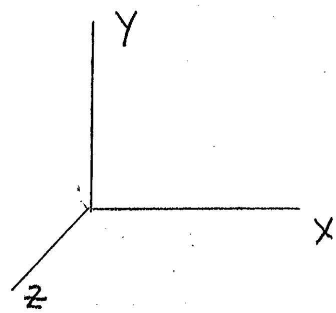
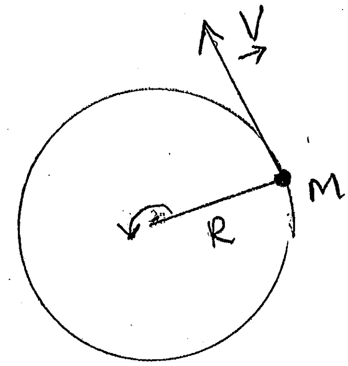
Conservation of Angular Momentum- Keplers Laws

A single mass m moving on a circle of radius R at a uniform velocity has a tangential velocity

$$\vec{v} = R\omega\hat{\tau}$$





It therefore has a linear momentum

$$\vec{p} = MR\omega\hat{\tau}$$

The angular momentum of this object is defined by $\vec{\ell} = \vec{r} \times \vec{p}$

where $\vec{r} = R\hat{r}$, so $\vec{\ell}$ is $\perp$ to the plane of the circle and will be along $\pm\hat{z}$

$$\vec{\ell} = \pm MR^2\omega\,\hat{z}$$

If a tangential force is applied to M

$$M\vec{a_t} = \vec{F_t}$$

and there will be a torque about z, $\vec{\tau} = \vec{r} \times \vec{F_t}$, and it will have an angular acceleration $\vec{\alpha}$

$$\vec{a_t} = R\alpha\,\hat{\tau}$$

Now

$$\vec{\tau} = \pm R\,Ma\,\hat{z} = \pm MR^2\alpha\,\hat{z}$$

$$= \pm MR^2\frac{\Delta\omega}{\Delta t}\hat{z} = \frac{\Delta\vec{\ell}}{\Delta t}$$

That is, if you want angular momentum to change with time you must apply a torque-Newton's Law for rotation in terms of angular momentum.

Next, apply it to a rigid body rotation

$\vec{w}$ and $\vec{\alpha}$ are common but i th mass has

$$\vec{v_i} = r\,i\,\omega\hat{\tau}$$

i th mass has angular momentum

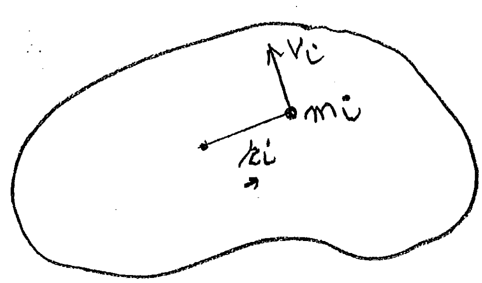
$$\vec{\ell_i} = m_i r_i^2\omega\,\hat{z} \quad \text{for C.C.W. rotation}$$

right hand rule:
- $\vec{r_i}$ along thumb
- $\vec{p_i}$ " fingers
- $\vec{\ell_i}$ $\perp$ palm

Total angular momentum of Rigid Body

$$\vec{L} = \Sigma m_i r_i^2\,\vec{\omega}$$

$$= I\vec{\omega}$$



compare this to the total linear momentum

$$\vec{p} = M\vec{v}$$

So again I replaces M and w replaces v.

Conservation Laws

Linear Mom^m

$\underset{\rightarrow}{F_{ext}} = 0$

$\underset{\rightarrow}{p} = const.$

Angular Mom^m

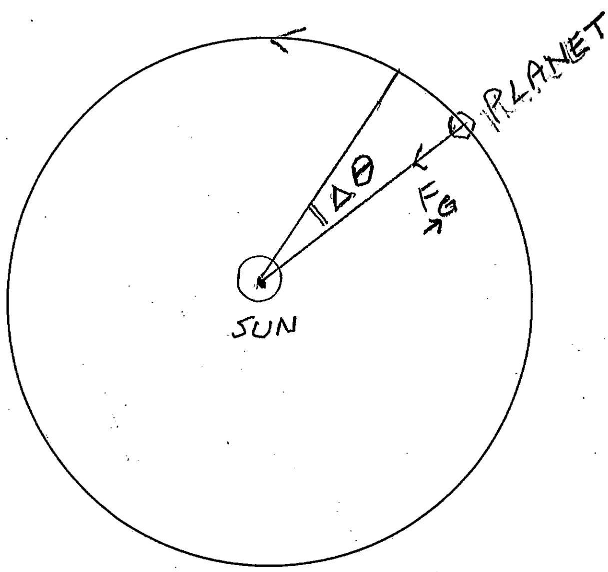
$\underset{\rightarrow}{\tau_{ext}} = 0$

$\underset{\rightarrow}{L} = const.$

Let us apply this to motion of planets around sun in circular orbits kepler's law: (i) PLANETS MOVE IN PLANAR ORBITS. (ii) As planet goes around the sun, the radius sweeps out equal areas in equal intervals of time.

radius vector

$\underset{\rightarrow}{r} = r_p$



The only force aching on the planet is the Gravitational force due to the sun

$$\underset{\rightarrow}{F_G} = -\frac{GM_sM_p}{r_p^2}\hat{r}$$

If we take the torque about an axis through the sun

$$\underset{\rightarrow}{\tau_p} = \underset{\rightarrow}{r} \times \underset{\rightarrow}{F_G} = 0 \qquad \text{because } [\hat{r} \times \hat{r}] = 0$$

Hence angular momentum of planet around this axis must be constant

$$\underset{\rightarrow}{L_p} = M_p r_p^2 w_p \hat{z}$$

Since $\underset{\rightarrow}{Lp}$ cannot change direction, orbit must lie in xy- plane.

[*It is also a plane because* $\underset{\rightarrow}{F_G}$ *is only along* $\hat{r}$]. Next, consider that the radius rotates through angle $\Delta\theta$ in time Δt.

Area swept out by $\underset{\rightarrow}{r}$ becomes

$$\Delta A = \frac{1}{2} r_p^2 \Delta\theta$$

and area swept per second

$$\frac{\Delta A}{\Delta t} = \frac{1}{2} \gamma_p^2 \frac{\Delta \theta}{\Delta t}$$

$$= \frac{1}{2} r_p^2 \omega = \frac{1}{2} \frac{L_p}{M_p}$$

$$= \text{const.}$$

Because magnitude of L_p is constant.