

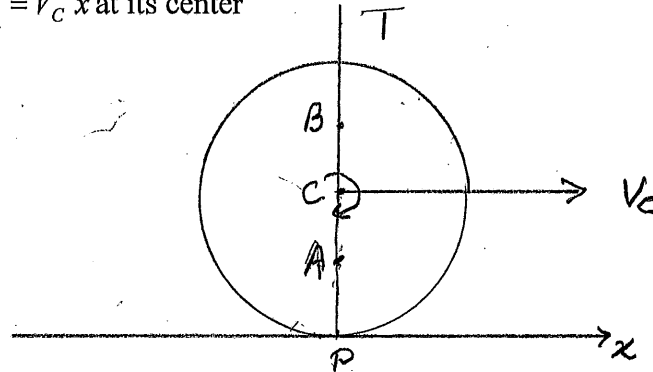
Rolling Without Slip; etc

A particularly interesting case of rotation arises when a ring or disk or cylinder or sphere rolls along a solid surface.

Case I: Rolling without slipping

Consider the case where surface is horizontal and the roller has a constant velocity $\underline{V}_C = V_C \hat{x}$ at its center

CP = R



\underline{V}_C is constant so acceleration $\underline{a} = 0$. No force involved. If there is no SLIP, the velocity at the point of contact P must be ZERO at ALL times. That is, the point on the circle which comes into contact with the surface changes with time but at the instant of contact $\underline{V}_P = 0$ always.

To achieve this, the object must have an angular velocity $\underline{\omega}$ such that the tangential velocity \underline{V}_t at P, due to the rotation, is exactly equal and opposite to \underline{V}_C .

$$\begin{aligned} \text{This will ensure that } \underline{V}_P = \underline{V}_C + \underline{V}_t &= 0 \\ \underline{V}_C \hat{x} - R\omega \hat{x} &= 0 \\ \omega &= \frac{V_C}{R} \end{aligned}$$

and for the case shown in the figure

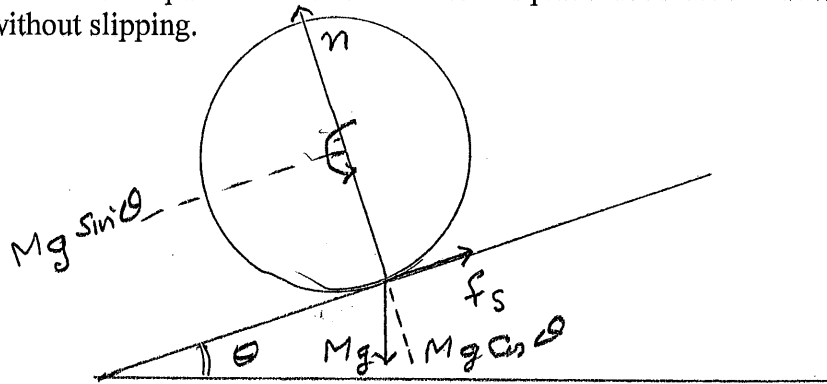
$$\underline{\omega} = -\frac{V_C}{R} \hat{z}$$

$\underline{\omega}$ is constant so $\underline{\alpha} = 0$ [NO TORQUE]

It is interesting to ask what are the velocities at the points A, C, B and T in the roller.

$$\begin{aligned} \left(AC = \frac{R}{2}\right) \quad \underline{V}_A &= \underline{V}_C - \frac{R\omega}{2} \hat{x} = \frac{V_C}{2} \hat{x} \\ \underline{V}_C &= \underline{V}_C \\ \left(BC = \frac{R}{2}\right) \quad \underline{V}_B &= \underline{V}_C - \frac{R\omega}{2} \hat{x} = \frac{3}{2} V_C \hat{x} \\ \underline{V}_T &= \underline{V}_C + R\omega \hat{x} = 2 V_C \hat{x} \end{aligned}$$

Case II: Let us put our roller on an inclined plane and let it roll down the incline without slipping.



Now it will have both a linear acceleration and an angular acceleration. We have drawn all the effective forces acting on the roller.

For the linear acceleration

$$(M\vec{a} = \sum \vec{F}_i) \quad -Ma = -Mg \sin\theta + f_s \quad \rightarrow (1)$$

For the angular acceleration

$$(I\vec{\alpha} = \sum \vec{\tau}_i) \quad -I\alpha = -R f_s \quad \rightarrow (2)$$

Since there is no slip, velocity and acceleration at P must be ZERO at all times and this requires

$$\alpha = \frac{a}{R} \quad \rightarrow (3)$$

From (2) and (3)

$$f_s = \frac{I\alpha}{R} = \frac{Ia}{R^2}$$

and substituting in (1)

$$Ma = Mg \sin\theta - \frac{Ia}{R^2}$$

$$a = \frac{g \sin\theta}{1 + \frac{I}{MR^2}} \quad \rightarrow (4)$$

Moments of Inertia

Ring $I = MR^2$

Disk $I = \frac{MR^2}{2}$

Cylinder $I = \frac{MR^2}{2}$

Sphere (hollow) $I = \frac{2}{3}MR^2$

Sphere (solid) $I = \frac{2}{5}MR^2$

Hence a is independent of M and R. It only depends on how mass is distributed around the axis of rotation.

Clearly, the ring has the smallest acceleration

$$a_{ring} = \frac{-g \sin \theta}{2} \hat{x}$$

and the solid sphere has the largest acceleration

$$a_{s.s.} = \frac{-g \sin \theta}{1.4} \hat{x}$$

Next, it must be realized that the static friction force cannot exceed $\mu_s n$

Because $f_s \leq \mu_s n$

So $f_s \leq \mu_s Mg \cos \theta$

From Eq (1) and Eq (4)

$$\begin{aligned} f_s &= Mg \sin \theta - Ma \\ &= Mg \sin \theta \left[1 - \frac{1}{1 + \frac{I}{MR^2}} \right] \\ &= Mg \sin \theta \left[\frac{\frac{I}{MR^2}}{1 + \frac{I}{MR^2}} \right] \end{aligned}$$

So if we start increasing θ eventually f_s becomes equal to its largest value and the roller will slip

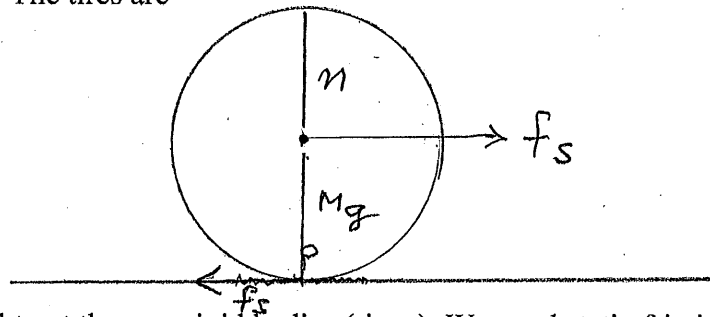
$$\begin{aligned} Mg \sin \theta \left[\frac{\frac{I}{MR^2}}{1 + \frac{I}{MR^2}} \right] &= \mu_s Mg \cos \theta \\ \tan \theta &= \mu_s \left[\frac{1 + \frac{I}{MR^2}}{\frac{I}{MR^2}} \right] \end{aligned}$$

Ring will be the first to slip $[\tan \theta = 2\mu_s]$

Note

In the above motion, the force of gravity provided the linear acceleration and f_s provided the torque.

Case III: It is interesting to compare this with the way your automobile gets going on a horizontal surface. The tires are



fairly complex but we will treat them as rigid bodies (rings). We need static friction (as anyone who has tried to get going on an icy road knows, the tires spin in place). But now the Torque is provided by the engine (as you engage the gear) and the tire pushes back on the road with f_s and by Newton's Third Law the road pushes the car forward. Again

$$f_s \leq \mu_s n \quad (n = Mg)$$

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and as always $M\vec{a} = \vec{F} = \vec{f}_s$

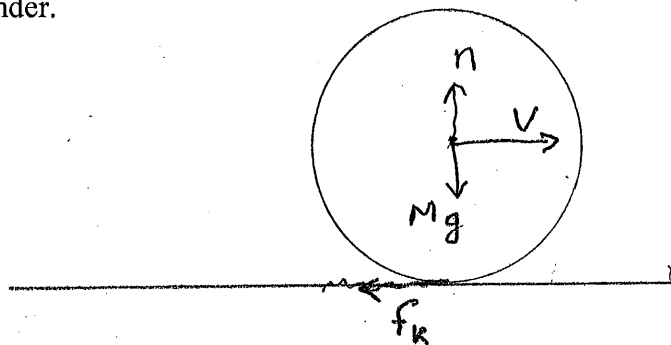
so

$$a \leq \mu_s g$$

Maximum acceleration is $\mu_s g$ in magnitude.

Case IV: After comparing case I and case IV you can begin to understand why while driving on a slippery road it is recommended that one maintains a constant speed ($\vec{F} = 0$) and definitely must avoid excessive use of acceleration/ brake ($\vec{F} \neq 0$).

Case V: When you go bowling you throw the ball so that when it arrives on the Shute surface it has a linear velocity $V_i \hat{x}$ and it slips along the surface. However, once it touches the surface kinetic friction comes into play. Let us see how this leads to rolling without slip. We will take the ~~more~~ general case of the roller being sphere, ring, and cylinder.



There is only one force acting on the roller

$$\vec{f}_k = -\mu_k Mg \hat{x} \quad [n - Mg = 0]$$

so

$$\vec{a} = -\mu_k g \hat{x}$$

and

$$v = (v_i - \mu_k g t) \hat{x}$$

However, now there is also a torque about the axis through the center

$$\vec{\tau} = -R f_k \hat{z}$$

so there is an angular acceleration

$$[I\vec{\alpha} = \vec{\tau}] \quad \vec{\alpha} = \frac{-R f_k \hat{z}}{I}$$

where I is the moment of Inertia.

The angular velocity

$$\vec{\omega} = 0 - \frac{R f_k t \hat{z}}{I}$$

$$\vec{\omega} = -\frac{\mu_k M g R t \hat{z}}{I}$$

and to get the condition for case I $\left[\omega = \frac{V}{R} \right]$, we can look for time t_1 when

$$v_i - \mu_k g t_1 = + \frac{\mu_k M g R^2 t_1}{I}$$

$$t_1 = \frac{v_i}{\mu_k g \left[1 + \frac{M R^2}{I} \right]}$$

Notice t_1 , is also independent of M and R since

$I = (\text{const}) \times M R^2$ for all rollers.

At later times we have pure roll, $\dot{y} = \text{const.}$, $\dot{\omega} = \text{const.}$ and there is no force or torque on the roller (case I).

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At later times we have pure roll, $\gamma = \text{const.}$, $\vec{\omega} = \text{const.}$ and there is no force or torque on the roller (case I).