CONSERVATION OF LINEAR MOMENTUM

An object of mass M travelling at a velocity $\underset{\rightarrow}{v}$ is said to have a linear momentum given by the equation

$$\underset{\rightarrow}{p} = M\underset{\rightarrow}{v} \tag{1}$$

[*Lin Mom*^{m} $\quad MLT^{-1}$ $\quad kg-m/\sec$ $\quad$ VECTOR]

The immediate consequences of defining $\underset{\rightarrow}{p}$ are that Newton's Laws should be read as:

First law: Objects do not change their linear momentum spontaneously

Second Law: If the linear momentum $\underset{\rightarrow}{p}$ varies with time there must be a net force present at that point at the time. That is

$$\frac{\underset{\rightarrow}{p_f} - \underset{\rightarrow}{p_i}}{t_f - t_i} = < \Sigma F_i > \quad \text{(Average)} \tag{2}$$

$$\lim_{\Delta t \to 0} \frac{\Delta \underset{\rightarrow}{p}}{\Delta t} = \Sigma \underset{\rightarrow}{F_i} \quad \text{(Instantaneous)} \tag{3}$$

Also, the Kinetic Energy should be written

$$\boxed{K = \frac{1}{2}Mv^2 = \frac{p^2}{2M}} \tag{4}$$

Note: if two objects have the same momentum (magnitude) the smaller M has a larger K!

One can turn Eqn. (2) around to define a vector quantity called impulse, $\underset{\rightarrow}{J}$, which is the change in momentum caused by the application of a large force over a short time interval

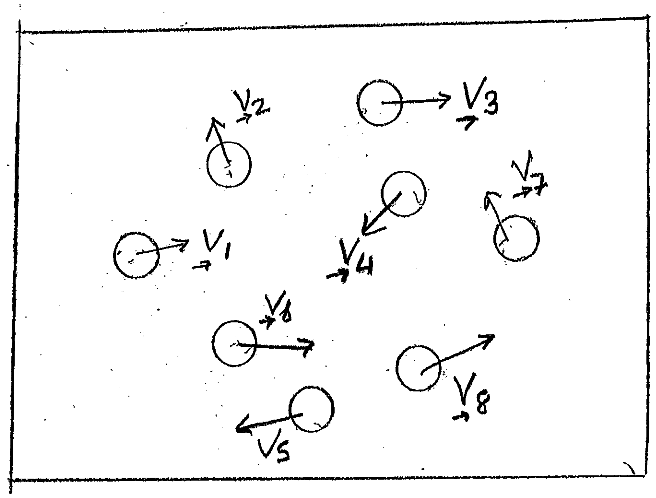
If $\underset{\rightarrow}{F}$ is constant

$$\underset{\rightarrow}{J} = \underset{\rightarrow}{p_f} - \underset{\rightarrow}{p_i} = \underset{\rightarrow}{F}\Delta t \tag{5}$$

If $\underset{\rightarrow}{F}$ varies with time then to calculate $\underset{\rightarrow}{J}$ you draw F as a function of time and calculate the area under the F vs. t graph to determine $\underset{\rightarrow}{J}$.

So much for single particles. To formulate the principle of conservation of momentum ($\underset{\rightarrow}{p}$) we need to consider a system consisting of many (at the very least two) objects and they cannot be point particles because point particles will not "collide" and we need thr objects to collide. So now our system, in a "box" containing many objects of masses $M_1, M_2, \ldots\ldots$ with velocities $\underset{\rightarrow}{v_1}, \underset{\rightarrow}{v_2}, \ldots\ldots$ and we can write

$$\underset{\rightarrow}{p_i} = M_i \underset{\rightarrow}{v_i}$$



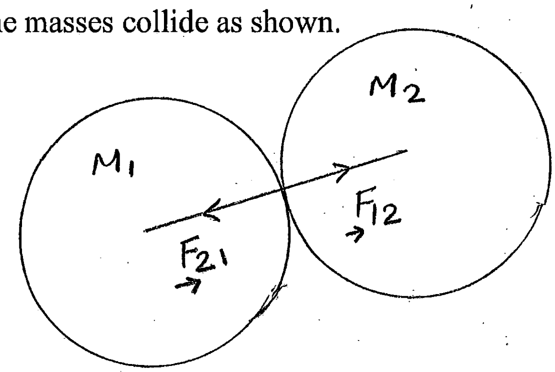
clearly, the system will have a totally mass

$$M = \Sigma M_i$$

and a totally momentum

$$P = \Sigma M_i \underrightarrow{v_i}$$

Now suppose two of the masses collide as shown.



At the instant of collision, Newton's 3rd Law tells us that the force $\underrightarrow{F_{21}}$ on M_1 due to M_2 must be equal and opposite to the force $\underrightarrow{F_{12}}$ ~~and~~ on M_2 due to M_1 that is

$$\underrightarrow{F_{12}} + \underrightarrow{F_{21}} = 0 \longrightarrow \text{CRUCIAL POINT}$$

If the collision lasts for Δt seconds, the impulse on M_1 is

$$\underrightarrow{J_1} = \underrightarrow{F_{21}} \Delta t$$

while

$$\underrightarrow{J_2} = \underrightarrow{F_{12}} \Delta t$$

and therefore

$$\underrightarrow{J_1} + \underrightarrow{J_2} = 0$$

But $\underrightarrow{J_1}$ = change in momentum of M_1

$\underrightarrow{J_2}$ = change in momentum of M_2

So this equation tells us that whatever vector momentum M_1 gains (loses) must be lost (gained) by M_2. So no matter how many collisions occur, if there is no external force the internal forces always come in action-reaction pairs and therefore the total vector momentum $\underrightarrow{p}$ of the system cannot change.

PRINCIPLE OF CONSERVATION OF LINEAR MOMENTUM SAYS: If $\underrightarrow{F_{ext}} = 0$, total vector momentum of a system is constant:

$$\boxed{\text{If } \underrightarrow{F_{ext}} = 0,\ \Sigma p_i = \text{Constant}} \tag{6}$$

In considering the motion of the entire system a useful concept is that of the Center of Mass. Let our masses M_1 be located at (x_i, y_i) in the xy-plane, the coordinates of the center of mass are

$$x_{CM} = \frac{\Sigma M_i x_i}{\Sigma M_i},\ y_{CM} = \frac{\Sigma M_i y_i}{\Sigma M_i}$$

[Near Earth $\quad x_{CM} = x_{CG},\ y_{CM} = y_{CG}$]

If the masses are moving, the displacements will be $\quad \underrightarrow{\Delta x_i}\ and\ \underrightarrow{\Delta y_i}$

$$M\,\underrightarrow{\Delta x_{CM}} = \Sigma M_i \frac{\Delta x_i}{\Delta t}$$

$$M\,\underrightarrow{\Delta y_{CM}} = \Sigma M_i \frac{\Delta y_i}{\Delta t}$$

Indeed $\qquad Mv_{CM} = \Sigma M_i \underrightarrow{v_i} = \underrightarrow{p}$

So conservation law says if $\underrightarrow{F_{ext}} = 0$, velocity of center of mass is CONSTANT.

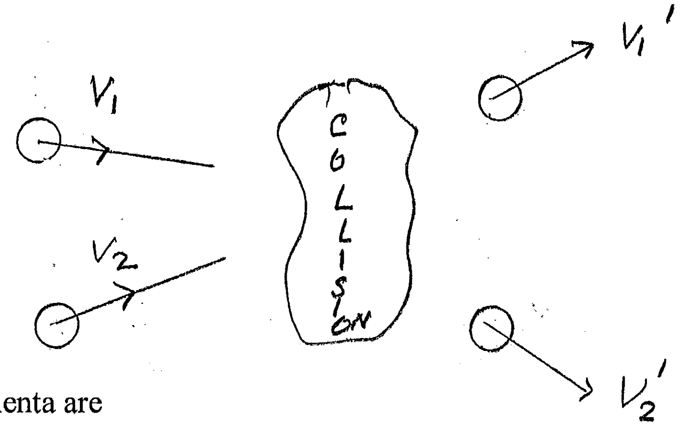
If $\underrightarrow{F_{ext}} \neq 0$

$$\left.\begin{array}{c} M\,\underrightarrow{a_{CM}} = \Sigma \underrightarrow{F_{ext}} \\ \text{Or} \\ \dfrac{\underrightarrow{\Delta p}}{\Delta t} = \Sigma \underrightarrow{F_{ext}} \end{array}\right\}\ \text{Newton's Law}$$

That is, one can pretend that the total mass M is located at the Center of Mass and treat it as a "translation" of the "box" as a whole.

TWO-BODY COLLISIONS

This is the experiment you performed a while ago. You took two pucks and put them on a horizontal frictionless surface thereby making $\underrightarrow{F_{ext}} = 0$ because firstly $(n - Mg) = 0$ and also $f_k = 0$. The pucks were given velocities $\underrightarrow{v_1}$ and $\underrightarrow{v_2}$, allowed to collide and emerge with velocities $\underrightarrow{v_1}'$ and $\underrightarrow{v_2}'$



The corresponding momenta are

Before	After
$\underrightarrow{p_1} = M_1 \underrightarrow{v_1}$	$\underrightarrow{p_1}' = M_1 \underrightarrow{v_1}'$
$\underrightarrow{p_2} = M_2 \underrightarrow{v_2}$	$\underrightarrow{p_2}' = M_2 \underrightarrow{v_2}'$

and the conservation law requires

$$\underrightarrow{p_1}' + \underrightarrow{p_2}' = \underrightarrow{p_1} + \underrightarrow{p_2} \tag{7}$$

(Total Vector Momentum After) = (Total Vector Momentum Before)

Experimentally, you have checked this relationship. The question we need to answer is: Given M_1, M_2 and $\underrightarrow{v_1}$, $\underrightarrow{v_2}$ do we have enough information to figure out $\underrightarrow{v_1}'$ and $\underrightarrow{v_2}'$. The answer is <u>No</u>. Why?

Let us put the objects in the xy-plane.
Eqn. (7) yields

$$M_1 v_{1x}' + M_2 v_{2x}' = M_1 v_{1x} + M_2 v_{2x} \tag{8}$$

$$M_1 v_{1y}' + M_2 v_{2y}' = M_1 v_{1y} + M_2 v_{2y} \tag{9}$$

The problem is that we have only two equations but there are 4 unknowns [$v_{1x}', v_{2x}', v_{1y}', v_{2y}'$] and therefore no unique solution is possible. We need to add further specification to the type of collision in order to get a solution.

We consider two special cases:

Type I Totally Inelastic Collision

The two objects stick together after the collision

$$\vec{v}_1' = \vec{v}_2' \text{ [Totally Inelastic Collision]}$$

And now we can use Eqn. (8) and Eqn. (9) to get precise answers.

Type II Totally Elastic Collision

Kinetic Energy is also conserved:

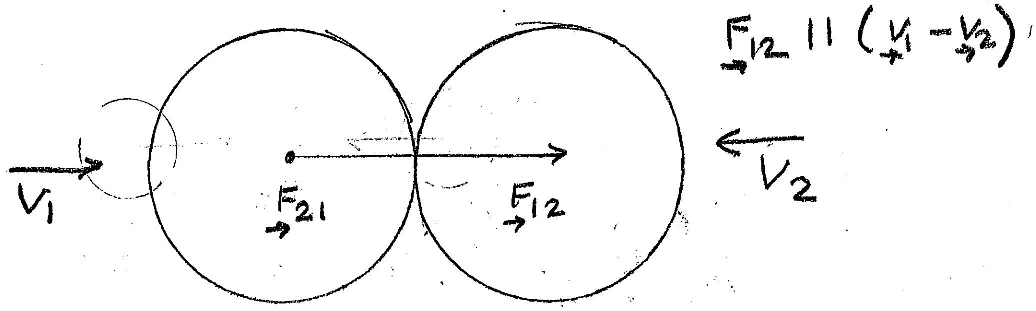
(Total Kinetic Energy After) = (Total Kinetic Energy Before)

This gives us another equation

$$\frac{1}{2}Mv_1'^2 + \frac{1}{2}Mv_2'^2 = \frac{1}{2}Mv_1^2 + \frac{1}{2}Mv_2^2 \qquad (10)$$

Now we have 3 equations (8), (9), (10) and we still have a problem because we have four unknowns.

We simplify further by specifying that the collision is "head-on"



Now the forces $\vec{F}_{12}$ and $\vec{F}_{21}$ are parallel to the relative velocity $(\vec{v}_1 - \vec{v}_2)$ so it becomes essentially a one-dimensional problem.

We can take all the vectors to be along the x-axis, $\vec{v}_1 = v_1\hat{x}$, $\vec{v}_2 = v_2\hat{x}$, $\vec{v}_1' = v_1'\hat{x}$, $\vec{v}_2' = v_2'\hat{x}$ and the conservation equations become

Lin. Mom^m $$M_1v_1' + M_2v_2' = M_1v_1 + M_2v_2 \qquad (11)$$

Kin. En. $$M_1\frac{v_1'^2}{2} + M_2\frac{v_2'^2}{2} = M_1\frac{v_1^2}{2} + M_2\frac{v_2^2}{2} \qquad (12)$$

Now we can use algebra to solve for v_1' and v_2'

Rewrite Eqns. (11) and (12) as

$$(v_1' - v_1) = \frac{M_1}{M_2}(v_2 - v_2') \qquad (11')$$

$$(v_1'^2 - v_1^2) = \frac{M_1}{M_2}(v_2^2 - v_2'^2) \qquad (12')$$

Divide Eqn. (12') by Eqn. (11')

$$(v_1 + v_1') = (v_2 + v_2') \qquad (13)$$

or

$$\longrightarrow \boxed{(v_1' - v_2') = (v_2 - v_1) = -(v_1 - v_2)} \qquad (A)$$

IN TOTALLY ELASTIC HEAD ON COLLISION THE RELATIVE VELOCITY REVERSES DIRECTION AS A RESULT OF THE COLLISION.

Next take Eqn. (11) write	$M_1 v_1' = M_1 v_1 + M_2 v_2 - M_2 v_2'$
Next take Eqn. (13)	$= M_1 v_1 + M_2 v_2 - M_2 (v_1 + v_1' - v_2)$
Rearrange	$(M_1 + M_2) v_1' = (M_1 - M_2) v_1 + 2M_2 v_2$

Yielding
$$v_1' = \frac{M_1 - M_2}{M_1 + M_2} v_1 + \frac{2M_2 v_2}{M_1 + M_2} \quad \text{(B)}$$

Similarly
$$v_2' = \frac{M_2 - M_1}{M_1 + M_2} v_2 \frac{2M_1 v_1}{M_1 + M_2} \quad \text{(C)}$$

Or respecting the vector nature of the velocities $(\pm\hat{x})$ we write

$$\underset{\rightarrow}{v_1}' = \frac{M_1 - M_2}{M_1 + M_2} \underset{\rightarrow}{v_1} + \frac{2M_2}{M_1 + M_2} \underset{\rightarrow}{v_2}$$

$$\underset{\rightarrow}{v_2}' = \frac{M_2 - M_1}{M_1 + M_2} \underset{\rightarrow}{v_2} \frac{2M_1}{M_1 + M_2} \underset{\rightarrow}{v_1}$$

We will discuss the consequences of these equations in the class.