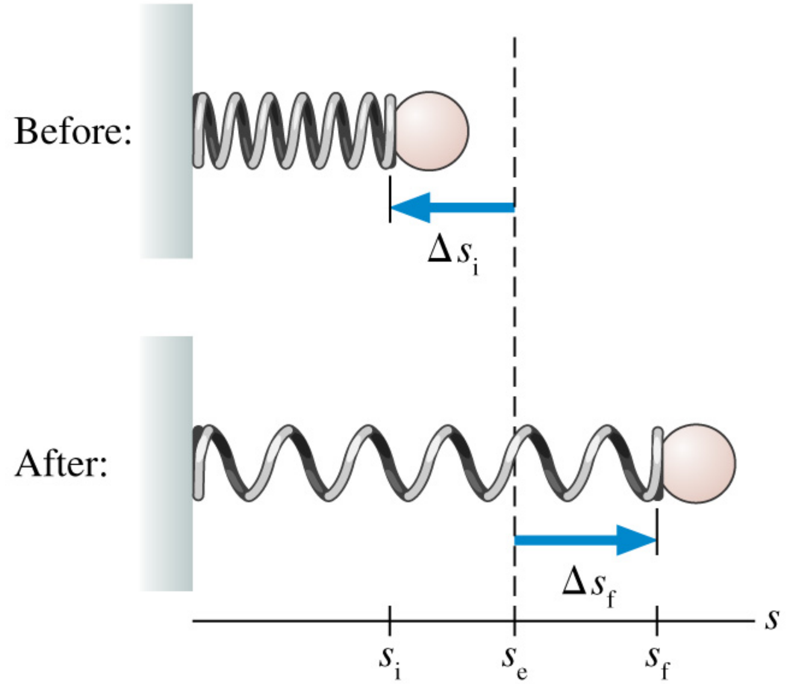
or $-k(s-s_e)ds = mv_s dv_s$

Define $u = s - s_e = \Delta s$, hence $du = ds$.

Integrating,

$$-\int_{s_i}^{s_f} k(s-s_e)ds = -\int_{\Delta s_i}^{\Delta s_f} kudu = -\frac{1}{2}ku^2\Big|_{\Delta s_i}^{\Delta s_f} = -\frac{1}{2}k(\Delta s_f)^2 + \frac{1}{2}k(\Delta s_i)^2$$

$$\int_{v_i}^{v_f} mv_s dv_s = \frac{1}{2}mv_f^2 - \frac{1}{2}mv_i^2 \Rightarrow \frac{1}{2}mv_i^2 + \frac{1}{2}k(\Delta s_i)^2 = \frac{1}{2}mv_f^2 + \frac{1}{2}k(\Delta s_f)^2$$



Elastic Potential Energy

- Potential energy of an object attached to a spring is

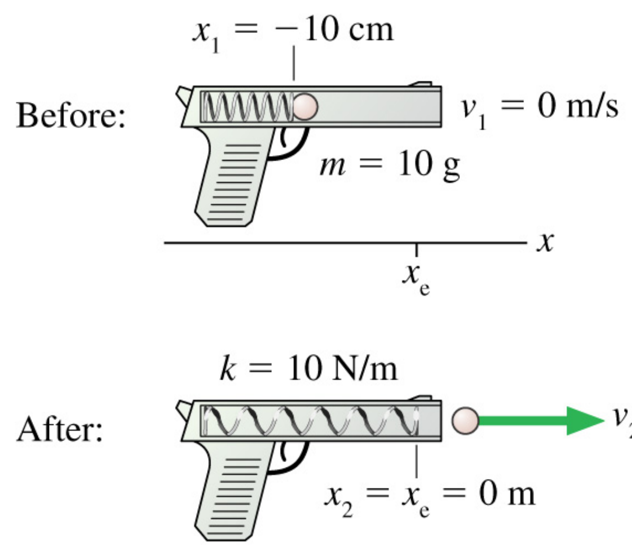
$$U_s = \tfrac{1}{2}\, k(\Delta s)^2$$

  - The elastic potential energy depends on the *square* of the displacement from the spring's equilibrium position
  - The same amount of potential energy can be converted to kinetic energy if the spring is *extended* or *contracted* by the same displacement

Example 5: Spring Gun Problem

A spring-loaded toy gun launches a 10 g plastic ball. The spring, with spring constant 10 N/m, is compressed by 10 cm as the ball is pushed into the barrel. When the trigger is pulled, the spring is released and shoots the ball out.

What is the ball's speed as it leaves the barrel? Assume friction is negligible.



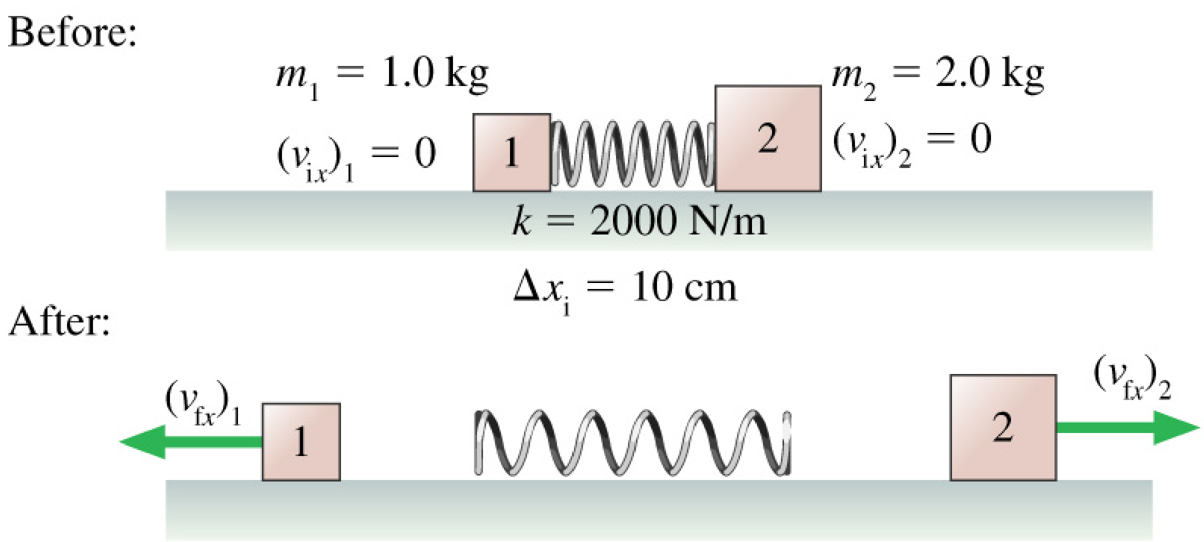
$$\frac{1}{2}mv_i^2 + \frac{1}{2}k\left(\Delta x_i\right)^2 = \frac{1}{2}mv_f^2 + \frac{1}{2}k\left(\Delta x_f\right)^2, \quad 0 + \frac{1}{2}k\left(x_1 - x_e\right)^2 = \frac{1}{2}mv_f^2 + \ 0$$

$$v_f = \sqrt{\frac{k(x_1 - x_e)^2}{m}} = \sqrt{\frac{(10\ \text{N/m})(-0.10\ \text{m} - 0\ \text{m})^2}{0.010\ \text{kg}}} = 3.16\ \text{m/s}$$

Example 6: Pushing Apart

A spring with spring constant 2000 N/m is sandwiched between a 1.0 kg block and a 2.0 kg block on a frictionless table. The blocks are pushed together to compress the spring by 10 cm, then released.

What are the velocities of the blocks as they fly apart?



Example 6, cont

From energy conservation,

$$\tfrac{1}{2}mv_{1,i}^2+\tfrac{1}{2}mv_{2,i}^2+\tfrac{1}{2}k(\Delta x_i)^2=\tfrac{1}{2}mv_{1,f}^2+\tfrac{1}{2}mv_{2,f}^2+\tfrac{1}{2}k(\Delta x_f)^2$$

$$0+0+\tfrac{1}{2}k(\Delta x_i)^2=\tfrac{1}{2}mv_{1,f}^2+\tfrac{1}{2}mv_{2,f}^2+0$$

From momentum conservation,

$$m_1v_{1,i}+m_2v_{2,i}=m_1v_{1,f}+m_2v_{2,f},\ 0=m_1v_{1,f}+m_2v_{2,f}\Rightarrow v_{1,f}=-\frac{m_2}{m_1}v_{2,f}$$

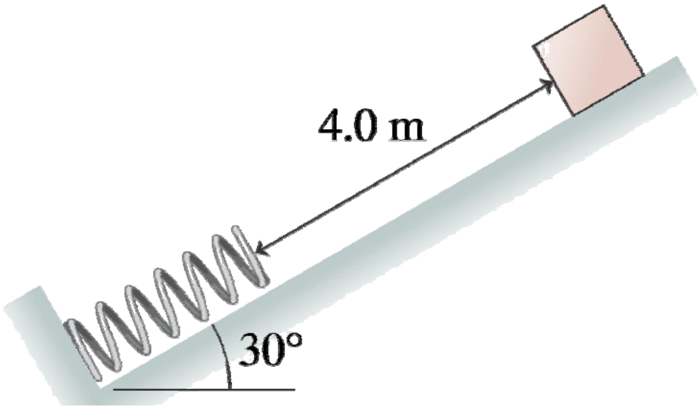
Substituting this into the energy equation,

$$\tfrac{1}{2}m_1\left(\frac{m_2}{m_1}v_{2,f}\right)^2+\tfrac{1}{2}m_2v_{2,f}^2=\tfrac{1}{2}k(\Delta x_1)^2,\ m_2\left(1+\frac{m_2}{m_1}\right)v_{2,f}^2=k(\Delta x_1)^2$$

$$v_{2,f}=\sqrt{\frac{k(\Delta x_1)^2}{m_2(1+m_2/m_1)}}=1.8\text{ m/s},\ v_{1,f}=-\frac{m_2}{m_1}v_{2,f}=-3.6\text{ m/s}$$

Example 7: Spring and Gravity

A 10 kg box slides 4.0 m down the frictionless ramp shown in the figure, then collides with a spring whose spring constant is 250 N/m.

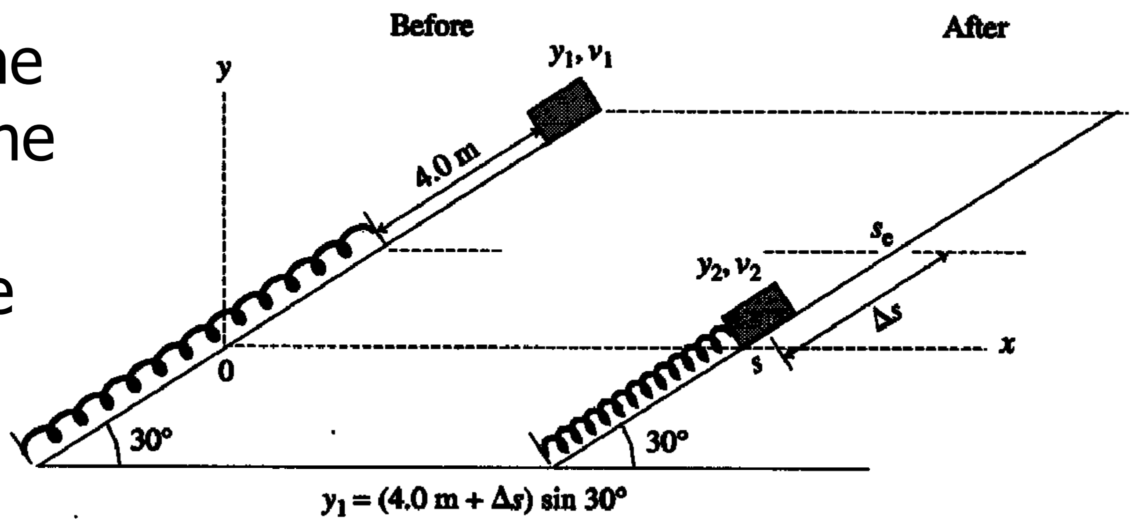


(a) What is the maximum compression of the spring?

(b) At what compression of the spring does the box have its maximum velocity?

Example 7, cont

(a) Choose the origin of the coordinate system at the point of maximum compression. Take the coordinate along the ramp to be *s*.



$$K_2 + U_{s2} + U_{g2} = K_1 + U_{s1} + U_{g1}$$

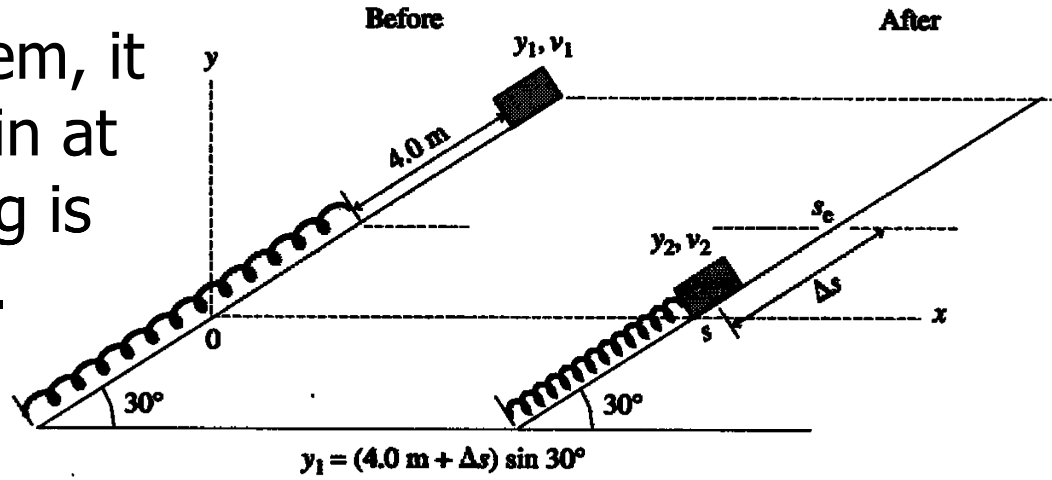
$$0 + \tfrac{1}{2}k(\Delta s)^2 + 0 = 0 + 0 + mg(4.0\text{ m} + \Delta s)\sin 30^\circ$$

$$(125\text{ N/m})(\Delta s)^2 - (49\text{ kg m/s}^2)\Delta s - 196\text{ kg m}^2/\text{s}^2 = 0$$

$$\Delta s = 1.46\text{ m}, \quad -1.07\text{ m (unphysical)}$$

Example 7, cont

(b) For this part of the problem, it is easiest to take the origin at the point where the spring is at its equilibrium position.



$$K_2 + U_{s2} + U_{g2} = K_1 + U_{s1} + U_{g1}$$

$$\tfrac{1}{2}mv^2 + mgy + \tfrac{1}{2}k(\Delta s)^2 = 0 + mgy_1 + 0$$

$$\tfrac{1}{2}k(\Delta s)^2 - (mg\sin 30^\circ)(\Delta s) + \tfrac{1}{2}mv^2 - mg(4.0\text{ m}\sin 30^\circ) = 0$$

To find v_{max} wrt Δs, take the derivative of this equation and put $\frac{dv}{d\Delta s} = 0.$

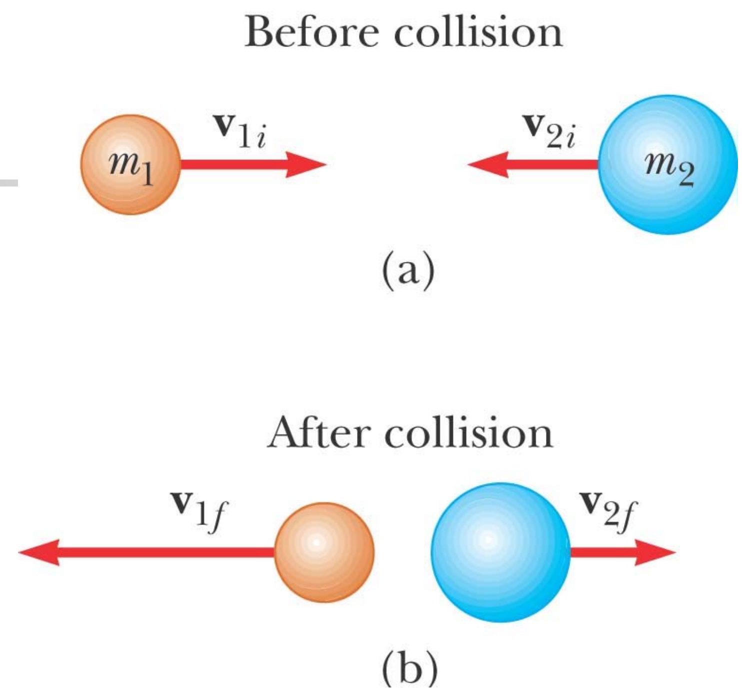
$$k\Delta s - mg\sin 30^\circ + mv\frac{dv}{d\Delta s} = 0,\ \ \Delta s = \frac{mg\sin 30^\circ}{k} = 0.196\text{ m}$$

Elastic Collisions

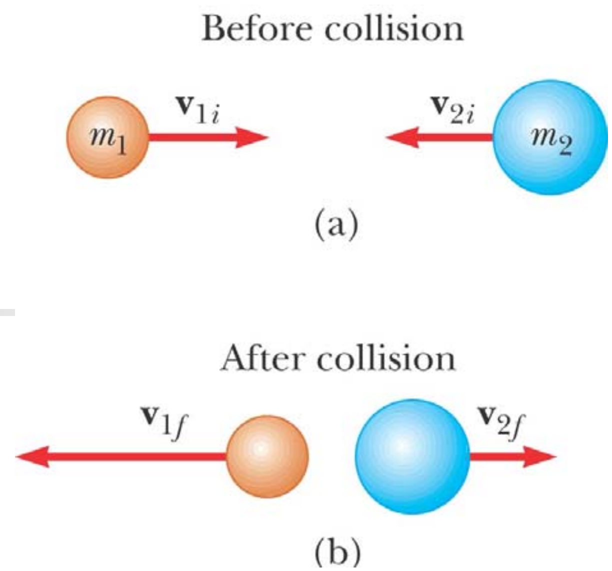
- Both momentum *and* kinetic energy are conserved ($U = 0$)

$$m_1\mathbf{v}_{1i} + m_2\mathbf{v}_{2i} = m_1\mathbf{v}_{1f} + m_2\mathbf{v}_{2f}$$

$$\frac{1}{2}m_1v_{1i}^2 + \frac{1}{2}m_2v_{2i}^2 = \frac{1}{2}m_1v_{1f}^2 + \frac{1}{2}m_2v_{2f}^2$$



- Typically, there are two unknowns ($\mathbf{v}_{2f}$ and $\mathbf{v}_{1f}$), so you need both equations: $\mathbf{p}_i = \mathbf{p}_f$ and $K_i = K_f$
  - The kinetic energy equation can be difficult to use
  - With some algebraic manipulation, the two equations can be used to solve for the two unknowns



1-D Elastic Collisions

$$K_f = K_i : \quad \frac{1}{2}m_1 v_{1f}^2 + \frac{1}{2}m_2 v_{2f}^2 = \frac{1}{2}m_1 v_{1i}^2 + \frac{1}{2}m_2 v_{2i}^2$$

$$\Rightarrow \quad m_1(v_{1f} - v_{1i})(v_{1f} + v_{1i}) = -m_2(v_{2f} - v_{2i})(v_{2f} + v_{2i})$$

$$p_f = p_i : \quad m_1 v_{1f} + m_2 v_{2f} = m_1 v_{1i} + m_2 v_{2i}$$

$$\Rightarrow \quad m_1(v_{1f} - v_{1i}) = -m_2(v_{2f} - v_{2i})$$

Combine the two equations, $(v_{1f} + v_{1i}) = (v_{2f} + v_{2i})$

Solve for v_{1f} or v_{2f} and substituting into the momentum equation,

$$v_{1f} = \left(\frac{m_1 - m_2}{m_1 + m_2}\right)v_{1i} + \left(\frac{2m_2}{m_1 + m_2}\right)v_{2i}, \quad v_{2f} = \left(\frac{2m_1}{m_1 + m_2}\right)v_{1i} + \left(\frac{m_2 - m_1}{m_1 + m_2}\right)v_{2i}$$

1-D Elastic Collisions, cont

$$(1)\ m_1 = m_2:$$

$$v_{1\mathrm{f}} = (0)v_{1\mathrm{i}} + (1)v_{2\mathrm{i}} = v_{2\mathrm{i}}, \quad v_{2\mathrm{f}} = (1)v_{1\mathrm{i}} + (0)v_{2\mathrm{i}} = v_{1\mathrm{i}}$$

$\Rightarrow$ Particles exchange velocities.

$$(2)\ m_1 << m_2, v_{2i} = 0:$$

$$v_{1\mathrm{f}} \approx (-1)v_{1\mathrm{i}} + (2)0 \approx -v_{1\mathrm{i}}, \; v_{2\mathrm{f}} \approx (0)v_{1\mathrm{i}} + (1)0 = 0$$

$\Rightarrow$ m_1 bounces back while m_2 remains stationary

$$(3)\ m_1 >> m_2, v_{2i} = 0:$$

$$v_{1\mathrm{f}} \approx (1)v_{1\mathrm{i}} + (0)0 = v_{1\mathrm{i}}, \; v_{2\mathrm{f}} \approx (2)v_{1\mathrm{i}} + (-1)0 = 2v_{1\mathrm{i}}$$

$\Rightarrow$ m_1 continues to move at $v_{1\mathrm{i}}$ while m_2 moves at $2v_{1\mathrm{i}}$

2-D Elastic Collisions

- Energy conservation:

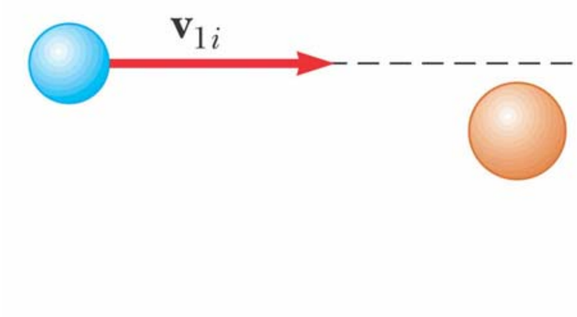
  $\frac{1}{2} m_1 v_{1i}^2 = \frac{1}{2} m_1 v_{1f}^2 + \frac{1}{2} m_2 v_{2f}^2$

- Momentum conservation:

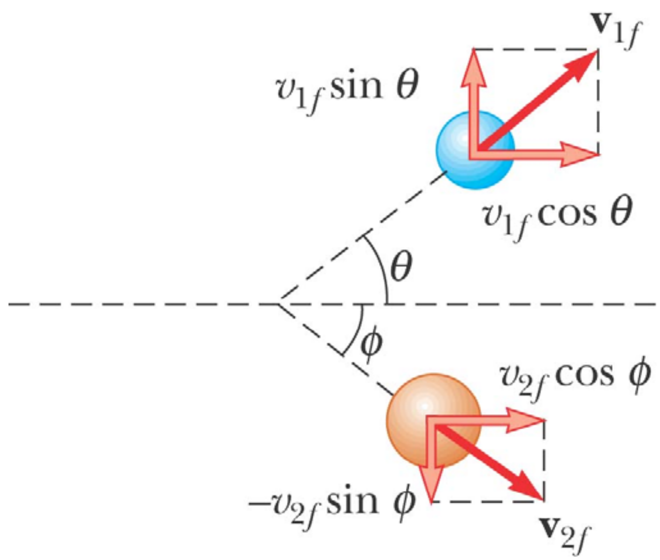
  x: $m_1 v_{1i} = m_1 v_{1f} \cos\theta + m_2 v_{2f} \cos\phi$

  y: $0 = m_1 v_{1f} \sin\theta + m_2 v_{2f} \sin\phi$

- Note that we have 3 equations and 4 unknowns
  - We cannot solve for the unknowns *unless* we have one final angle or final velocity given



(a) Before the collision

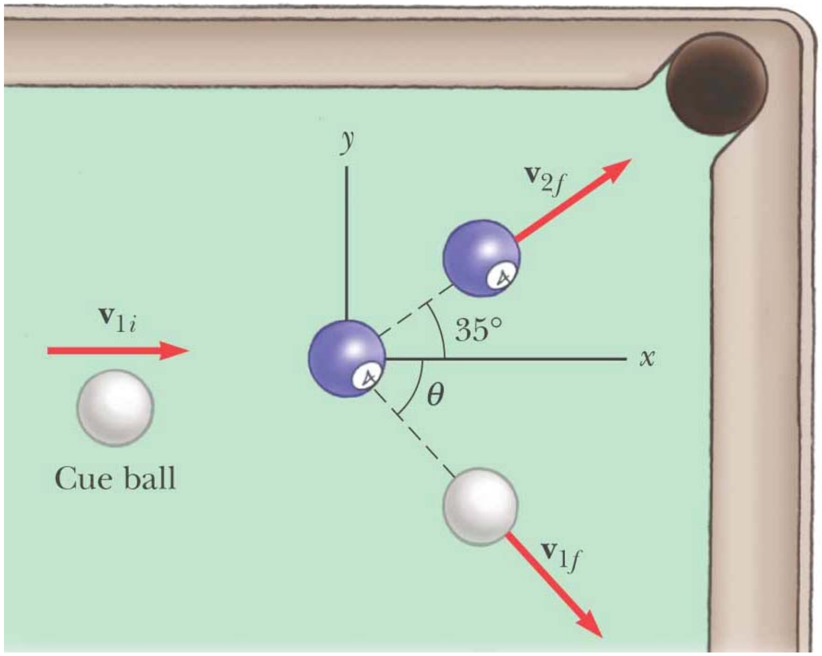


(b) After the collision

Example 8: Billiard Ball

A player wishes to sink target ball in the corner pocket. The angle to the corner pocket is 35°.

At what angle is the cue ball deflected?



Energy conservation :

$$\frac{1}{2}m_1v_{1i}^2 = \frac{1}{2}m_1v_{1f}^2 + \frac{1}{2}m_2v_{2f}^2$$

Momentum conservation :

$$m_1v_{1i} = m_1v_{1f}\cos\theta + m_2v_{2f}\cos 35°$$

$$0 = -m_1v_{1f}\sin\theta + m_2v_{2f}\sin 35°$$

Example 8, cont

Since $m_1 = m_2$,

$$v_{1i}^2 = v_{1f}^2 + v_{2f}^2, \; v_{1i} = v_{1f}\cos\theta + v_{2f}\cos 35°, \; 0 = -v_{1f}\sin\theta + v_{2f}\sin 35°$$

From the third equation, $v_{1f} = v_{2f}\dfrac{\sin 35°}{\sin\theta}$.

Substituting this into the first and second equations,

$$v_{1i}^2 = \left(\frac{\sin^2 35°}{\sin^2\theta} + 1\right) v_{2f}^2, \; v_{1i} = \left(\frac{\sin 35°}{\sin\theta}\cos\theta + \cos 35°\right) v_{2f}$$

From these, $\left(\dfrac{\sin 35°}{\sin\theta}\cos\theta + \cos 35°\right)^2 = \dfrac{\sin^2 35°}{\sin^2\theta} + 1.$

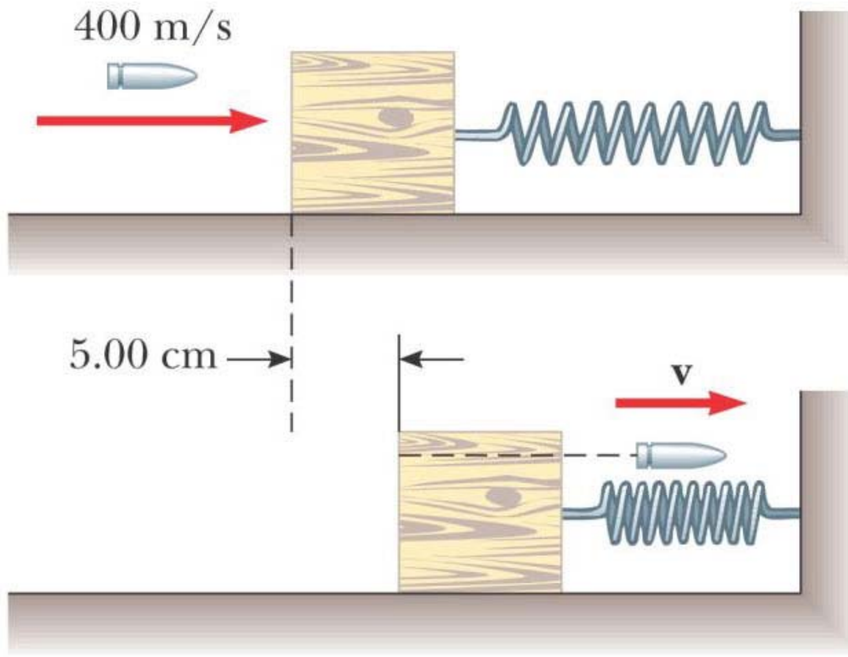
This leads to $\cot\theta = \tan 35°$ or $\theta = 55°$.

Internal Energy

- So far we have considered situations where there is no friction
- In the absence of friction, the mechanical energy is conserved: $E = K + U =$ constant
- *Where friction is present*, some energy is converted to internal energy, usually heat
  - Here $K + E \neq$ constant

Example 9: Bullet Fired into Block

A 5.00-g bullet moving with an initial speed of 400 m/s is fired into and passes through a 1.00-kg block. The block, initially at rest on a frictionless, horizontal surface, is connected to a spring of force constant 900 N/m. The block moves 5.00 cm to the right after impact.



Find (a) the speed at which the bullet emerges from the block and (b) the mechanical energy converted into internal energy in the collision.

Example 9, cont

(a) Energy conservation for the spring: $\frac{1}{2}MV_i^2 = \frac{1}{2}kx^2$

$$V_i = \sqrt{\frac{kx^2}{M}} = \sqrt{\frac{(900\text{ N/m})(0.05\text{ m})^2}{1.00\text{ kg}}} = 1.50\text{ m/s}$$

Momentum conservation in the collision : $mv_i = MV_i + mv$

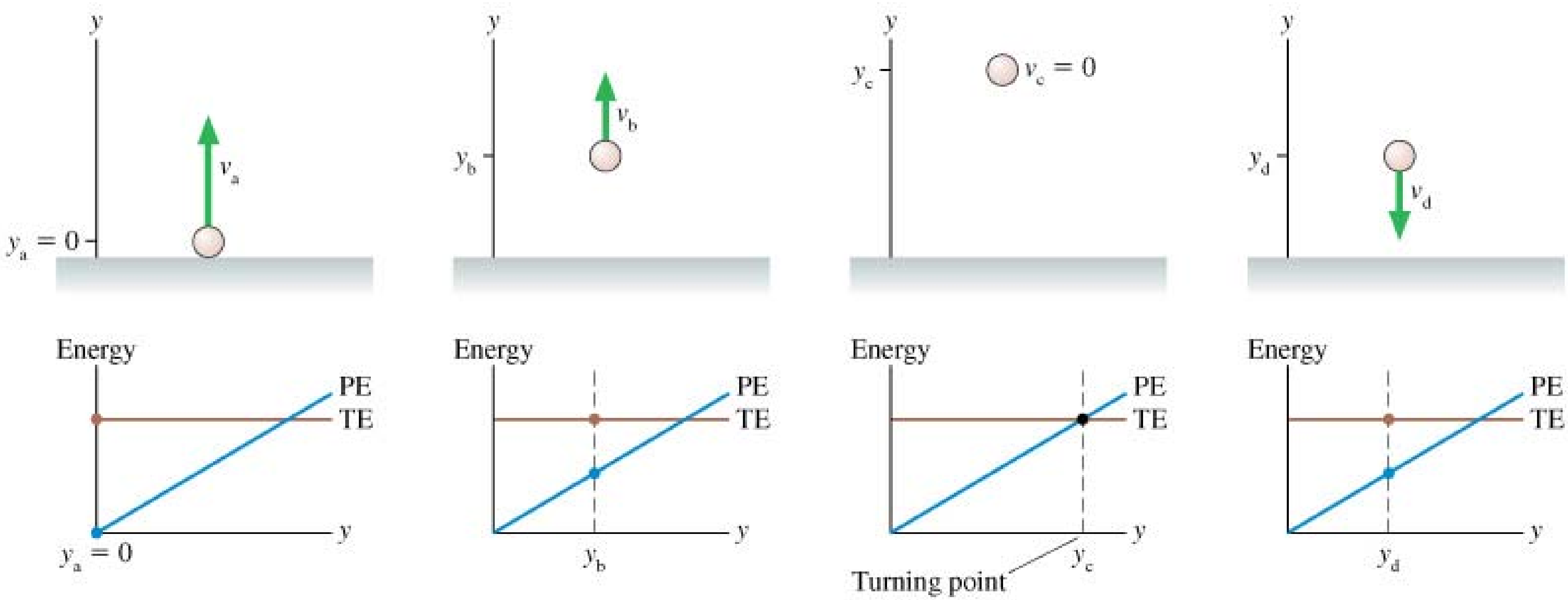
$$v = \frac{mv_i - MV_i}{m} = \frac{(5.00\times10^{-3}\text{kg})(400\text{ m/s}) - (1.00\text{ kg})(1.5\text{ m/s})}{5.00\times10^{-3}\text{kg}} = 100\text{ m/s}$$

(b) $$\Delta E = \Delta K + \Delta U = \tfrac{1}{2}\left(5.00\times10^{-3}\text{ kg}\right)\left[(100\text{ m/s})^2 - (400\text{ m/s})^2\right] + \tfrac{1}{2}\left(900\text{ N/m}\right)\left[\left(5.00\times10^{-2}\text{m}\right)^2 - (0\text{ m})^2\right] = -374\text{ J}$$

The lost energy is due to the friction between the bullet and block. The block heats up a little.

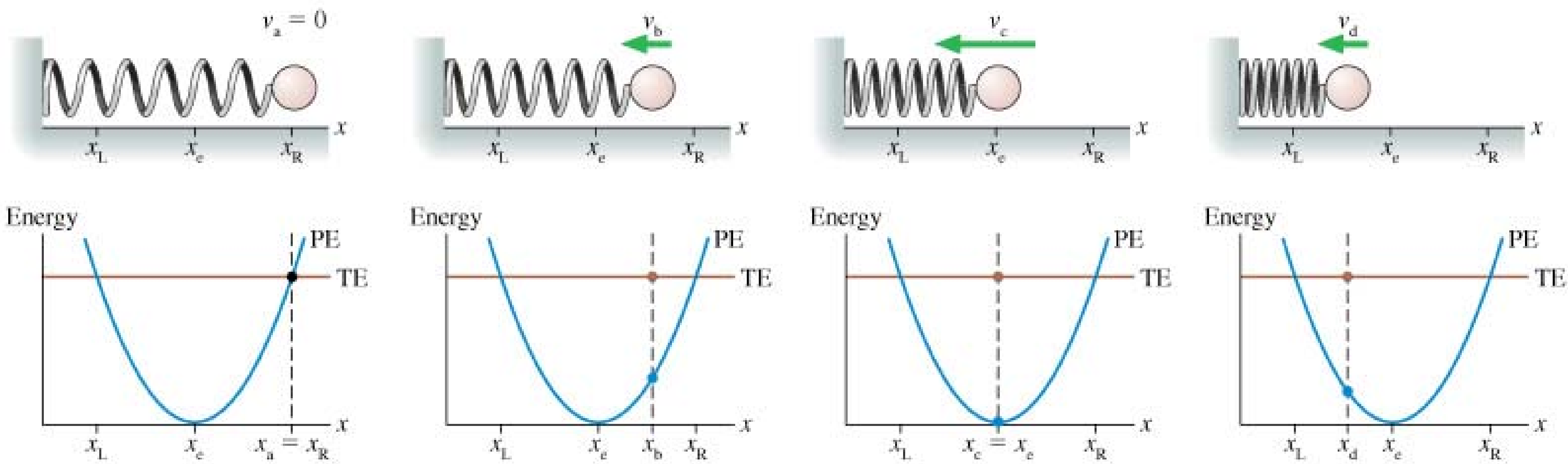
Energy Diagrams

- Particle under a free fall

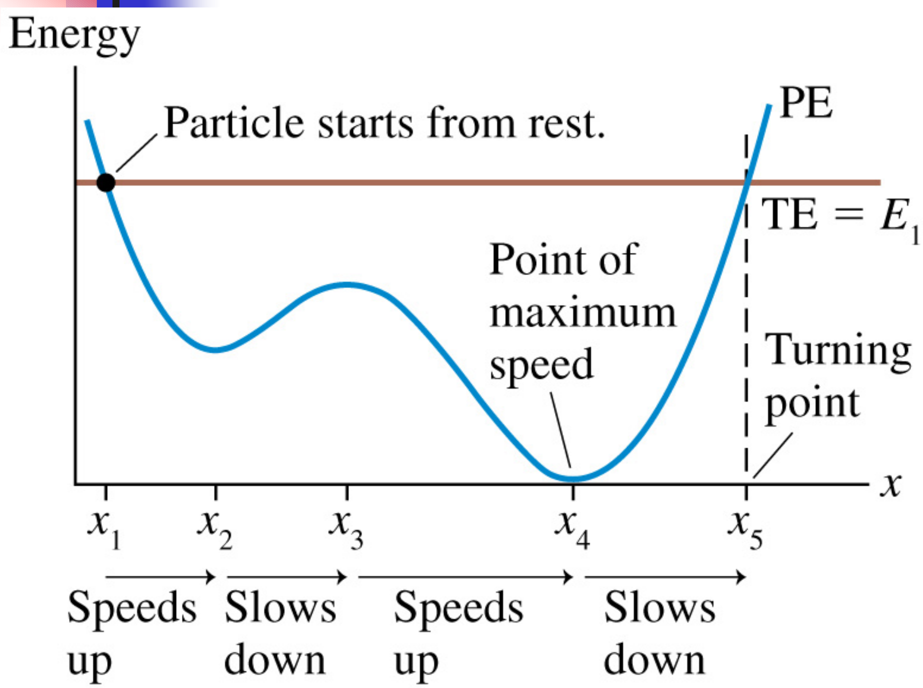


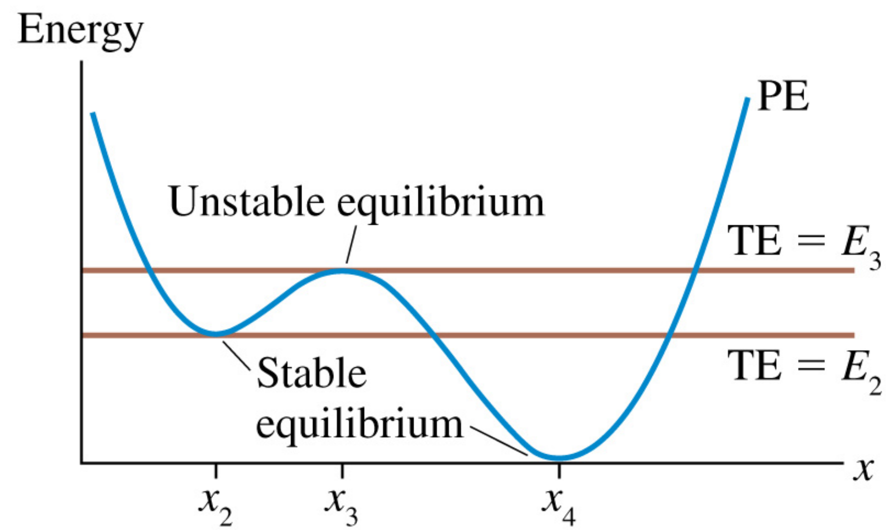
Energy Diagrams, cont

- Particle attached to a spring



General Energy Diagram



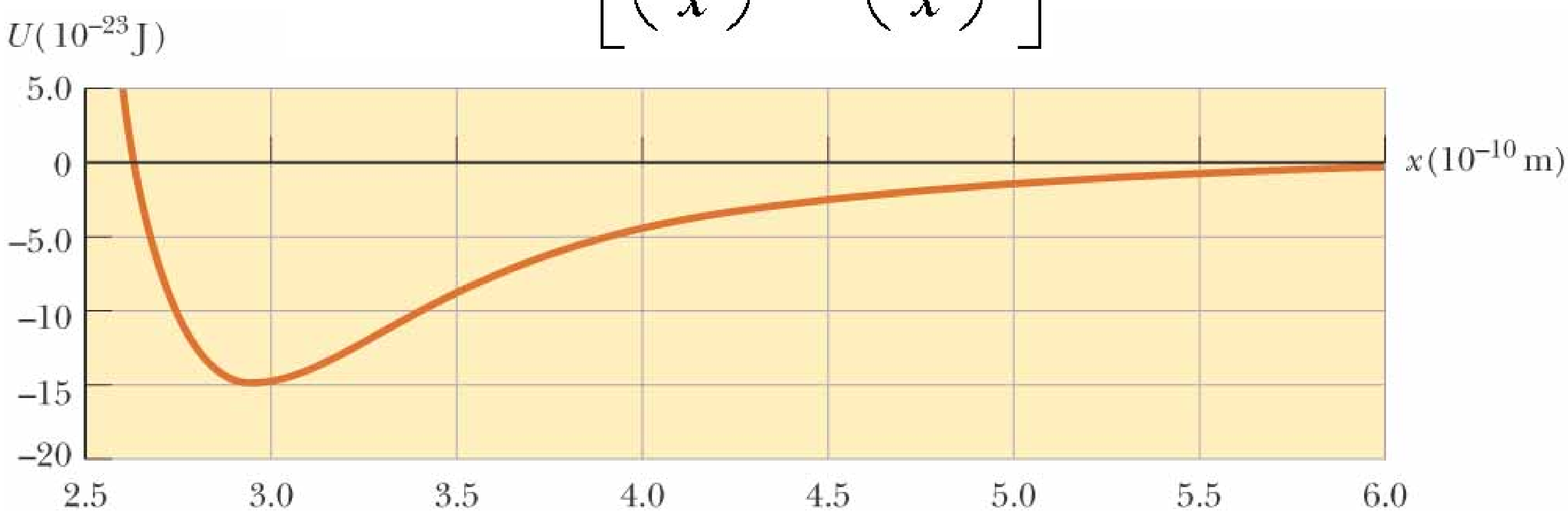


Equilibria are at points where $dU/dx = 0$. At equilibria, $F = 0$. Away from equilibria, $dU/dx \neq 0$ and $F \neq 0$.

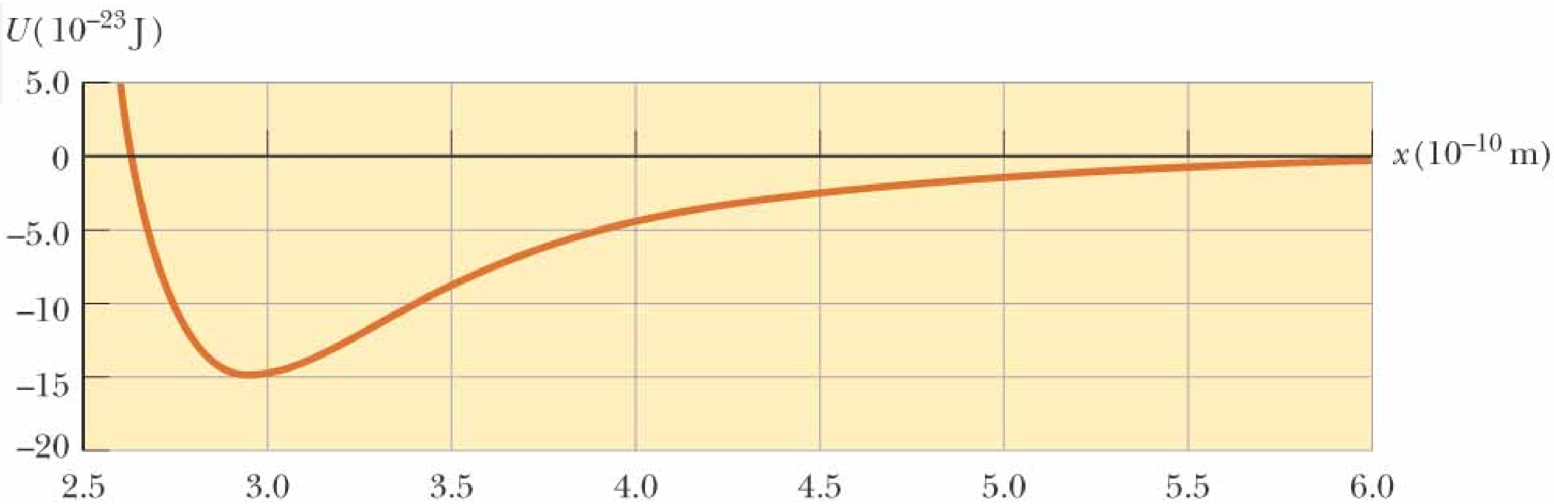
Molecular Bonding

- Potential energy associated with the force between two neutral atoms in a molecule can be modeled by the *Lennard-Jones* function:

$$U(x) = 4\in\left[\left(\frac{\sigma}{x}\right)^{12} - \left(\frac{\sigma}{x}\right)^{6}\right]$$



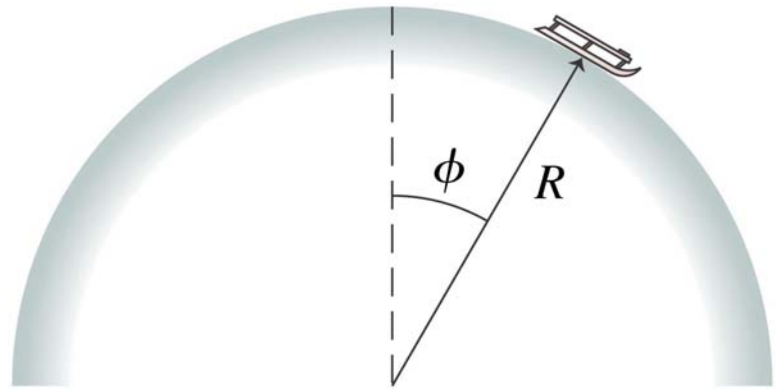
Force Acting in a Molecule



- The force is repulsive (positive) at small separations
- The force is zero at the point of stable equilibrium
  - This is the most likely separation between the atoms in the molecule (2.9×10^{-10} m at minimum energy)
- The force is attractive (negative) at large separations
- At great distances, the force approaches zero

Example 10: Hemispherical Hill

A sled starts from rest at the top of the frictionless, hemispherical, snow-covered hill shown in the figure.

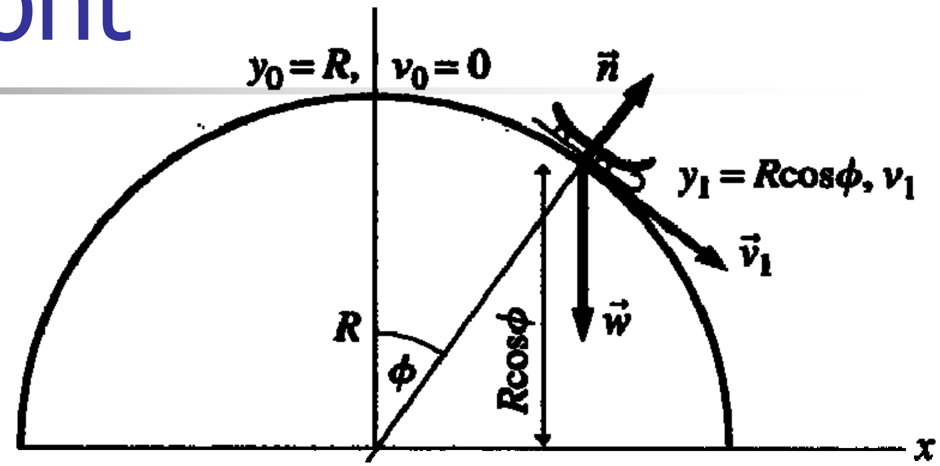


(a) Find an expression for the sled's speed when it is at angle ϕ.

(b) Use Newton's laws to find the maximum speed the sled can have at angle ϕ without leaving the surface.

(c) At what angle ϕ_{max} does the sled "fly off" the hill?

Example 10, cont



(a) $K_1 + U_1 = K_0 + U_0$

$\frac{1}{2}mv_1^2 + mgy_1 = \frac{1}{2}mv_0^2 + mgy_0$

$\frac{1}{2}mv_1^2 + mgR\cos\phi = \frac{1}{2}mv_0^2 + mgR$

$v_1 = \sqrt{2gR(1-\cos\phi)}$

(b) $\sum F_r = ma_r,\quad n - mg\cos\phi = -\dfrac{mv^2}{R} \Rightarrow n = m\left(g\cos\phi - \dfrac{v^2}{R}\right)$

n decreases as v increases. When $n = 0$, the sled leaves the hill.

$n \geq 0 \Rightarrow \quad v_{max} = \sqrt{gR\cos\phi_{max}}$

(c) We have v for an arbitrary angle and v_{max}. Equating the two,

$\sqrt{2gR(1-\cos\phi_{max})} = \sqrt{gR\cos\phi_{max}},\ \cos\phi_{max} = \dfrac{2}{3} \Rightarrow \phi_{max} = \cos^{-1}\dfrac{2}{3} = 48.2^\circ$