VECTOR ALGEBRA / TRIG. IDENTITIES.

VECTOR ($\underset{\rightarrow}{V}$): A mathematical object which has both a magnitude and a direction

SCALAR Has Magnitude only (S)

I If you multiply a vector $\underset{\rightarrow}{V}$ by a scalar S you get a Vector $\underset{\rightarrow}{V}' = S\underset{\rightarrow}{V}$ such that $\underset{\rightarrow}{V}' \parallel \underset{\rightarrow}{V}$ and has magnitude sV.
This property allows us to express any vector as a product of a scalar (magnitude) and a unit vector (magnitude 1, direction only). Hence, we have written

$$\underset{\rightarrow}{A} = A\hat{x}$$

as a vector of magnitude A in $+x$ direction
Indeed, a vector along any direction $\hat{d}$ can be written as

$$\underset{\rightarrow}{V} = V\hat{d}$$

II Addition of vectors. Given vectors $\underset{\rightarrow}{V_1}$ and $\underset{\rightarrow}{V_2}$, we want to determine

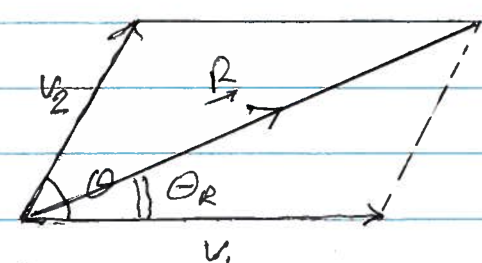
$$\underset{\rightarrow}{R} = \underset{\rightarrow}{V_1} + \underset{\rightarrow}{V_2} \qquad -(1).$$

There are Three methods for doing This:

i) Geometry

Choose a scale to represent $\underset{\rightarrow}{V_1}$ and $\underset{\rightarrow}{V_2}$ and draw a parallelogram.

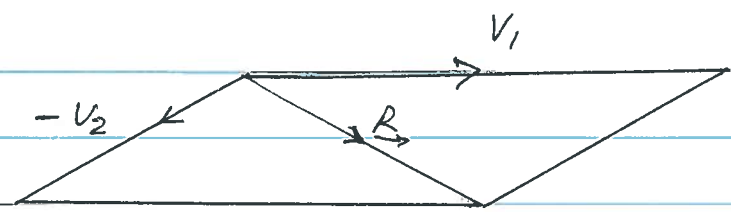
The long diagonal gives you $\underset{\rightarrow}{R} = \underset{\rightarrow}{V_1} + \underset{\rightarrow}{V_2}$.



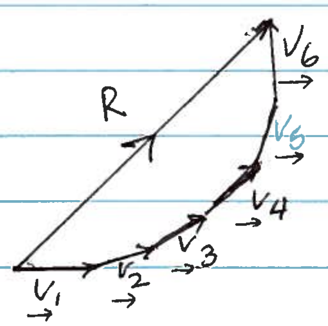
You can get magnitude of R by using scale and of course measure θ_R

Also,

$\vec{R} = \vec{V}_1 - \vec{V}_2$



is determined by the short diagonal. Repeated application of this construct will allow you to add as many vectors



$$\vec{R} = \vec{V}_1 + \vec{V}_2 + \vec{V}_3 + \vec{V}_4 + \vec{V}_5 + \vec{V}_6$$

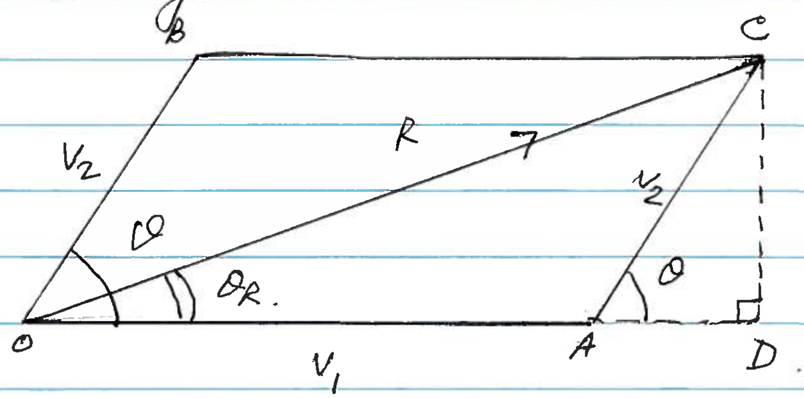
as the vector which connects the "tail" of $\vec{V}_1$ to the "head" of $\vec{V}_6$.

Further, it immediately follows that if all the vectors are parallel to one another

$$\vec{R} = V_1 \hat{d} + V_2 \hat{d} + V_3 \hat{d} - V_4 \hat{d} \ldots$$
$$= (V_1 + V_2 + V_3 - V_4 + \ldots)\,\hat{d}$$

(ii) Algebra / Trig.

We want to calculate R, so as shown drop a $\perp$ from C to OA extended. Clearly,



$$\frac{CD}{V_2} = \sin\theta$$

$$\frac{AD}{V_1} = \cos\theta$$

Using Pythagoras' Theorem

$$R^2 = OD^2 + CD^2$$

$$= (V_1 + V_2 \cos\theta)^2 + (V_2 \sin\theta)^2$$

$$= V_1^2 + V_2^2 \cos^2\theta + 2 V_1 V_2 \cos\theta + V_2^2 \sin^2\theta.$$

That is

$$R = \sqrt{V_1^2 + V_2^2 + 2 V_1 V_2 \cos\theta} \qquad [2]$$

Also

$$\tan\theta_R = \frac{CD}{OD} = \frac{V_2 \sin\theta}{V_1 + V_2 \cos\theta} \qquad [3]$$

So indeed we have determined both the magnitude (Eq 2) and direction [Eq 3] of the vector $\underset{\rightarrow}{R} = \underset{\rightarrow}{V_1} + \underset{\rightarrow}{V_2}$

Again, if we have more than 2 vectors we can use Eqs. [2] and [3] repeatedly to arrive at $\underset{\rightarrow}{R} = \underset{\rightarrow}{V_1} + \underset{\rightarrow}{V_2} + \underset{\rightarrow}{V_3} + \cdots$.

(iii) The method of components.

This is the most elegant procedure for adding (or subtracting) many vectors.

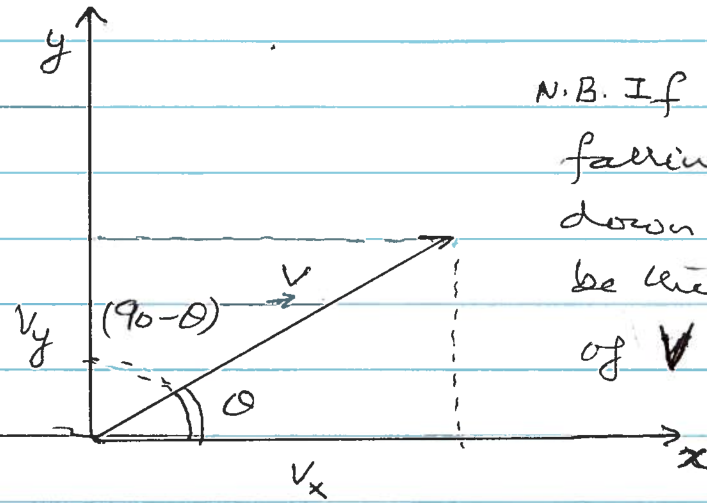
We begin by defining

Component of a vector $\underset{\rightarrow}{V}$ along any direction $\hat{d}$ is a SCALAR quantity

$$V_d = V \cos(\underset{\rightarrow}{V}, \hat{d})$$

That is, V_d = [magnitude of V] × [Cosine of angle between $\underset{\rightarrow}{V}$ and $\hat{d}$].

Let us put our vector $\underset{\rightarrow}{V}$ in the x-y coordinate system and we see immediately that



N.B. If light were falling straight down V_x would be the "shadow" of **V** along x.

$$V_x = V \cos\theta$$

$$V_y = V \cos(90-\theta) = V \sin\theta$$

And clearly $V = \sqrt{V_x^2 + V_y^2}$ or $\underset{\rightarrow}{V} = V_x \hat{x} + V_y \hat{y}$

$$\tan\theta = \frac{V_y}{V_x}$$

This tells us that a vector can be specified either by writing magnitude (V) and direction (θ) or by writing the magnitudes of its components.

So now if we have many vectors:

$$\underset{\rightarrow}{V_1} = V_{1x}\hat{x} + V_{1y}\hat{y}$$

$$\underset{\rightarrow}{V_2} = V_{2x}\hat{x} + V_{2y}\hat{y}$$

$$\vdots$$

$$\underset{\rightarrow}{V_i} = V_{ix}\hat{x} + V_{iy}\hat{y}$$

$$\underset{\rightarrow}{R} = \sum \underset{\rightarrow}{V_i} = \sum V_{ix}\hat{x} + \sum V_{iy}\hat{y} \quad \rightarrow [4]$$

$$= R_x\hat{x} + R_y\hat{y} \quad \rightarrow [4']$$

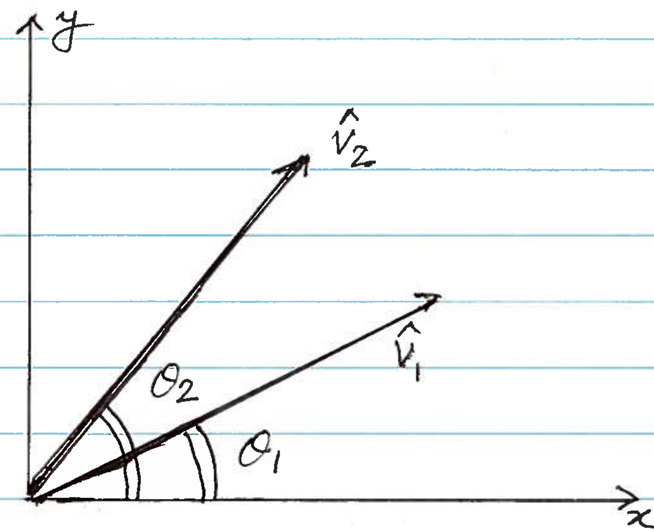
And hence $R = \sqrt{R_x^2 + R_y^2}$ [5]

$\tan \theta_R = \dfrac{R_y}{R_x}$ [6]

where θ_R is angle between $\vec{R}$ and $\hat{x}$.

TRIG IDENTITIES.

Take two unit vectors $\hat{V}_1$ and $\hat{V}_2$ making angles θ_1 and θ_2 with the axis of x as shown.



$\vec{R} = \hat{V}_1 + \hat{V}_2$

FROM Eq (1) $R = \sqrt{1 + 1 + 2\cos(\theta_2 - \theta_1)}$ (7)

Also $\hat{V}_1 = \cos\theta_1 \hat{x} + \sin\theta_1 \hat{y}$

$\hat{V}_2 = \cos\theta_2 \hat{x} + \sin\theta_2 \hat{y}$

so $R_x = (\cos\theta_1 + \cos\theta_2)$

$R_y = (\sin\theta_1 + \sin\theta_2)$

$$R = \sqrt{(\cos\theta_1 + \cos\theta_2)^2 + (\sin\theta_1 + \sin\theta_2)^2}$$

$$= \sqrt{\cos^2\theta_1 + \cos^2\theta_2 + 2\cos\theta_1\theta_2 + \sin^2\theta_1 + \sin^2\theta_2 + 2\sin\theta_1\sin\theta_2}$$

$$= \sqrt{1 + 1 + 2[\cos\theta_1\cos\theta_2 + \sin\theta_1\sin\theta_2]} \quad (8)$$

Compare Eqs (7) and [8] and you get the Trig Identity:

$$\cos(\theta_1 - \theta_2) = \cos\theta_1 \cos\theta_2 + \sin\theta_1 \sin\theta_2 \rightarrow I_1$$

Next, let $\theta_1 = (\frac{\pi}{2} - \theta_3.)$

$$\cos\left(\frac{\pi}{2} - \theta_3 - \theta_2\right) = \sin(\theta_3 + \theta_2)$$

$$= \cos(\frac{\pi}{2} - \theta_3) \cos\theta_2 + \sin(\frac{\pi}{2} - \theta_3) \sin\theta_2$$

Which gives another identity

$$\sin(\theta_3 + \theta_2) = \sin\theta_3 \cos\theta_2 + \cos\theta_3 \sin\theta_2 \rightarrow I_2$$

If in I_1 you put $\theta_4 = -\theta_2$. and remember that $\sin(-\theta) = -\sin\theta$.

you get

$$\cos(\theta_1 + \theta_4) = \cos\theta_1 \cos\theta_4 - \sin\theta_1 \sin\theta_4) \rightarrow I_3.$$

and similarly

$$\sin(\theta_3 - \theta_5) = \sin\theta_3 \cos\theta_5 - \sin\theta_5 \cos\theta_3 \rightarrow I.$$