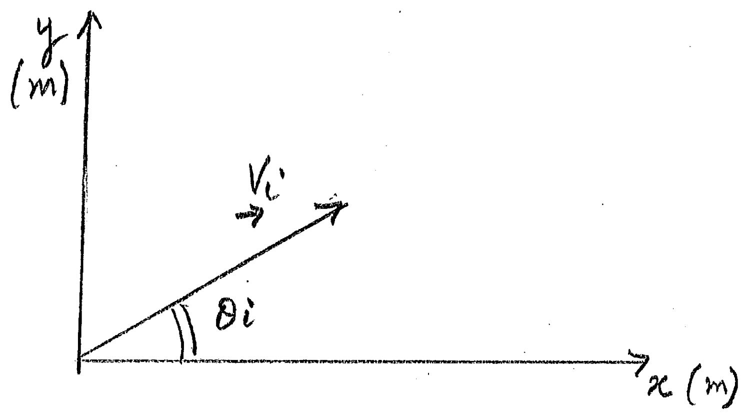
KINEMATICS – TWO DIMENSIONS – PROJECTILE MOTION

At $t = 0$ a projectile is launched from the origin $(x_i = 0, y_i = 0)$ with a velocity of v_i m/sec at angle of Θ_i above the horizon (x-axis). What are the equations which describe its motion in the xy–plane? It is best to write down the components and then the vectors.



	x-component	y-component	Vector
Acceleration	0	$-9.8 m/\sec^2$	$\underset{\to}{a} = O\hat{x} - 9.8 m/s^2 \hat{y}$ → (1)
Velocity	$v_i \cos\Theta_i$	$v_i \sin\Theta_i - 9.8t$	$\underset{\to}{v} = (v_i \cos\Theta_i)\hat{x} - (v_i \sin\Theta_i - 9.8t)\hat{y}$ → (2)
Position	$(v_i \cos\Theta_i)\ t$	$(v_i \sin\Theta_i)\ t - 4.9t^2$	$\underset{\to}{r} = (v_i \cos\Theta_i)t\hat{x} + \left[(v_i \sin\Theta_i)t - 4.9t^2\right]\hat{y}$ → (3)

We can also write for the y-velocity

$$v_y^{\ 2} = (v_i \sin\Theta_i)^2 - 19.6y \qquad \to (4)$$

and we use Eq(4) when t is not known.

Questions

1. What is its path in the xy-plane as we saw the parabola in the water stream. To derive it note that

$$y = (v_i \sin\Theta_i)t - 4.9t^2$$

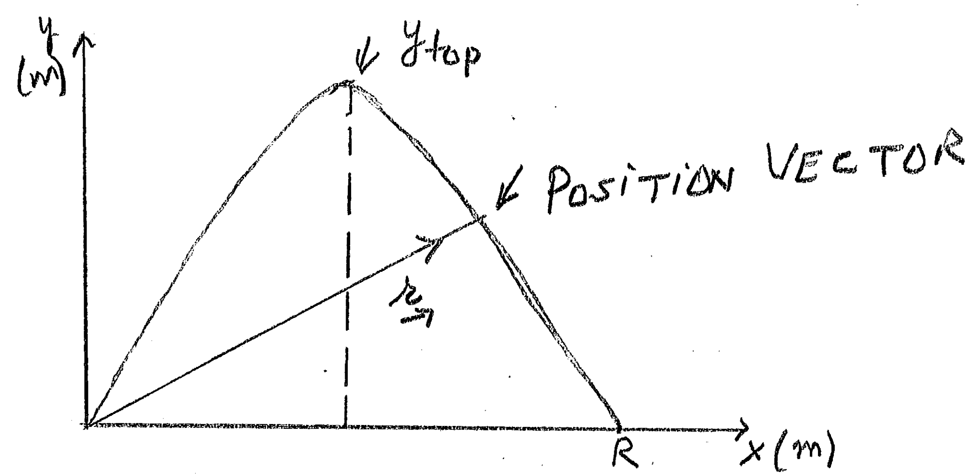
and $$x = (v_i \cos\Theta_i)t$$

so one can write

$$y = \frac{(v_i \sin\Theta_i)x}{(v_i \cos\Theta_i)} - 4.9\left(\frac{x}{v_i \cos\Theta_i}\right)^2$$

$$= x \tan\Theta_i - 4.9\left(\frac{x^2}{v_i \cos\Theta_i}\right) \qquad \to (5)$$

This is a very useful equation. Do not need to know t, relates y to x and v_i. See plot along side. It helps to define two quantities



y_{top} = highest point during flight
R = range; distance travelled before returning to Earth.

2. Why does it stop rising? Because the y velocity goes to zero. Using Eq(4) we write

$$y_{top} = \frac{v_i^2 \sin^2 \Theta_i}{19.6} \quad \rightarrow (6)$$

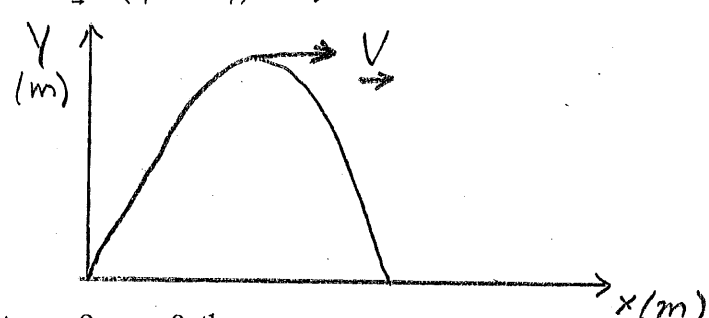
3. What is its acceleration while it is in the air? At all points $y \neq 0$

$$!\rightarrow \quad \underset{\rightarrow}{a} = -9.8 m/s^2 \hat{y} \quad \leftarrow!$$

fixed by the Earth.

4. Velocity at y_{top}, $v_y = 0$, $v_x = v_i \cos\Theta_i$

$$\underset{\rightarrow}{v} = (v_i \cos\Theta_i)\hat{x} + 0\hat{y}$$



5. When does it get to y_{top}? $v_y = 0$ there

So we use

$$v_y = v_i \sin\Theta_i - 9.8t$$

And get

$$t_{top} = \frac{v_i \sin\Theta_i}{9.8} \qquad \rightarrow (7)$$

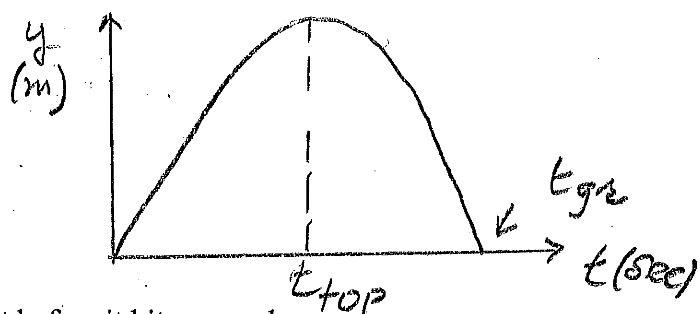
6. When does it return to ground ($y = 0$)?

   Use $\quad y = (v_i \sin\Theta_i)t - 4.9t_{gr}^{\,2}$

   So

$$O = (v_i \sin\Theta_i)t_{gr} - 4.9t_{gr}^{\,2}$$

$$t_{gr} = \frac{v_i \sin\Theta_i}{4.9} = 2t_{top} \qquad \rightarrow (8)$$



7. What is its velocity just before it hits ground

$$v_x = v_i \cos\Theta_i$$

$$v_y = v_i \cos\Theta_i - 2\frac{v_i \sin\Theta_i}{9.8} \times 9.8$$

$$= -v_i \sin\Theta_i$$

   Hence $\quad \underset{\rightarrow}{v} = (v_i \cos\Theta_i)\hat{x} - (v_i \sin\Theta_i)\hat{y} \qquad \rightarrow (9)$

   That is, x-component of velocity is same as at the start, y-component is reversed.

8. What is the range?

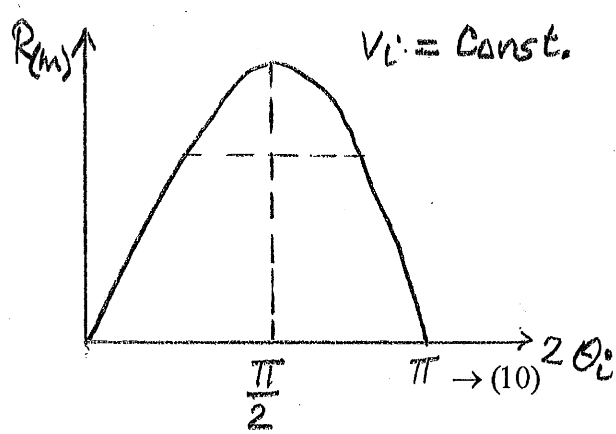
$$x = (v_i \cos\Theta_i)t$$

   And to get to R

$$t = t_{gr} = \frac{2v_i \sin\Theta_i}{9.8}$$

$$R = \frac{(v_i \cos\Theta_i)(2v_i \sin\Theta_i)}{9.8}$$

$$= \frac{v_0^{\,2} \sin 2\Theta_i}{9.8} \qquad \rightarrow (10)$$



9. For a given v_i what launch angle will give you maximum range R?
   (Galileo's findings)

   Eq(10) says

$$R = \frac{v_0^2 \sin 2\Theta_i}{9.8}$$

Maximum value of $\sin 2\Theta_i = 1$ when $2\Theta_i = \frac{\pi}{2}$. Hence maximum range when $\Theta_i = 45°$.

Also, note that there are two angles for which R is the same.

$$2\Theta_2 = \frac{\pi}{2} + \alpha$$

$$2\Theta_1 = \frac{\pi}{2} - \alpha$$

$$\Theta_1 + \Theta_2 = \frac{\pi}{2}$$

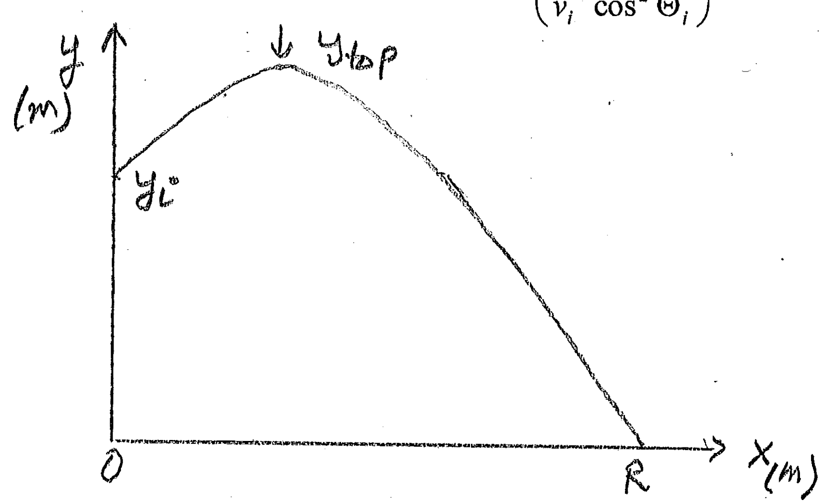
So Θ_1 and Θ_2 are complementary angles.

10. What happens if projectile is launched at $x = 0,\ y = y_i$. In that case

$$y_{top} = y_i + \frac{v_i^2 \sin^2 \Theta_i}{19.6}$$

and R is obtained by solving the quadratic equation

$$O = y_i + R \tan \Theta_i - 4.9\left(\frac{R^2}{v_i^2 \cos^2 \Theta_i}\right)$$



$$R = \frac{-v_i^2 \sin \Theta_i \cos \Theta_i \pm v_i \cos \Theta_i \sqrt{v_i^2 \sin^2 \Theta_i + 19.6 y_i}}{-9.8}$$

Not surprisingly, the projectile travels farther before returning to ground. This is what led Newton to suggest that if one goes high up and uses a large enough initial speed one can get the projectile to go around the Earth.