VECTOR ALGEBRA/ TRIG. IDENTITIES

Vector ($\underset{\rightarrow}{V}$) : A mathematical object which has both a magnitude and a direction.
Scalar (S): Has magnitude only

I. If you multiply a vector $\underset{\rightarrow}{V}$ by a scalar S you get a vector $\underset{\rightarrow}{V}' = S\underset{\rightarrow}{V}$ such that $\underset{\rightarrow}{V}' \parallel \underset{\rightarrow}{V}$ and has magnitude SV. This property allows us to express any vector as a product of a scalar (magnitude) and a unit vector (magnitude 1, direction only). Hence, we have written:

$$\underset{\rightarrow}{A} = A\hat{x}$$

as a vector of magnitude A in the $+x$ direction. Indeed, a vector along any direction $\hat{d}$ can be written as:

$$\underset{\rightarrow}{V} = V\hat{d}$$

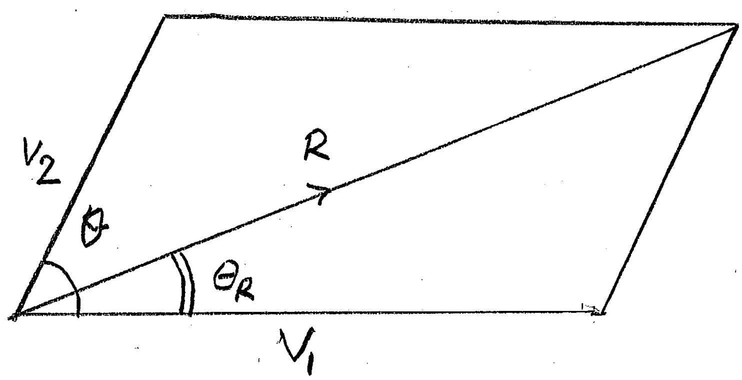
II. Addition of Vectors. Given vectors $\underset{\rightarrow}{V_1}$ and $\underset{\rightarrow}{V_2}$ we want to determine the Resultant Vector

$$\underset{\rightarrow}{R} = \underset{\rightarrow}{V_1} + \underset{\rightarrow}{V_2}$$

There are three methods for doing this:

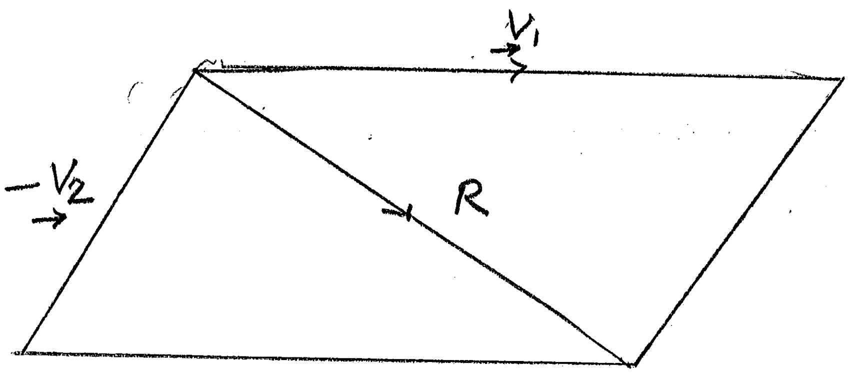
(i) Geometry

Choose a scale to represent $\underset{\rightarrow}{V_1}$ and $\underset{\rightarrow}{V_2}$, and draw a parallelogram.



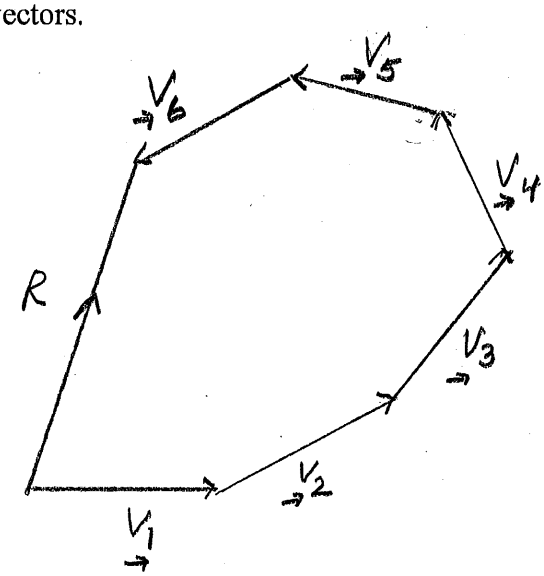
The long diagonal gives you $\underset{\rightarrow}{R} = \underset{\rightarrow}{V_1} + \underset{\rightarrow}{V_2}$

You can get magnitude of R by using a scale, and of course measure Θ_R with a protractor.



Also,
$\underset{\rightarrow}{R} = \underset{\rightarrow}{V_1} - \underset{\rightarrow}{V_2}$

is determined by the short diagonal. Repeated application of this construct will allow you to add many vectors.



$\underset{\rightarrow}{R} = \underset{\rightarrow}{V_1} + \underset{\rightarrow}{V_2} + \underset{\rightarrow}{V_3} + \underset{\rightarrow}{V_4} + \underset{\rightarrow}{V_5} + \underset{\rightarrow}{V_6}$
as the vector which connects the "tail of $\underset{\rightarrow}{V_1}$
to the head of $\underset{\rightarrow}{V_6}$.

Further, it immediately follows that if all the vectors are parallel to one another
$$\underset{\rightarrow}{R} = V_1\hat{d} + V_2\hat{d} + V_3\hat{d} - V_4\hat{d}...$$

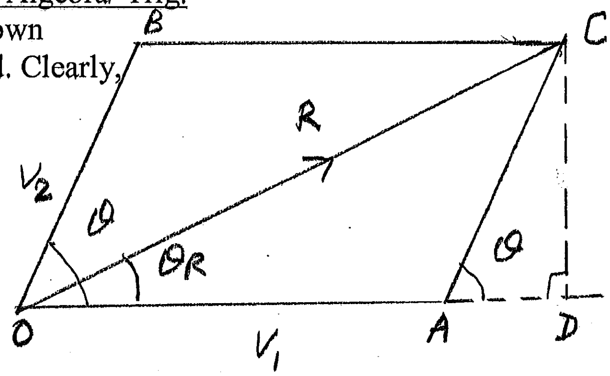
$$=(V_1 + V_2 + V_3 - V_4 + ...)\hat{d}$$

<u>(ii) Algebra/ Trig.</u>

We want to calculate R, so as shown drop a $\perp$ from C to OA extended. Clearly,

$$\frac{CD}{V_2} = Sin\Theta$$

$$\frac{AD}{V_2} = Cos\Theta$$



using Pythagoras' Theorem

$$R^2 = OD^2 + CD^2$$
$$=(V_1 + V_2 Cos\Theta)^2 + (V_2 Sin\Theta)^2$$
$$=V_1 + V_2 Cos^2\Theta + 2V_1V_2Cos\Theta + V_2^2 Sin^2\Theta$$

That is

$$R = \sqrt{V_1^2 + V_2^2 + 2V_1V_2Cos\Theta} \qquad [2]$$

Also

$$\tan\Theta_R = \frac{CD}{OD} = \frac{V_2 Sin\Theta}{V_1 + V_2 Cos\Theta} \qquad [3]$$

So indeed we have determined both the magnitude [Eq2] and direction [Eq3] of the vector $\underline{R} = (\underline{V_1} + \underline{V_2})$

Again, if we have more than 2 vectors we can use Eq. [2] and [3] repeatedly to arrive at $\underline{R} = \underline{V_1} + \underline{V_2} + \underline{V_3} + ...$

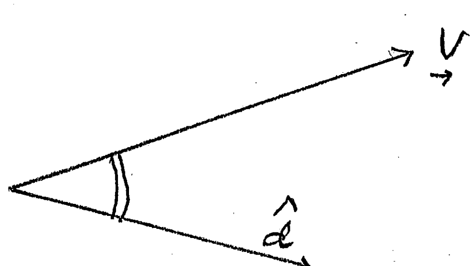
<u>(iii) The Method of Components</u>

This is the most elegant procedure for adding (or subtracting) many vectors.

We begin by defining that the component of a vector $\underline{V}$ along any direction $\hat{d}$ is a <u>Scalar</u> quantity.

$$V_d = V\cos(\underline{V}, \hat{d})$$

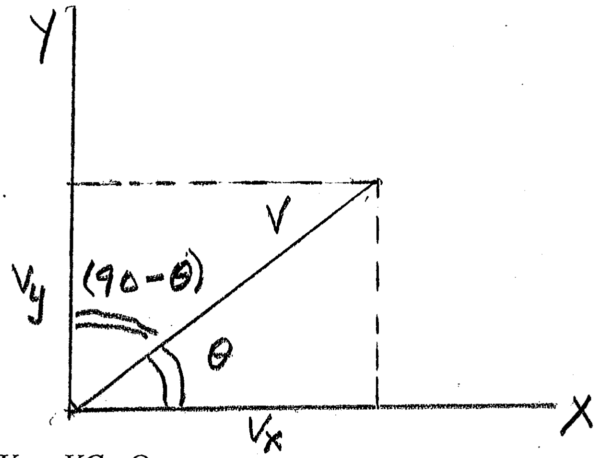
That is, V_d = [magnitude of V] $\times$ [Cosine of angle between $\underline{V}$ and $\hat{d}$]



Let us put our vector $\underline{V}$ in the x-y coordinate

N. B. If light were falling straight down system and we see immediately that:

V_x would be the "shadow" of V along x.



$V_x = VCos\Theta$

$V_y = V\cos(90-\Theta) = VSin\Theta$

and clearly $V = \sqrt{V_x^2 + V_y^2}$

or $\underrightarrow{V} = V_x\hat{x} + V_y\hat{y}$

$\tan\Theta = \frac{V_y}{V_x}$

This tells us that a vector can be specified either by writing magnitude (V) and direction (ϑ) or by writing the magnitudes of its components.
So now if we have many vectors:

$\underrightarrow{V_1} = V_{1x}\hat{x} + V_{1y}\hat{y}$

$\underrightarrow{V_2} = V_{2x}\hat{x} + V_{2y}\hat{y}$

.

.

$\underrightarrow{V_i} = V_{ix}\hat{x} + V_{iy}\hat{y}$

$\underrightarrow{R} = \Sigma\underrightarrow{V_i} = \Sigma V_{ix}\hat{x} + \Sigma V_{iy}\hat{y}$ → [4]

$= R_x\hat{x} + R_y\hat{y}$ → [4']

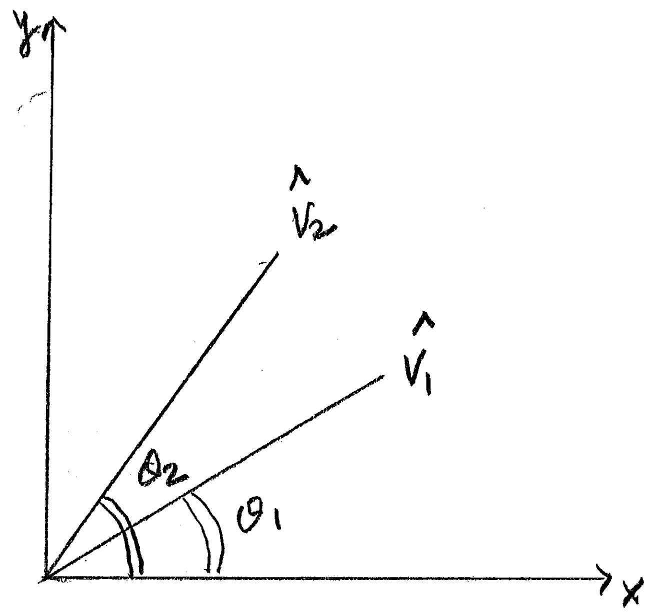
and hence $R = \sqrt{R_x^{\ 2} + R_y^{\ 2}}$ [5]

$\tan\Theta R = \frac{R_y}{R_x}$ [6]

where Θ_r is the angle between $\underrightarrow{R}$ and $\hat{x}$.

TRIG IDENTITIES

Take two unit vectors $\hat{V}_1$ and $\hat{V}_2$ making angles ϑ_1 and ϑ_2 with the axis of x as shown.



$\underset{\rightarrow}{R} = \hat{V}_1 + \hat{V}_2$

From Eq(1) $\qquad R = \sqrt{1+1+2Cos(\Theta_{2_1} - \Theta_1)}$ $\qquad$ [7]

Also $\qquad \hat{V}_1 = Cos\Theta_1\hat{x} + Sin\Theta_1\hat{y}$

$\qquad \hat{V}_2 = Cos\Theta_2\hat{x} + Sin\Theta_2\hat{y}$

so $\qquad R_x = (Cos\Theta_1 + Cos\Theta_2)$

$\qquad R_y = (Sin\Theta_1 + Sin\Theta_2)$

$$R = \sqrt{(Cos\Theta_1 + Cos\Theta_2)^2 + (Sin\Theta_1 + Sin\Theta_2)^2}$$
$$= \sqrt{Cos^2\Theta_1 + Cos^2\Theta_2 + 2Cos\Theta_1\Theta_2 + Sin^2\Theta_1 + Sin^2\Theta_2 + 2Sin\Theta_1 Sin\Theta_2}$$
$$= \sqrt{1+1+2[Cos\Theta_1 Cos\Theta_2 + Sin\Theta_1 Sin\Theta_2]} \qquad [8]$$

Compare Eqs [7] and [8] and you get the trig identity:

$$Cos(\Theta_1 - \Theta_2) = Cos\Theta_1 Cos\Theta_2 + Sin\Theta_1 Sin\Theta_2 \rightarrow I_1$$

Next, let $\Theta_1 = (\frac{\pi}{2} - \Theta_3)$

$$Cos(\frac{\pi}{2} - \Theta_3 - \Theta_2) = Sin(\Theta_3 + \Theta_2)$$

$$= Cos(\frac{\pi}{2} - \Theta_3)Cos\Theta_2 + Sin(\frac{\pi}{2} - \Theta_3)Sin\Theta_2$$

Which gives another identity

$$Sin(\Theta_3 + \Theta_2) = Sin\Theta_3 Cos\Theta_2 + Cos\Theta_3 Sin\Theta_2 \rightarrow I_2$$

if in I_1 you put $\vartheta_4 = -\vartheta_2$ and remember that

$$Sin(-\Theta) = -Sin\Theta$$

you get

$$Cos(\Theta_1 + \Theta_4) = Cos\Theta_1 Cos\Theta_4 - Sin\Theta_1 Sin\Theta_4) \rightarrow I_3$$

and similarly

$$Sin(\Theta_3 - \Theta_5) = Sin\Theta_3 Cos\Theta_5 - Sin\Theta_5 Cos\Theta_3 \rightarrow I_4$$