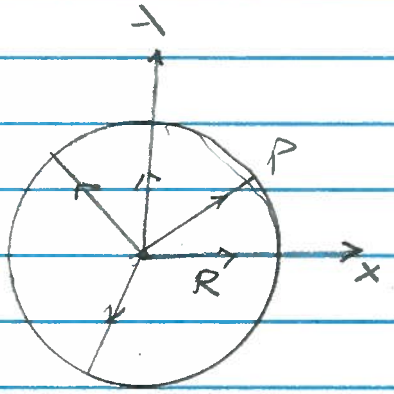
UNIFORM CIRCULAR MOTION
— KINEMATICS and DYNAMICS

A particle is moving on a circle of radius R at a constant speed S. First, we begin by describing the motion precisely — kinematics.

Th Let us put the Circular orbit in the xy-plane with the center of the ⊙ at $x=0, y=0$.



The very first quantity we define is the Period: Time taken to go around once, T.

The speed can then be immediately written as

$$S = \frac{2\pi R}{T}$$

As you can see when the particle moves around the ⊙ the radius rotates as a function of time. That is why it is customary to describe the motion in terms of revolutions per sec (n_s) (rps) so $T = \frac{1}{n_s}$ sec

or revolutions per minute (n_m), (rpm) $T = \frac{60}{n_m}$ sec.

For instance, 15 rpm means $T = 4$ sec.

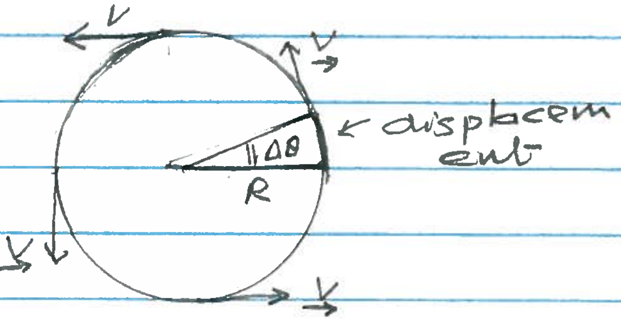
Speed is an interesting concept but as before it is rather limiting. We need to look deeper.

Position Vector: We notice that the particle moves at fixed distance away from the center but the radius rotates. Hence, its position vector will be written as

$$\underset{\rightarrow}{r} = R\hat{r} \qquad \rightarrow ①$$

where $\hat{r}$ is a unit vector along the radius which rotates so as to go around once in time T.

Velocity Vector: Velocity is defined as rate of change of position vector So we need to find the displacement vector. Consider a time interval Δt during which $\hat{r}$ rotates by angle $\Delta\theta$.



displacement during Δt is $R\Delta\theta$ so magnitude of instantaneous velocity is

$$V = \frac{R\Delta\theta}{\Delta t} \qquad (\Delta t \to 0)$$

Notice, direction of displacement is perpendicular to $\hat{r}$ so direction of velocity is along the tangent to the circle. We define $\hat{\tau}$ unit vector along tangent and write

$$\underset{\rightarrow}{V} = \frac{R\Delta\theta}{\Delta t}\hat{\tau}$$

We will soon introduce a formal definition for rate of change of angle with time, for now let us introduce a new symbol (greek letter omega)

$$\omega = \frac{\Delta\theta}{\Delta t}$$

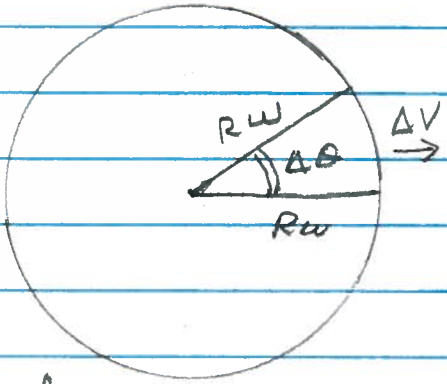
and note $\underset{\rightarrow}{V} = R\omega\hat{\tau}$ → ②

and $\hat{\tau}$ rotates with time

For uniform case rate of rotation is constant so Eq.(2) tells us that magnitude of V is CONSTANT. DIRECTION CHANGES!.

<u>acceleration vector</u>:

Since the velocity vector is rotating the object has an acceleration. Again we need to calculate change in $\underset{\rightarrow}{V}$ and divide by Δt.



Change in magnitude of V:

$$\Delta V = R\omega\Delta\theta$$

So magnitude of acceleration is

$$a = R\omega\frac{\Delta\theta}{\Delta t} = R\omega^2$$

and $\underset{\rightarrow}{a}$ must be perpendicular to $\hat{\tau}$. If you look at the $\underset{\rightarrow}{V}$ it is continuously turning TOWARD the center so $\underset{\rightarrow}{a}$ is along $-\hat{r}$

SO $\underset{\rightarrow}{a} = -R\omega^2\hat{r}$ —
+ $\hat{r}$ rotates

So $\underset{\rightarrow}{a}$ is constant in magnitude but also rotates.

This is a special case so this acceleration has a special name: CENTRIPETAL ACCELERATION

FINALLY, we go back and look at

$$\omega = \frac{\Delta\theta}{\Delta t}$$

this is the rate at which the radius vector sweeps out an angle as it rotates so it is not surprising that we call it

ANGULAR VELOCITY

Question? What is the direction of $\underset{\rightarrow}{\omega}$.

Well  is a positive angle

and  is a negative angle

and rotation is about an axis perpendicular to ⊙ so it makes sense to say that $\underset{\rightarrow}{\omega}$ is perpendicular to plane of circle. In our case ⊙ is in xy-plane so $\underset{\rightarrow}{\omega} \parallel \pm\hat{z}$; $+\hat{z}$ for counter-clockwise (+ive θ's) $-\hat{z}$ for clockwise (−ive θ's). This is summarized by Right Hand Rule: Curl fingers of right-hand along direction of motion on the ⊙, extend your thumb, it points in direction of $\underset{\rightarrow}{\omega}$

$$\vec{\omega} = \frac{\Delta\theta}{\Delta t}\hat{z}$$

Table → ANGULAR VELOCITY L^0T^{-1} rad/sec Vector

So to summarize kinematics:

<u>Position</u> $\vec{r} = R\hat{r}$ rotates by ω rad/sec. (1)

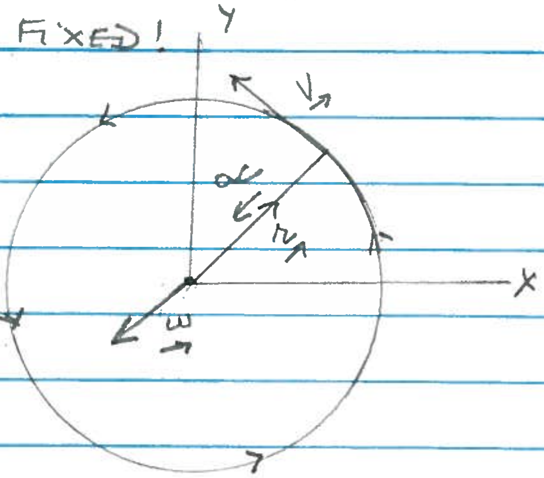
<u>velocity</u> $\vec{v} = R\omega\hat{\tau}$ rotates by ω rad/sec. (2)

<u>centripetal acceleration</u>

$$\vec{a_c} = -R\omega^2\hat{r} = -\frac{V^2}{R}\hat{r} \qquad (3)$$

rotates by ω rad/sec.

<u>angular velocity</u> $\vec{\omega} = \pm\frac{\Delta\theta}{\Delta t}\hat{z}$ (4) FIXED!



<u>Dynamics</u> A particle moving on a ⊙ of radius R at a constant angular velocity $\vec{\omega}$ has a centripetal acceleration

$$\vec{a_c} = -R\omega^2\hat{r} = -\frac{V^2}{R}\hat{r}$$

Newton's law $M\vec{a} = \Sigma\vec{F}$ requires that for this motion to occur we must provide a CENTRIPETAL FORCE

$$\vec{F_c} = -MR\omega^2\hat{r} = -\frac{MV^2}{R}\hat{r} \rightarrow (5)$$

It is to be noted that $\vec{F_c}$ must come from one or more of the available forces: Weight, normal force, tension, spring force, friction

<u>Note</u> $\vec{F_c}$ CANNOT BE DRAWN ON A DIAGRAM.