# CONSERVATION OF LINEAR MOMENTUM

An object of mass M travelling at a velocity $\underset{\sim}{v}$ is said to have a linear momentum $\vec{p}$ given by the equation

$$\vec{p} = M\vec{v} \qquad (1).$$

[Lin Mom$^{M.}$ $MLT^{-1}$ kg-m/sec VECTOR]

The immediate consequence of defining $\vec{p}$ are that Newton's Laws should be read as:

FIRST LAW Objects donot change their linear momentum spontaneously.

SECOND LAW If the linear momentum $\vec{p}$ varies with time there must be a net force present at that point at that time: That is

$$\frac{\vec{p}_f - \vec{p}_i}{t_f - t_i} = \langle \Sigma F_i \rangle \text{ (Average)} \qquad (2)$$

$$\lim_{\Delta t \to 0} \frac{\Delta \vec{p}}{\Delta t} = \Sigma \vec{F}_i \text{ (Instaneous)} \qquad (3)$$

Also, the kinetic Energy should be written

$$\boxed{K = \frac{1}{2}Mv^2 = \frac{p^2}{2M}} \qquad (4)$$

Note: If two objects have the same momentum (magnitude) the smaller M has a larger K!

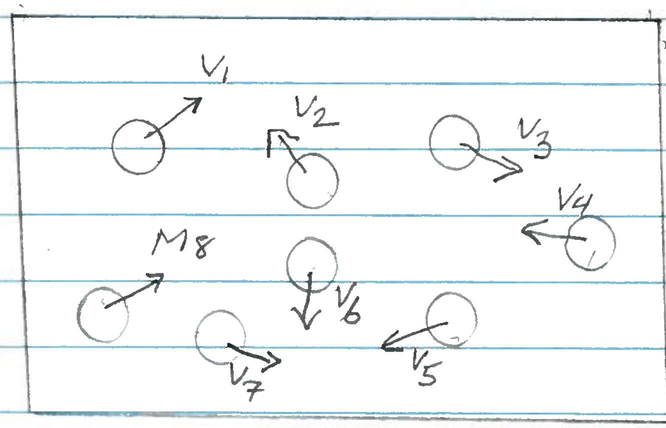
One can turn Eqns (2) around to define a vector quantity called impulse, $\vec{J}$, which is the change in momentum caused by the application of a large force over a time interval.

If $\vec{F}$ is constant

$$\vec{J} = \vec{p}_f - \vec{p}_i = \vec{F}\,\Delta t \qquad (5)$$

If $\vec{F}$ varies with time then to calculate $\vec{J}$ you draw F as a function of time and calculate the area under the F vs t graph to determine $\vec{J}$.

So much for single particles. To formulate the principle of conservation of momentum ($\vec{p}$) we need to consider a system consisting of many (at the very least two) objects and they cannot be point particles because point particles will not "collide" and we need the objects to collide. So now our system is a "box" containing many objects of masses $M_1, M_2, \ldots$ with velocities $\vec{v}_1, \vec{v}_2, \ldots$ and we can write



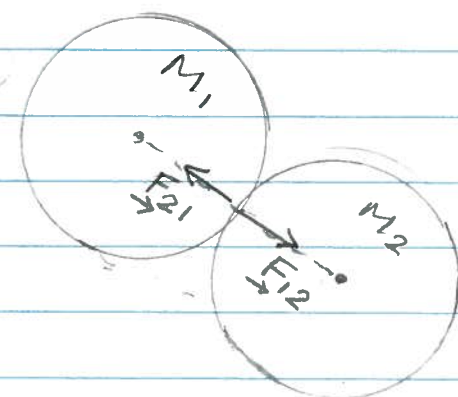

$$\vec{p}_i = M_i \vec{v}_i$$

Clearly, the system will have a

total mass $M = \sum M_i$

and a total momentum $P = \sum M_i \underset{\to}{v_i}$

Now suppose two of the masses collide as shown. At the instant of collision. Newton's 3rd Law tells us that the force $\underset{\to}{F_{21}}$ on $M_1$ due to $M_2$ must be equal and opposite to the force $\underset{\to}{F_{12}}$ on $M_2$ due to $M_1$, that is




$$\underset{\to}{F_{12}} + \underset{\to}{F_{21}} = 0 \longrightarrow \text{CRUCIAL FACT.}$$

If the collision last for $\Delta t$ secs, the Impulse on $M_1$ is

$$\underset{\to}{J_1} = \underset{\to}{F_{21}}\, \Delta t$$

while

$$\underset{\to}{J_2} = \underset{\to}{F_{12}}\, \Delta t$$

and therefore

$$\underset{\to}{J_1} + \underset{\to}{J_2} = 0$$

But $\underset{\to}{J_1}$ = change in momentum of $M_1$
$\underset{\to}{J_2}$ " " " " $M_2$

So this equation tells us that whatever vector momentum $M_1$ gains (loses) must be lost (gained) by $M_2$. So no matter how many collisions occur, if there is no external force the internal forces always come in action-reaction pairs and therefore the total vector momentum $\vec{P}$ of the system cannot change.

PRINCIPLE OF CONSERVATION OF LINEAR MOMENTUM SAYS: If $\vec{F}_{ext} = 0$ Total vector momentum of a system is CONSTANT:

$$\boxed{\text{If } \vec{F}_{ext} = 0, \quad \sum \vec{p}_i = \text{Constant}} \quad (6)$$

In considering the motion of the entire system a useful concept is that of the center of mass. Let our masses $M_i$ be located at $(x_i, y_i)$ in the xy-plane, the coordinates of the center of mass are

$$x_{CM} = \frac{\sum M_i x_i}{\sum M_i}, \quad y_{CM} = \frac{\sum M_i y_i}{\sum M_i}$$

[Near Earth $x_{CM} = x_{CG}$, $Y_{CM} = Y_{CG}$]

If the masses are moving, the displacements will be $\Delta \vec{x}_i$, $\Delta \vec{y}_i$

$$M \, \Delta \vec{x}_{CM} = \sum M_i \, \Delta \vec{x}_i$$

$$M \, \Delta Y_{CM} = \sum M_i \, \Delta \vec{y}_i$$

and the velocity components become

$$M \, \frac{\Delta \vec{x}_{CM}}{\Delta t} = \sum M_i \frac{\Delta x_i}{\Delta t}$$

$$M \, \frac{\Delta \vec{Y}_{CM}}{\Delta t} = \sum M_i \frac{\Delta y_i}{\Delta t}$$

Indeed $\qquad M\,\vec{V}_{CM} = \sum M_i \vec{v}_i = \vec{P}$.

So conservation law says if $\vec{F}_{ext} = 0$, velocity of center of mass is CONSTANT.

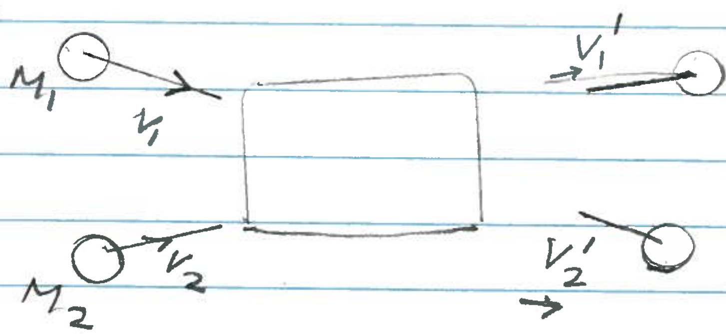
If $\vec{F}_{ext} \neq 0$

$$\left.\begin{aligned} M\,\vec{a}_{CM} &= \sum \vec{F}_{ext} \\ \text{or} \quad \frac{\Delta \vec{P}}{\Delta t} &= \sum \vec{F}_{ext} \end{aligned}\right\} \text{Newton's Law}$$

That is, one can pretend ~~that~~ the total mass M is located at the C.M. and treat it as a "translation" of the "box" but this is not a rigid body.

## Two – Body Collisions

This is the experiment you performed a while ago. You took two pucks and put them on a horizontal frictionless surface thereby making $\vec{F}_{ext} = 0$ because firstly $(n - Mg) = 0$ and also $f_k = 0$. The pucks were given velocities $\vec{V}_1$ and $\vec{V}_2$, allowed to collide and emerge with velocities $\vec{V}_1'$ and $\vec{V}_2'$



The corresponding momenta are

| Before | After |
| --- | --- |
| $\vec{p}_1 = M_1 \vec{V}_1$ | $\vec{p}_1' = M_1 \vec{V}_1'$ |
| $\vec{p}_2 = M_2 \vec{V}_2$ | $\vec{p}_2' = M_2 \vec{V}_2'$ |

and conservation law required

$$\vec{p}_1' + \vec{p}_2' = \vec{p}_1 + \vec{p}_2 \qquad (7)$$

(Total Vector Mom$^m$ After) = (Total Vector Mom$^m$ before)

Experimentally, you have checked this relationship. The question we need to answer is: Given $M_1$, $M_2$ and $\vec{V}_1$, $\vec{V}_2$ do we have enough information to figure out $\vec{V}_1'$ and $\vec{V}_2'$. The answer is No. Why?

Let us put the objects in the xy-plane. Eq. (7) yields

$$M_1 V_{1x}' + M_2 V_{2x}' = M_1 V_{1x} + M_2 V_{2x} \qquad (8)$$

$$M_1 V_{1y}' + M_2 V_{2y}' = M_1 V_{1y} + M_2 V_{2y} \qquad (9)$$

The problem is that we have only two equations but there are 4 unknowns [$V_{1x}'$, $V_{2x}'$, $V_{1y}'$, $V_{2y}'$] and therefore no unique solution is possible. We need to add further specifications to the type of collision in order to get a solution.

We consider two special cases:

Type I  Totally inelastic Collision. The two objects stick together after the Collision

$$\vec{V}_1' = \vec{V}_2' \quad \text{[Totally Inelastic Coll.]}$$

and now we can use (8) and (9) to get precise answers.

Type II  Totally Elastic Collision Kinetic Energy is also conserved:

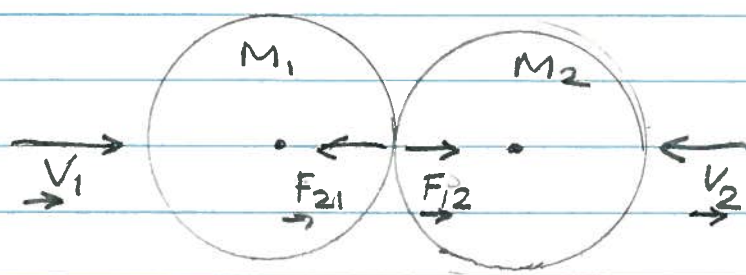
(Total Kin Energy After) = (Total kin. Energy Before)

This gives us another equation

$$\frac{1}{2} M V_1'^2 + \frac{1}{2} M V_2'^2 = \frac{1}{2} M_1 V_1^2 + \frac{1}{2} M_2 V_2^2 \qquad (10)$$

Now we have 3 eqns (8), (9), (10) and we still have a problem because we have four unknowns.

We simplify further by specifying that the collision is "head-on"




$\vec{F}_{12} \parallel (\vec{V}_1 - \vec{V}_2)$

Now the force $\vec{F}_{12}$ and $\vec{F}_{21}$ are parallel to the relative velocity $(\vec{V}_1 - \vec{V}_2)$ so it becomes essentially a one-dimensional problem.

We can take all the vectors to be along the $x$-axis, $\vec{v}_1 = v_1\hat{x}$, $\vec{v}_2 = v_2\hat{x}$, $\vec{v}_1' = v_1'\hat{x}$, $\vec{v}_2' = v_2'\hat{x}$ and the conservation Eqns become.

Lin. mom^m.
$$M_1 v_1' + M_2 v_2' = M_1 v_1 + M_2 v_2 \qquad (11)$$
$$M_1 \frac{v_1'^2}{2} + M_2 \frac{v_2'^2}{2} = M_1 \frac{v_1^2}{2} + M_2 \frac{v_2^2}{2} \qquad (12)$$

Now we can use algebra to solve for $v_1'$ and $v_2'$

Rewrite Eqs. (11) and (12) as
$$v_1' - v_1 = \frac{M_2}{M_1}(v_2 - v_2') \qquad (11')$$

$$v_1'^2 - v_1^2 = \frac{M_2}{M_1}(v_2^2 - v_2'^2) \qquad (12')$$

Divide Eq (12') by Eq (11')
$$v_1 + v_1' = v_2 + v_2' \qquad (13)$$

or

$$\rightarrow \boxed{(v_1' - v_2') = (v_2 - v_1) = -(v_1 - v_2)} \qquad (A)$$

IN A TOTALL ELASTIC HEAD ON COLLISION THE RELATIVE VELOCITY REVERSES DIRECTION AS A RESULT OF THE COLLISION.

Next take (11) write $M_1 v_1' = M_1 v_1 + M_2 v_2 - M_2 v_2'$

Next use Eq 13 $\qquad = M_1 v_1 + M_2 v_2 - M_2(v_1 + v_1' - v_2)$

Rearrange $\qquad (M_1 + M_2) v_1' = (M_1 - M_2) v_1 + 2M_2 v_2$

Yielding
$$v_1' = \frac{M_1 - M_2}{M_1 + M_2} v_1 + \frac{2M_2 v_2}{M_1 + M_2} \qquad (B)$$

similarly
$$v_2' = \frac{M_2 - M_1}{M_1 + M_2} v_2 + \frac{2M_1 v_1}{M_1 + M_2} \qquad (C)$$

or respecting the vector nature of the velocities $(\pm \hat{x})$ we write

$$\vec{v}_1' = \frac{M_1 - M_2}{M_1 + M_2}\vec{v}_1 + \frac{2M_2}{M_1 + M_2}\vec{v}_2$$

$$\vec{v}_2' = \frac{M_2 - M_1}{M_1 + M_2}\vec{v}_2 + \frac{2M_1}{M_1 + M_2}\vec{v}_1 .$$

We will discuss the consequences of these equations in the class.