VECTOR ALGEBRA / TRIG. IDENTITIES.

VECTOR: ($\underset{\rightarrow}{V}$). A mathematical object which has both a magnitude and a direction

SCALAR Has Magnitude only (S)

I If you multiply a vector $\underset{\rightarrow}{V}$ by a scalar S you get a Vector $\underset{\rightarrow}{V}' = S\underset{\rightarrow}{V}$ such that $\underset{\rightarrow}{V}' \parallel \underset{\rightarrow}{V}$ and has magnitude SV.
This property allows us to express any vector as a product of a scalar (magnitude) and a unit vector (magnitude 1, direction only). Hence, we have written

$$\underset{\rightarrow}{A} = A\hat{x}$$

as a vector of magnitude A in $+x$ direction
Indeed, a vector along any direction $\hat{d}$ can be written as

$$\underset{\rightarrow}{V} = V\hat{d}$$

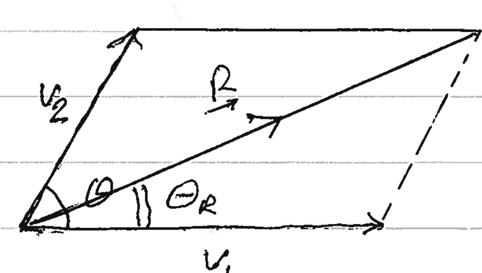
II <u>Addition of vectors.</u> Given vectors $\underset{\rightarrow}{V_1}$ and $\underset{\rightarrow}{V_2}$ we want to determine the <u>RESULTANT VECTOR</u>

$$\underset{\rightarrow}{R} = \underset{\rightarrow}{V_1} + \underset{\rightarrow}{V_2} \qquad -(1).$$

There are Three methods for doing This:

i) Geometry

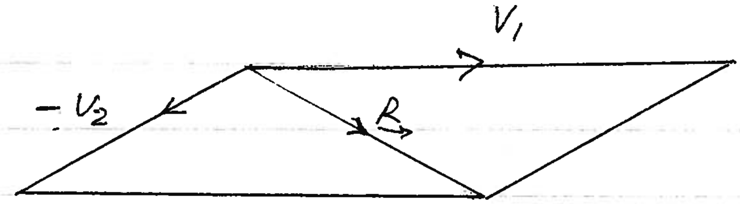
Choose a scale to represent $\underset{\rightarrow}{V_1}$ and V_2 and draw a parallelogram.
The long diagonal gives you $\underset{\rightarrow}{R} = \underset{\rightarrow}{V_1} + \underset{\rightarrow}{V_2}$.



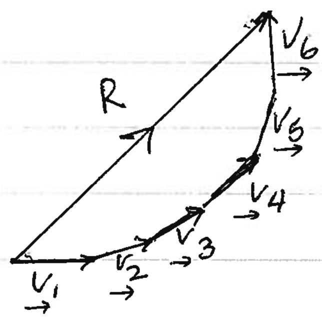
You can get magnitude of R by using a scale and of course measure θ_R

Also,

$\vec{R} = \vec{V_1} - \vec{V_2}$



is determined by the short diagonal.
Repeated application of this construct will allow you to add as many vectors



$\vec{R} = \vec{V_1} + \vec{V_2} + \vec{V_3} + \vec{V_4} + \vec{V_5} + \vec{V_6}$

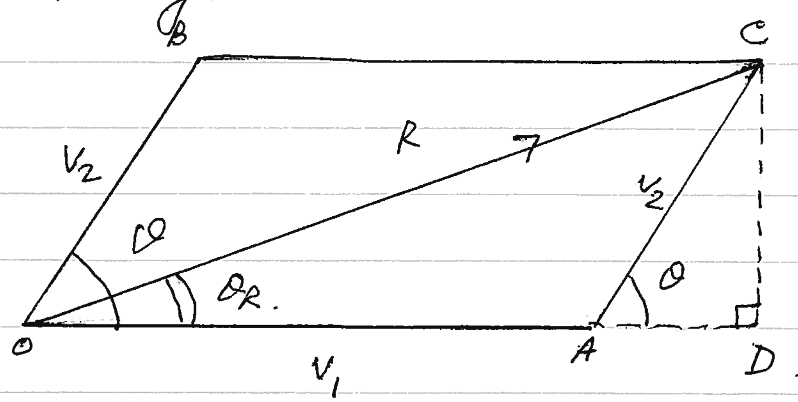
as the vector which connects the "tail" of $\vec{V_1}$ to the "head" of $\vec{V_6}$.

Further, it immediately follows that if all the vectors are parallel to one another

$$\vec{R} = V_1\hat{a} + V_2\hat{a} + V_3\hat{a} - V_4\hat{a} \cdots$$
$$= (V_1 + V_2 + V_3 - V_4 + \cdots)\,\hat{a}$$

(ii) Algebra/Trig.

We want to calculate R, so as shown, drop a $\perp$ from C to OA extended. Clearly,



$$\frac{CD}{V_2} = \sin\theta$$

$$\frac{AD}{V_1} = \cos\theta$$

Using Pythagoras' Theorem

$$R^2 = OD^2 + CD^2$$
$$= (V_1 + V_2 \cos\theta)^2 + (V_2 \sin\theta)^2$$
$$= V_1^2 + V_2^2 \cos^2\theta + 2 V_1 V_2 \cos\theta + V_2^2 \sin^2\theta.$$

That is

$$R = \sqrt{V_1^2 + V_2^2 + 2V_1 V_2 \cos\theta} \qquad [2]$$

Also

$$\tan\theta_R = \frac{CD}{OD} = \frac{V_2 \sin\theta}{V_1 + V_2 \cos\theta} \qquad [3]$$

So indeed we have determined both the magnitude (Eq 2) and direction [Eq 3] of the vector $\underset{\rightarrow}{R} = (\underset{\rightarrow}{V_1} + \underset{\rightarrow}{V_2})$

Again, if we have more than 2 vectors we can use Eqs. [2] and [3] repeatedly to arrive at $\underset{\rightarrow}{R} = \underset{\rightarrow}{V_1} + \underset{\rightarrow}{V_2} + \underset{\rightarrow}{V_3} + \cdots$.

(iii) The method of components.

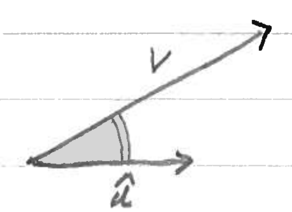
This is the most elegant procedure for adding (or subtracting) many vectors.

We begin by defining ~~that the~~

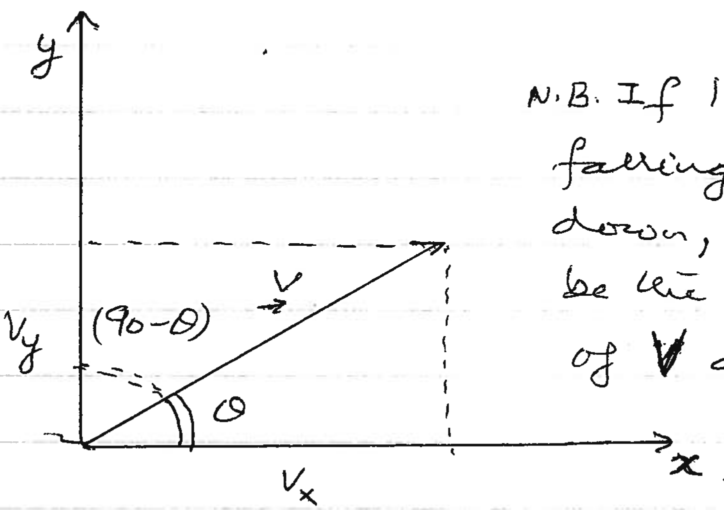
Component of a vector $\underset{\rightarrow}{V}$ along any direction $\hat{d}$ is a <u>SCALAR</u> quantity

$$V_d = V \cos(\underset{\rightarrow}{V}, \hat{d})$$

That is, V_d = [magnitude of V] × [Cosine of angle between $\underset{\rightarrow}{V}$ and $\hat{d}$]



Let us put our vector $\vec{V}$ in the x-y coordinate system and we see immediately that



N.B. If light were falling straight down, V_x would be the "shadow" of **V** along x.

$$V_x = V \cos\theta$$
$$V_y = V \cos(90-\theta) = V \sin\theta$$

(and clearly $\quad V = \sqrt{V_x^2 + V_y^2}$

$$\tan\theta = \frac{V_y}{V_x}$$

or $\vec{V} = V_x \hat{x} + V_y \hat{y}$

This tells us that a vector can be specified either by writing magnitude (V) and direction (θ) or by writing the magnitudes of its components.

So Now if we have many vectors:

$$\vec{V}_1 = V_{1x}\hat{x} + V_{1y}\hat{y}$$
$$\vec{V}_2 = V_{2x}\hat{x} + V_{2y}\hat{y}$$
$$\vdots$$
$$\vec{V}_i = V_{ix}\hat{x} + V_{iy}\hat{y}$$

$$\vec{R} = \sum \vec{V}_i = \sum V_{ix}\hat{x} + \sum V_{iy}\hat{y} \qquad \rightarrow [4]$$
$$= R_x\hat{x} + R_y\hat{y} \qquad \rightarrow [4']$$

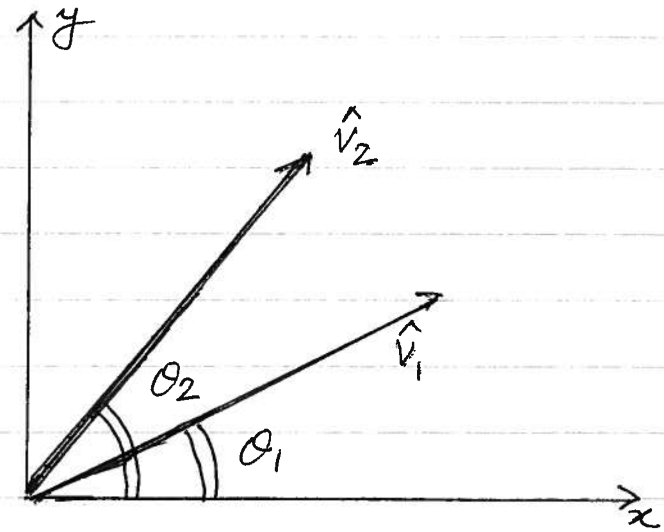
and hence $R = \sqrt{R_x^2 + R_y^2}$ [5]

$\tan\theta_R = \dfrac{R_y}{R_x}$ [6]

where θ_R is the angle between $\vec{R}$ and $\hat{x}$.

TRIG IDENTITIES.

Take two unit vectors $\hat{v}_1$ and $\hat{v}_2$ making angles θ_1 and θ_2 with the axis of x as shown.



$\vec{R} = \hat{v}_1 + \hat{v}_2$

FROM Eq (1) $R = \sqrt{1 + 1 + 2\cos(\theta_2 - \theta_1)}$ (7)

Also $\hat{v}_1 = \cos\theta_1\,\hat{x} + \sin\theta_1\,\hat{y}$

$\hat{v}_2 = \cos\theta_2\,\hat{x} + \sin\theta_2\,\hat{y}$

so $R_x = (\cos\theta_1 + \cos\theta_2)$

$R_y = (\sin\theta_1 + \sin\theta_2)$

$$R = \sqrt{(\cos\theta_1 + \cos\theta_2)^2 + (\sin\theta_1 + \sin\theta_2)^2}$$

$$= \sqrt{\cos^2\theta_1 + \cos^2\theta_2 + 2\cos\theta_1\theta_2 + \sin^2\theta_1 + \sin^2\theta_2 + 2\sin\theta_1\sin}$$

$$= \sqrt{1 + 1 + 2\left[\cos\theta_1\cos\theta_2 + \sin\theta_1\sin\theta_2\right]} \quad (8)$$

Compare Eqs (7) and [8] and you get the Trig Identity:

$$\cos(\theta_1 - \theta_2) = \cos\theta_1 \cos\theta_2 + \sin\theta_1 \sin\theta_2 \rightarrow I_1$$

Next, let $\theta_1 = \left(\frac{\pi}{2} - \theta_3\right)$

$$\cos\left(\frac{\pi}{2} - \theta_3 - \theta_2\right) = \sin(\theta_3 + \theta_2)$$

$$= \cos\left(\frac{\pi}{2} - \theta_3\right)\cos\theta_2 + \sin\left(\frac{\pi}{2} - \theta_3\right)\sin\theta_2$$

Which gives another identity

$$\sin(\theta_3 + \theta_2) = \sin\theta_3 \cos\theta_2 + \cos\theta_3 \sin\theta_2 \rightarrow I_2$$

(if in I_1 you put $\theta_4 = -\theta_2$ and remember that

$$\sin(-\theta) = -\sin\theta$$

you get

$$\cos(\theta_1 + \theta_4) = \cos\theta_1 \cos\theta_4 - \sin\theta_1 \sin\theta_4) \rightarrow I_3$$

and Similarly

$$\sin(\theta_3 - \theta_5) = \sin\theta_3 \cos\theta_5 - \sin\theta_5 \cos\theta_3 \rightarrow I_4$$