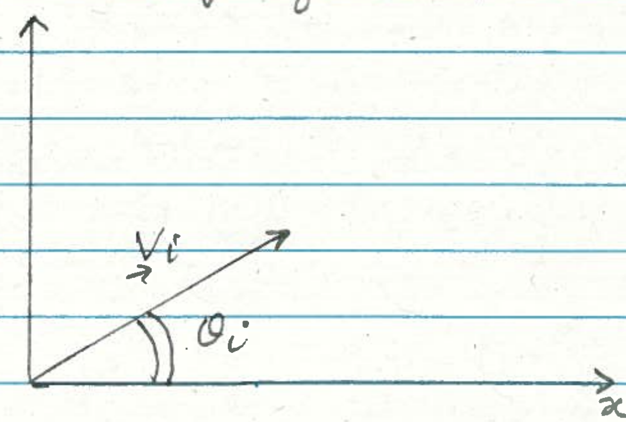
KINEMATICS — TWO DIMENSIONS — PROJECTILE MOTION

At $t=0$ a projectile is launched from the origin ($x_i=0$, $y_i=0$) with a velocity of V_i m/sec at angle of θ_i above the horizon ($x \Rightarrow$ axis). What are the equations which describe its motion in the xy-plane? It is best to write down the components and then the vectors.



	X-Component	Y-Comp.	Vector
acceleration	0	$9.8 m/s^2$	$\vec{a} = 0\hat{x} - 9.8 m/s^2\,\hat{y}$ — (1)
Velocity	$V_i \cos\theta_i$	$V_i \sin\theta_i - 9.8t$	$\vec{V} = (V_i \cos\theta_i)\,\hat{x} - (V_i \sin\theta_i - 9.8t)\,\hat{y}$ → (2)
Position	$(V_i \cos\theta_i)t$	$(V_i \sin\theta_i)t - 4.9t^2$	$\vec{r} = (V_i \cos\theta_i)\,t\,\hat{x} + [(V_i \sin\theta_i)t - 4.9t^2]\,\hat{y}$ → (3)

We can also write for the y-velocity

$$V_y^2 = (V_i \sin\theta_i)^2 - 19.6y \rightarrow (4)$$

and use Eq (4) when t is not known.

Questions ① What is its path in the xy-plane? We saw the parabola in the water stream. To derive it note that

$$y = (V_i \sin\theta_i)t - 4.9t^2$$

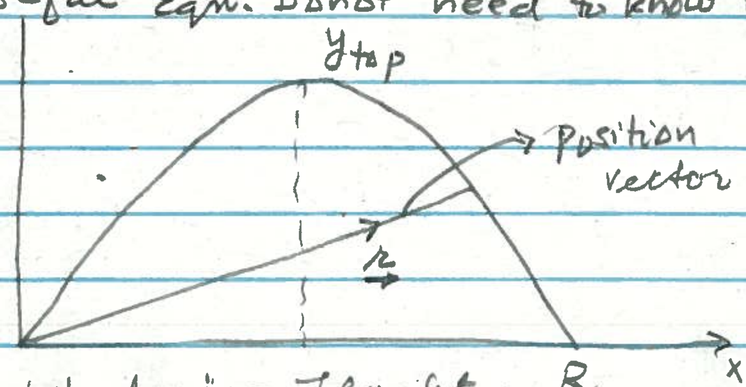
and $\quad x = (v_i \cos\theta_i)\, t$

So one can write

$$y = \frac{(v_i \sin\theta_i)\, x}{(v_i \cos\theta_i)} - 4.9\left(\frac{x}{v_i \cos\theta_i}\right)^2$$

$$= x \tan\theta_i - 4.9\left(\frac{x}{v_i \cos\theta_i}\right)^2 \rightarrow (5)$$

→ This is a very useful Eqn. Donot need to know t, Relates y to x and v_i. See plot alongside. It helps to define two quantities



y_{top} = highest point during flight

R = range; distance travelled before returning to Earth.

② Why does it stop rising? Because the y velocity goes to zero. Using Eq (4) we write

$$y_{top} = \frac{v_0^2 \sin^2\theta_0}{19.6} \rightarrow (6)$$

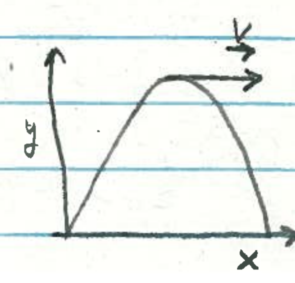
③ What is its acceleration while it is in the air? At all points $y \neq 0$

$$! \longrightarrow \underset{\rightarrow}{a} = -9.8\,\text{m/s}^2\, \hat{y} \longleftarrow !$$

fixed by the Earth.

④ Velocity at y_{top}. $v_y = 0$, $v_x = v_i \cos\theta_i$

$$\underset{\rightarrow}{v} = (v_i \cos\theta_i)\, \hat{x} + 0\, \hat{y}$$



(5) When does it get to y_{top}? $v_y = 0$ there

So we use

$$v_y = v_i \sin\theta_i - 9.8t$$

and get

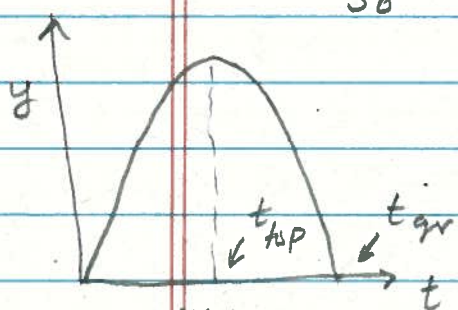
$$t_{top} = \frac{v_i \sin\theta_i}{9.8} \rightarrow (7)$$

(6) When does it return to ground $(y=0)$?

use $$y = (v_i \sin\theta_i)t - 4.9t^2$$

so

$$0 = (v_i \sin\theta_i)t_{gr} - 4.9\,t_{gr}^2$$



$$t_{gr} = \frac{v_i \sin\theta_i}{4.9} = 2t_{top} \quad (8)$$

(7) What is its velocity just before it hits ground

$$v_x = v_i \cos\theta_i$$

$$v_y = v_i \sin\theta_i - 2\frac{v_i \sin\theta_i}{9.8} \times 9.8$$

$$= -v_i \sin\theta_i$$

hence $$\underline{v} = (v_i \cos\theta_i)\,\hat{x} - (v_i \sin\theta_i)\,\hat{y} \rightarrow (9)$$

That is, x component of velocity is same as at the start, y component is reversed.

(8) What is the range?

$$x = (v_i \cos\theta_i)\, t$$

and to get to R

$$t = t_{gr} = \frac{2 v_i \sin\theta_i}{9.8}$$

$$R = \frac{(v_i \cos\theta_i)(2 v_i \sin\theta_i)}{9.8}$$

$$= \frac{v_i^2 \sin 2\theta_i}{9.8} \qquad (10)$$

(9) For a given v_i what launch angle will give you maximum range R? (Galileo's finding).

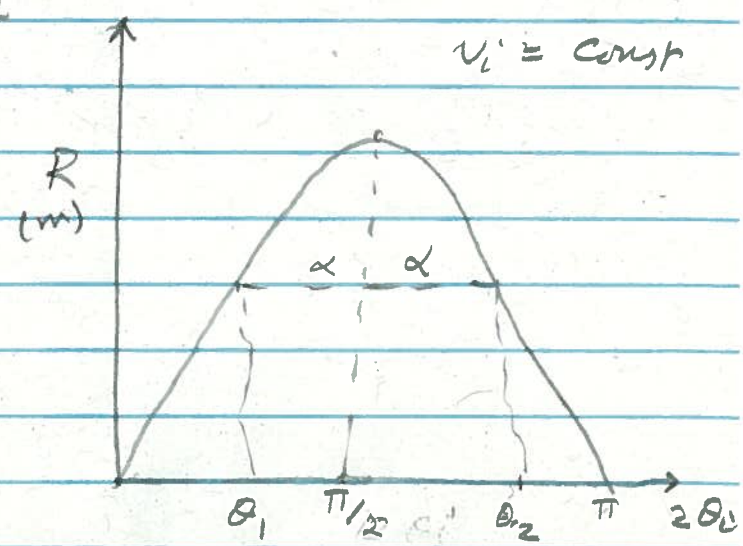
Eq (10) says

$$R = \frac{v_0^2 \sin 2\theta_i}{9.8}$$

Maximum Value of $\sin 2\theta_i = 1$ when $2\theta_i = \pi/2$.

Hence, maximum range when $\theta_i = 45°$

Also, note that there are two angles for which R is the same



$$2\theta_2 = \frac{\pi}{2} + \alpha$$

$$2\theta_1 = \pi/2 - \alpha$$

$$\theta_1 + \theta_2 = \pi/2.$$

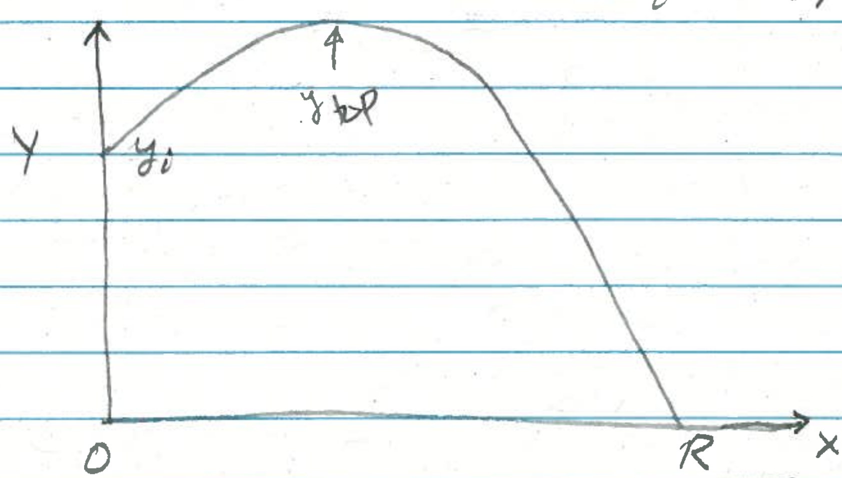
So θ_1 and θ_2 are complementary angles

(1i) What happens if projectile is launched at $x=0$, $y=y_i$. In that case

$$y_{top} = y_i + \frac{v_i^2 \sin^2\theta_i}{19.6}$$

and R is obtained by solving the quadratic eqn.

$$0 = y_i + R\tan\theta_i - 4.9\left(\frac{R^2}{v_i^2\cos^2\theta_i}\right)$$



$$R = \frac{-v_i^2\sin\theta_i\cos\theta_i \pm v_i\cos\theta_i\sqrt{v_i^2\sin^2\theta_i + 19.6\,y_i}}{-9.8}$$

Not surprisingly, the projectile travels farther before returning to ground. This is what led Newton to suggest that if one goes high up and uses a large enough initial speed one can get the projectile to go around the Earth