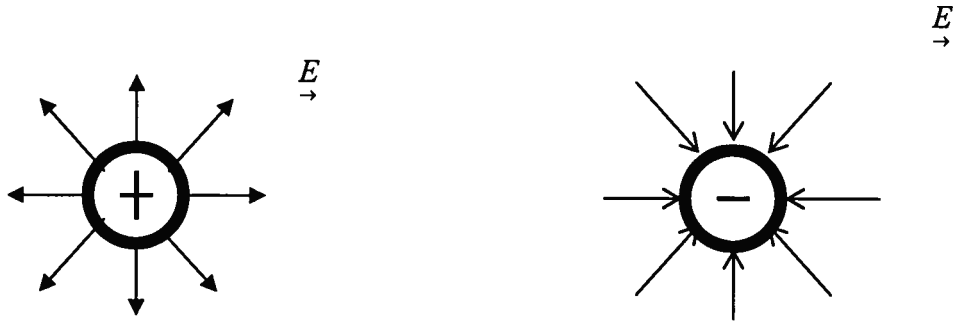


FLUX OF \vec{E} - FIELD: GAUSS' LAW

We have used the analogy of water flowing into (out of) a sink (source) to imagine the \vec{E} - field surrounding a point charge.



Just as we talk of the amount of water flowing through an area, we now talk of the FLUX of \vec{E} as the quantity

$$\Delta \Phi_E = \vec{E} \cdot \Delta \vec{A} = E \Delta A \cos(\vec{E}, \hat{n})$$

as a measure of the "amount" of \vec{E} - field "flowing" out of or into a surface of area $\Delta A = \Delta A \hat{n}$

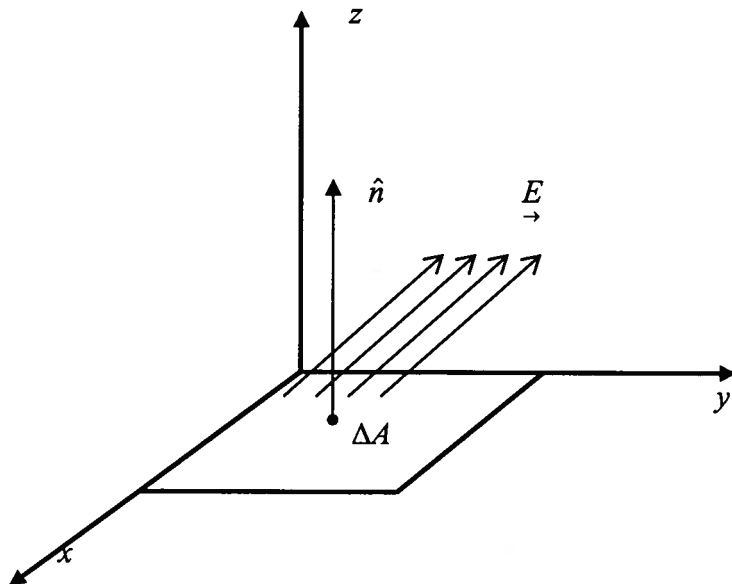
Example

Take area ΔA lying in xy -plane

$$\Delta \vec{A} = \Delta A \hat{n}$$

$$= \Delta A \hat{z}$$

$$\text{Take } \vec{E} = E_y \hat{y} + E_z \hat{z}$$



NOTICE:

FLUX Is MAXIMUM

when $\vec{E} \parallel \hat{n}$.

FLUX Is ZERO

when $\vec{E} \perp \hat{n}$

Thus

$$\Delta \Phi_E = E \Delta A \cos(\vec{E}, \hat{z})$$

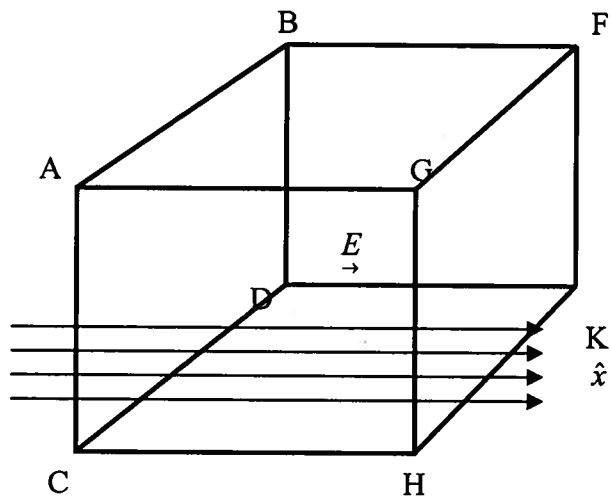
or

$$\Delta \Phi_E = (\Delta A) E_z$$

That is only component of \vec{E} parallel to \hat{z} contributes to flux of \vec{E} through ΔA . [To go through a door you must travel along its normal]

Example:

Consider a cube. Let $\vec{E} = E\hat{x}$



NO SOURCE OR SINK INSIDE CUBE

Flux of \vec{E} is non-zero only over faces ABCD and FGHE.

Over ABCD flux is into Cube

Over FGHE flux is out of Cube

TOTAL FLUX THROUGH CUBE=0 AS IT MUST BE BECAUSE THERE IS NO SOURCE OR SINK INSIDE IT. EVERY LINE THAT COMES IN MUST LEAVE [LINES STOP (START) AT SINKS (SOURCES) ONLY].

So it is not surprising that Gauss' Law says: THE TOTAL FLUX OF THE \vec{E} -FIELD THROUGH A CLOSED SURFACE IS DETERMINED SOLELY BY THE SOURCES (+CHARGES) AND SINKS (-CHARGES) IN THE VOLUME ENCLOSED BY THE SURFACE.

$$\text{Mathematically, } \sum_c \vec{E} \cdot \Delta \vec{A} = \sum_c E \perp \Delta A = \frac{1}{\epsilon_0} \sum Q_i \quad (I)$$

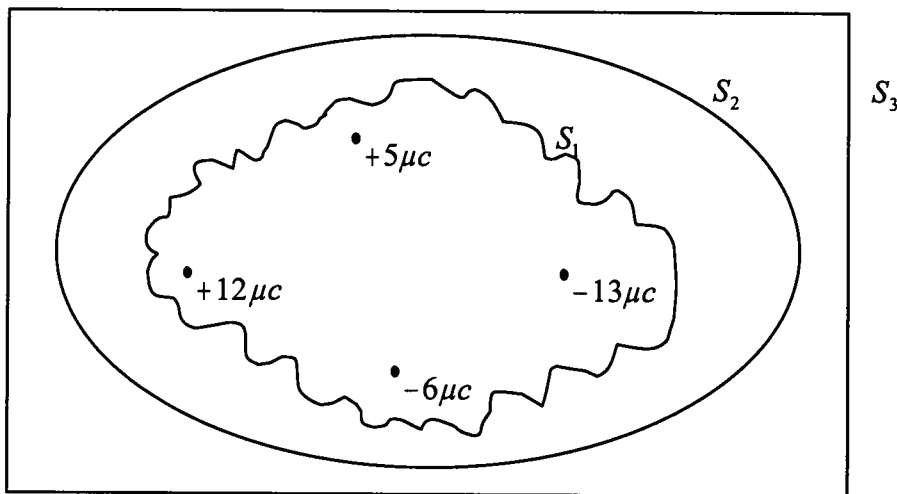
$$\text{Where } \frac{1}{4\pi\epsilon_0} = k_e = 9 \times 10^9 \frac{N \cdot m^2}{C^2}, \epsilon_0 = 9 \times 10^{-12} \frac{F}{m}$$

The \sum_c on the left hand side is sum of the flux over all parts of the closed surface. $\sum Q_i$ is the algebraic sum of all the enclosed charges.

Example:

In general, Gauss' law would be very difficult to use to calculate \vec{E} . All it can give us is the total flux once the Q_i 's are known. Here are 4 charges. If you draw any surface enclosing all of them we can write down the total flux of \vec{E} immediately.

S_1, S_2, S_3 are any three closed surfaces drawn around our four charges. In each case



$$\begin{aligned} \sum_c \vec{E} \cdot \Delta \vec{A} &= \frac{1}{9 \times 10^{-12}} [5 \times 10^{-6} + 12 \times 10^{-6} - 6 \times 10^{-6} - 13 \times 10^{-6}] \\ &= -2.2 \times 10^5 \frac{N \cdot m^2}{C} \end{aligned}$$

but it tells you nothing about the \vec{E} -field at any point on any surface.

However, there are some special cases where we can use this law to calculate \vec{E} -field in one step. [Please look at notes from Phys 121 to recall similar results for Gravitational field.]

The calculation depends crucially on recognizing the symmetry of the problem and choosing the Gaussian Surface in such a way that the sum on the left of Eq. I can be calculated right away.

Example 1:

Point charge $+Q$ located at the origin, $r=0$. Because of spherical symmetry about $r=0$, the \vec{E} -field cannot depend on angle. It must be radial (*along* \hat{r}) and be a function of r only. We should choose for our surface a sphere of radius r centered at $r=0$. Then I) E has the same magnitude at all points on the surface of the sphere and II)

$$\vec{E} \cdot \vec{\Delta A} = E(r)4\pi r^2 = \frac{Q}{\epsilon_0}$$

because

$$\vec{E} \parallel \hat{r} \text{ everywhere so } \vec{E} \cdot \hat{n} = E$$

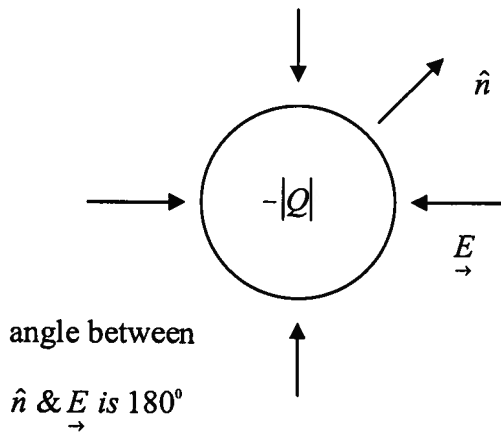
$$\vec{E} = \frac{Q}{4\pi\epsilon_0 r^2} \hat{r} = \frac{k_e Q}{r^2} \hat{r}$$

Example 2:

$-|Q|$ Located at $r=0$. Again \vec{E} can only be a function of r and directed along

$-\hat{r}$ so (*FLUX is inward*) $-E \cdot 4\pi r^2 = \frac{-|Q|}{4\pi\epsilon_0}$

$$\vec{E}(r) = \frac{-|Q|}{4\pi\epsilon_0} \hat{r}$$



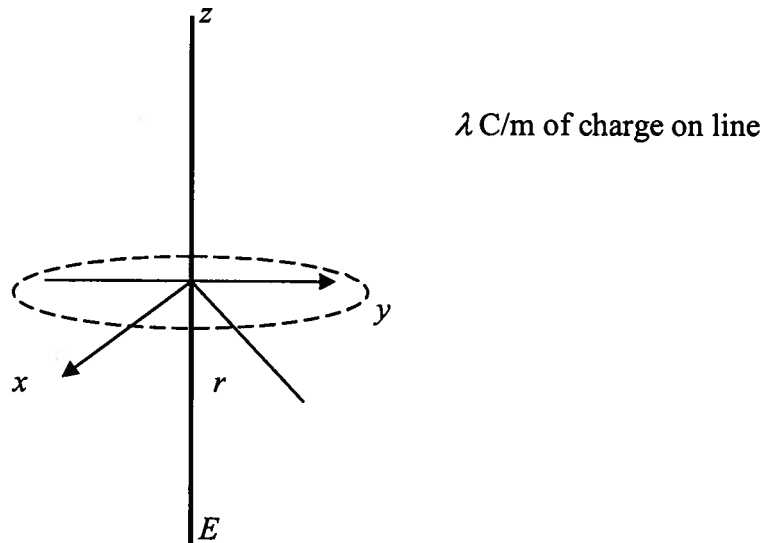
Example 3:

Long Line of charge: λ C/m along \hat{z} . Now there is cylindrical symmetry about z-axis.

Field cannot depend on z .

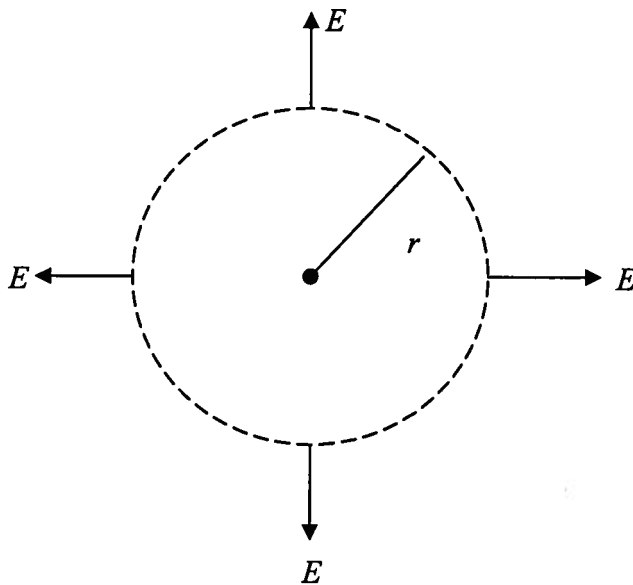
Field cannot depend on Angle.

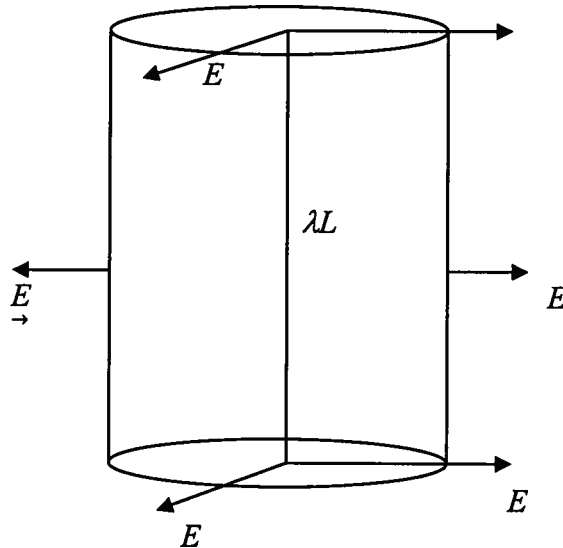
Must be a fn of r only and directed along r .



Choose as surface cylinder of length L , radius r , axis of cylinder on z -axis.

Then $\Sigma_c \vec{E} \cdot \vec{\Delta A} = E(r)2\pi rL$. No contribution from End surfaces as \vec{E} is parallel to them.





Also

$$\sum Q_i = \lambda L$$

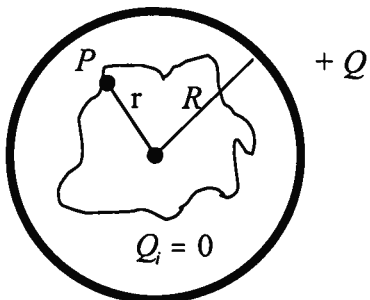
so

$$E(r) \cdot 2\pi r L = \lambda L$$

$$\vec{E}(r) = \frac{\lambda}{2\pi r \epsilon_0} \hat{r}$$

Example 4:

Hollow sphere (spherical shell) has charge $+Q$ on its surface, radius R and is centered at $r=0$. Again symmetry about $r=0$, requires that $\vec{E} \parallel \hat{r}$ and is a function r only.



First, consider $r < R$. Choose any surface through r . $r < R \implies \sum_c \vec{E} \cdot \Delta A = 0 \implies$ No enclosed charge.

If $\sum_c \vec{E} \cdot \Delta A = 0$ for any and every surface as long as $r < R$, it implies $\vec{E} = 0$

If $r > R$; the appropriate surface is a sphere of radius r and now $\sum_c \vec{E} \cdot \Delta \vec{A} = E(r)4\pi r^2$

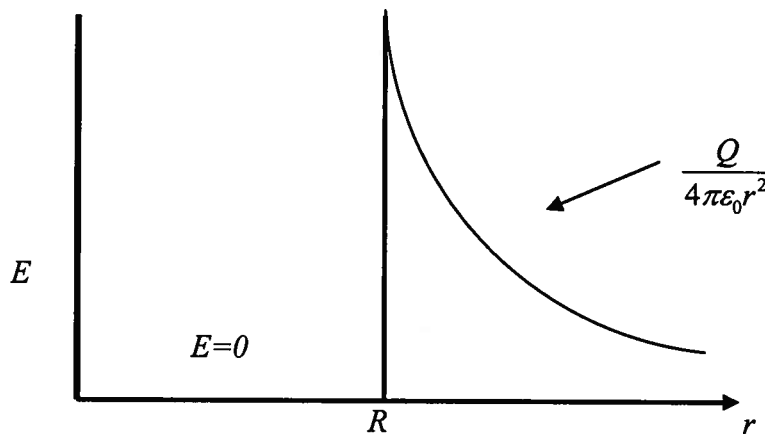
Hence,

$$\begin{aligned} \vec{E} &= \frac{Q}{4\pi\epsilon_0 r^2} \hat{r} \\ &= \frac{k_e Q}{r^2} \hat{r} \end{aligned}$$

$r > R$

as if the shell were replaced by a single charge Q located at its center ($r=0$). That is, r must be measured from the center of the shell.

Sph. Shell: Q on shell



Note: As one goes from $r < R$ to $r > R$, the \vec{E} -field jumps by

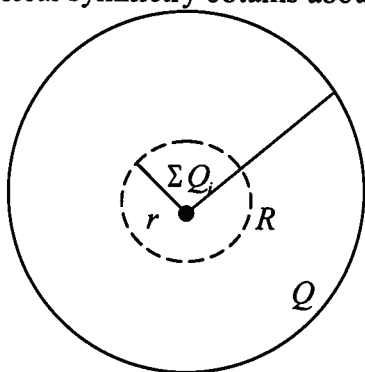
$$\begin{aligned} E &= \frac{Q}{4\pi\epsilon_0 r^2} \\ &= \frac{\sigma}{\epsilon_0} \end{aligned} \quad \left[\sigma = \frac{Q}{4\pi r^2} \right]$$

Where σ = charge density on the surface of the shell. Crossing a sheet of charge \vec{E} jumps!

Ex5: Insulating Solid Sphere

Charge Q distributed uniformly over a sphere of radius R which is located with its center at $r=0$. First define charge density $\rho = \frac{Q}{\frac{4\pi}{3} R^3}$ $[\rho = Rho]$

Again, spherical symmetry obtains about $r=0$.



Now, if $r < R$.

$$\sum_c \vec{E} \cdot \vec{\Delta A} = E(r) 4\pi r^2$$

$$\Sigma Q_i = \rho \cdot \frac{4\pi}{3} r^3$$

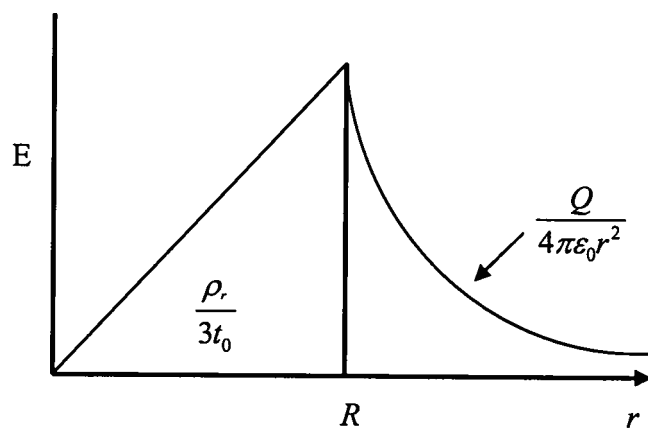
so for $r < R$.

$$E(r) 4\pi r^2 = \frac{\rho}{\epsilon_0} \cdot \frac{4\pi}{3} r^3$$

$$\vec{E} = \frac{\rho}{3\epsilon_0} r \hat{r} \quad r < R$$

If $r > R$ all of Q contributes so

$$\vec{E} = \frac{Q}{4\pi\epsilon_0 r^2} \hat{r} \quad r > R$$



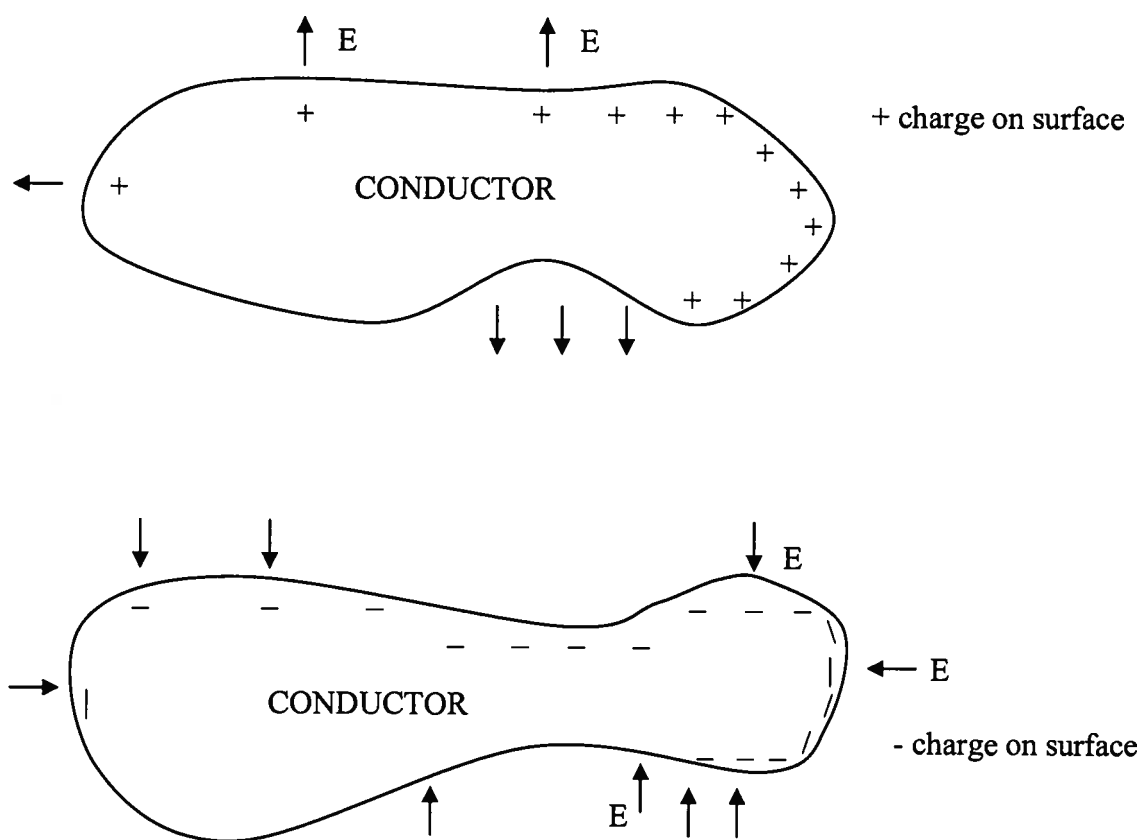
INSULATING SPHERE UNIFORMLY CHARGED with $\rho = \frac{Q}{\frac{4\pi}{3}R^3}$

Ex 6:

Conductor under stationary conditions: charge NOT allowed to move. If charge has to be immobile the field inside must be zero at every point. This is possible only if $Q=0$ at every point inside the conductor. So under stationary conditions charge can reside ONLY ON the surface of the conductor consequences:

Q on conducting sphere \equiv Hollow spherical charge

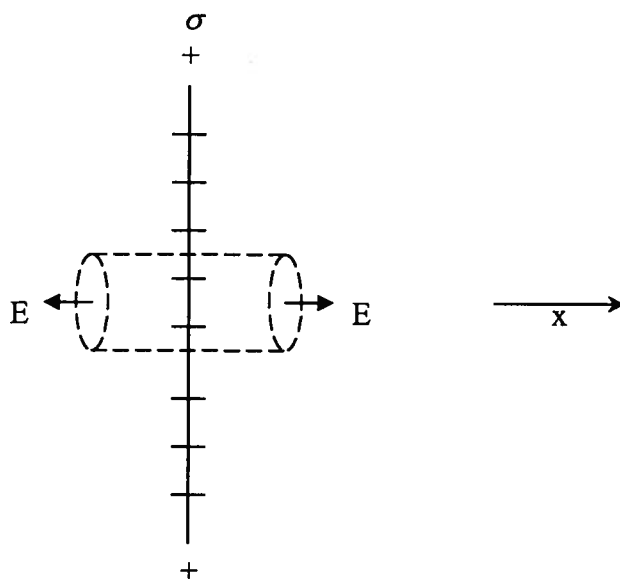
Further, \vec{E} at surface must be perpendicular to surface otherwise charges will start moving along surface.



Notice: In both cases force on charge is outward, that is, charge is bound to the surface.

Ex 7:

Sheet of charge $\perp x$ -axis carries $+\sigma \text{ C/m}^2$ of charge. Sheet located at $x=0$. Look at it end-on Recall that we have shown if two equal charges are at $+y$ and $-y$ the \vec{E} field is purely along \hat{x} .



So here \vec{E} along $+\hat{x}$ on right - \hat{x} on left. Choose cylinder as Gaussian Surface.

$$\begin{aligned}\Sigma_c \vec{E} \cdot \vec{\Delta A} &= E\pi r^2 + E \cdot \pi r^2 \\ &= E \cdot 2\pi r^2\end{aligned}$$

and

$$\Sigma Q_i = \sigma \pi r^2 \quad \text{[charge enclosed by cylinder]}$$

so

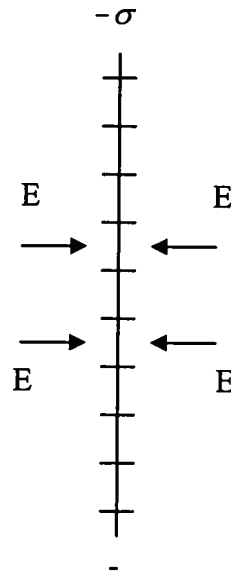
$$2E\pi r^2 = \sigma \cdot \pi r^2$$

$$\begin{aligned}\vec{E} &= +\frac{\sigma}{2\epsilon_0} \hat{x} \quad x > 0 \\ &= -\frac{\sigma}{2\epsilon_0} \hat{x} \quad x < 0\end{aligned}$$

Ex 7': Sheet carries $-\sigma C/m^2$

Then
$$\vec{E} = -\frac{\sigma}{2\epsilon_0} \hat{x} \quad x > 0$$

$$= +\frac{\sigma}{2\epsilon_0} \hat{x} \quad x < 0$$



Ex 8: Sheet at $x=0$, $\sigma C/m^2$

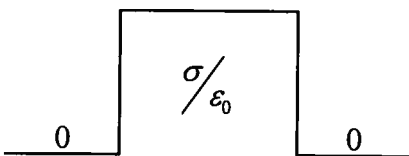
Sheet at $x=d$, $-\sigma C/m^2$

$$\vec{E} = 0, \quad x < 0$$

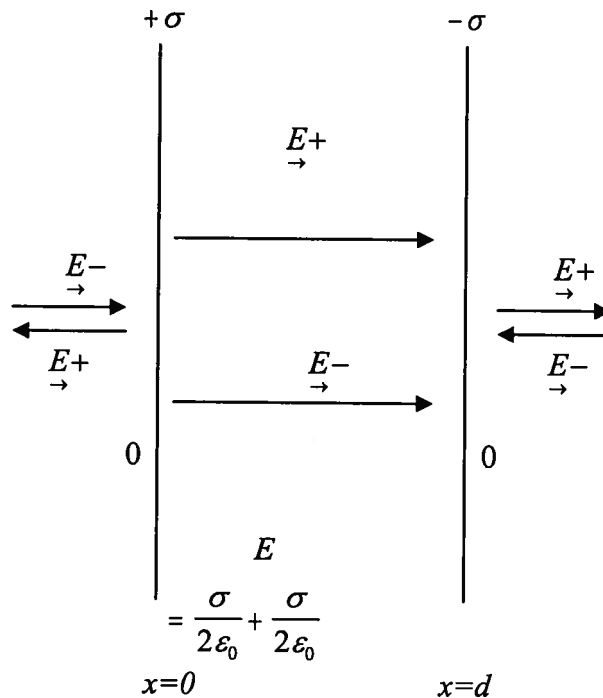
$$\vec{E} = \frac{\sigma}{\epsilon_0} \quad 0 < x < d$$

$$\vec{E} = 0 \quad x > d$$

Net \vec{E} -field:



Now the \vec{E}_+ and \vec{E}_- fields will add vertically. Hence



Again \vec{E} jumps on crossing sheet of charge