L,T,M

UNIVERSAL LAW OF GRAVITATION-FORCE

Experimental facts which led to Newton's postulate.

1. Near Earth all unsupported objects have an acceleration

$$\underset{\to}{a} = -9.8\frac{m}{s^2}\hat{r}$$

where $\hat{r}$ is the unit vector along the radius of the Earth.

2. Kepler's laws of planetary motion around the Sun:

   i) Planets go around the sun in <u>PLANE</u> elliptical orbits with the sun being located at one focus of the Ellipse. Actually, in most cases the Ellipses are very close to being circles.

   ii) The period of a planet T_P is proportional to $R_P^{3/2}$ where R_P is the semi-major axis of the Ellipse. That is

$$T_P^2 \; \alpha \; R_P^3$$

   iii) The line connecting the planet's position to that of the Sun sweeps out equal areas in equal intervals of time.

Newton's solution developed in several steps.

<u>FIRST</u>

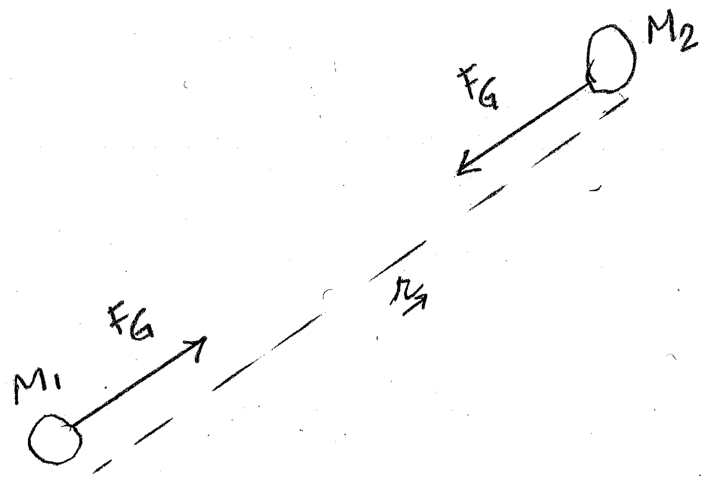
He postulated that if two point masses M_1 and M_2 are separated by a distance *r*, there exists a force between them given by the equation

$$\underset{\to}{F}_G = -\frac{GM_1M_2}{r^2}\hat{r}$$

Where *G* is a universal constant. Now we know that the value of *G* is about

$$6.7\times10^{-11}\frac{N-m^2}{(kg)^2}$$

$[DIM: M^{-1}L^3T^{-2}]$



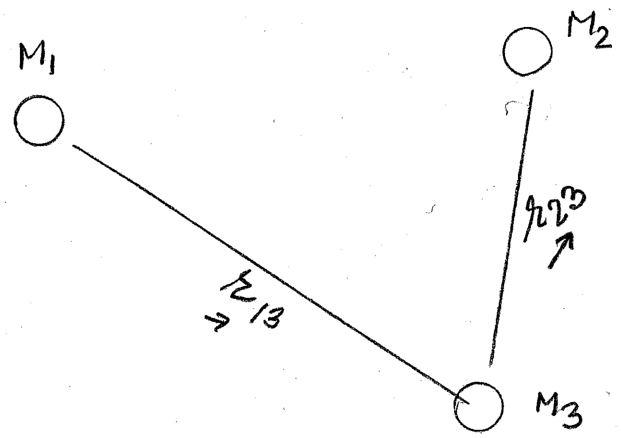
<u>Two Crucial Points:</u>

The force acts along the line joining M_1 and M_2; Hence $\hat{r}$

The force is <u>ATTRACTIVE</u>, hence the <u>MINUS</u> sign, M_1 and M_2 are being pulled <u>TOWARD</u> one another. As you can see, the equation represents two forces.

SECOND

He demonstrated the principle of super position. That is for 3 masses M_1, M_2, and M_3 located as shown.



The force on M_3 is

$$\underset{\rightarrow}{F}_G(3) = \frac{-GM_1M_3}{r_{13}}\hat{r}_{13} - \frac{GM_2M_3}{r_{23}}\hat{r}_{23}$$

M_3 is being pulled by both M_1 and M_2.

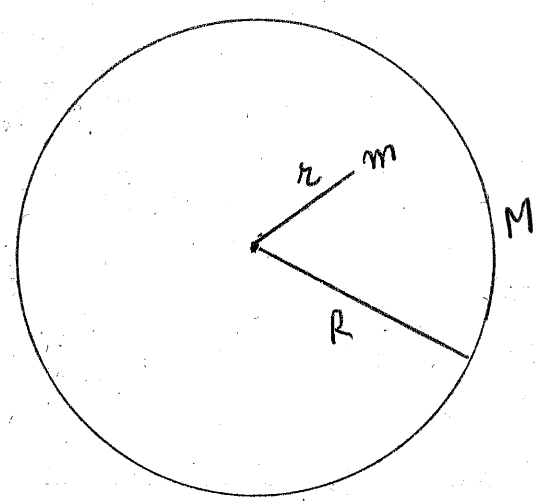
THIRD

He realized that real objects in nature are essentially made up of continuous distributions of matter so he derived the force between a point object located at $\underset{\rightarrow}{r}$ and a sphere of radius R located with its center at $r=0$.

In order to do so he discovered integral calculus and again had to get the result in two steps:

Step 1

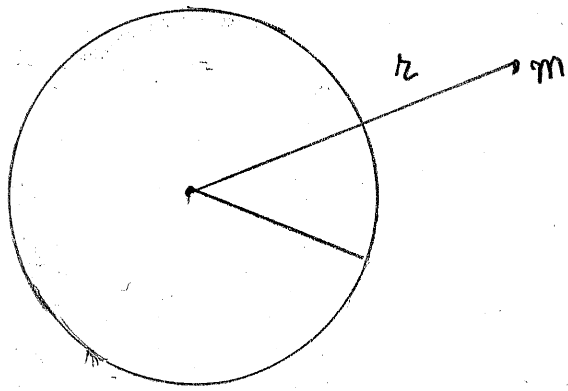
He calculated the force between a pt. mass m located at $\underset{\rightarrow}{r}$ and a spherical shell of mass M and radius R located with its center at $r=0$ [Note that for a shell all the mass M is located at the surface, the space between O and R is empty]



I: Amazingly, he found that if m is located inside the shell, that is $r<R$ as shown above, the Gravitational force ON IT IS IDENTICALLY EQUAL TO ZERO!

$$\text{If } r<R \quad \underset{\rightarrow}{F}_G \equiv 0$$

II. He found that if m Is located outside the shell ($r > R$) the force on m is

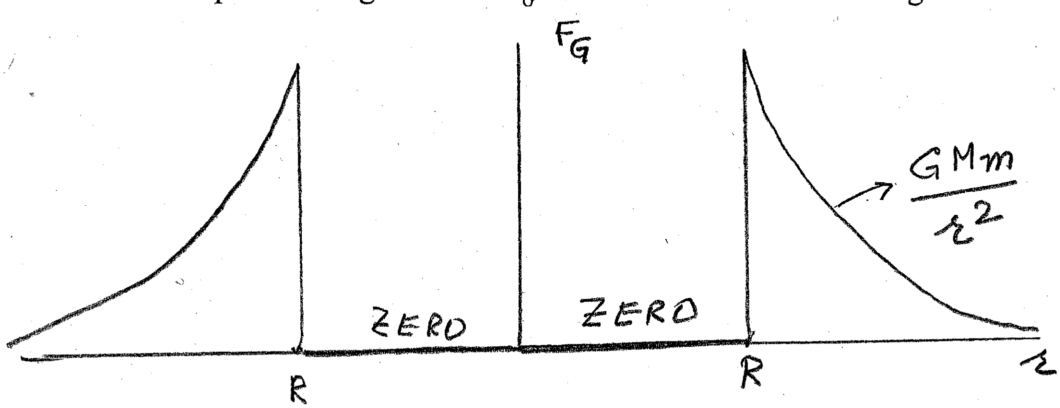


$$\underset{\rightarrow}{F}_G = \frac{-GMm}{r^2}\hat{r}$$

when $r>R$

In other words, at any point outside, the force on m is as if the entire mass of the shell was located at its center (r=0)!

So if we plot the magnitude of F_G as a function of r we would get



Maximum force on m is when it is just outside the shell.

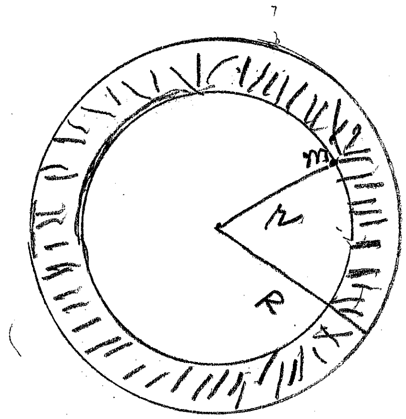
Step 2:

He used the results of Step 1 to calculate the force between a point mass m located at r and a solid sphere of mass M and radius R located with its center at r=0. He assumed that the mass of the sphere was distributed uniformly so that he could define the density.

$$d = \frac{M}{vol.\ of\ shpere} = \frac{M}{\frac{4\pi}{3}R^3}$$

[Density ML^{-3} $\quad \frac{kg}{m^3}$ scalar]

First, put m inside the sphere $r<R$. One can think of the solid sphere as if it consisted of a large number of concentric shells (an onion comes to mind) and realize that from the point of view of m, it lies inside all the shells in the shaded area and hence they contribute ZERO to the force on m.



The force on m is due to all the shells which are located between $r=0$ and $r=r$. Namely, for $r<R$

$$\underset{\rightarrow}{F}_G = \frac{-Gm \times (mass\ inside\ r)}{r^2}\hat{r}$$

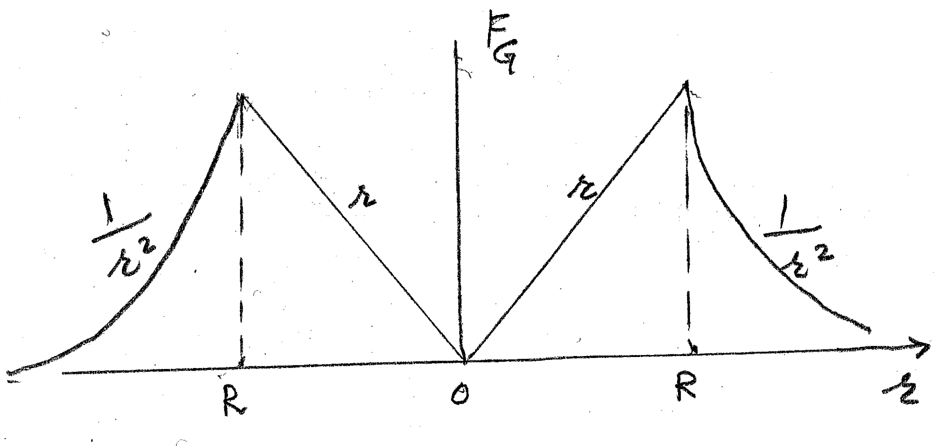
$$= \frac{-Gm \times (\frac{4\pi}{3} r^3 d)}{r^2}\hat{r}$$

$$\underset{\rightarrow}{F}_G = \frac{-4\pi}{3} GMrd\hat{r} \qquad r<R$$

If m is located outside ($r>R$) the entire mass M contributes so that

$$\underset{\rightarrow}{F}_G = -\frac{GmM}{r^2}\hat{r} \qquad r>R$$

Now, the plot of magnitude of F_G as a function of r looks like



NOTICE: FORCE IS MAXIMUM when m is at the surface.

Also if you think of the Earth as a solid sphere your weight reduces as you go in, being zero when you reach the center.

FINAL STEP

Follows from above discussion. Force between two uniform spheres of masses M_1 and M_2 is given by

$$\underset{\to}{F}_G = -\frac{GM_1 M_2}{r^2}\hat{r}$$

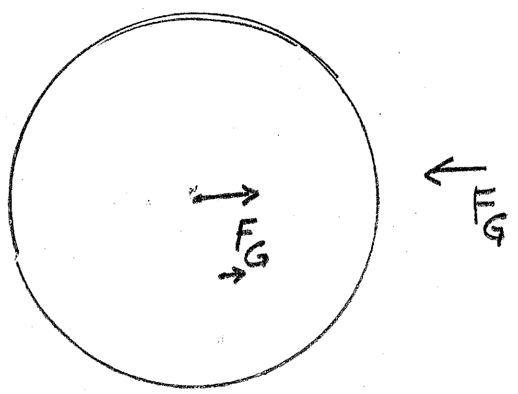
where $\underset{\to}{r}$ is measured from center to center.

APPLICATIONS

I Force on m near Earth's surface

$$R_E \cong 6.4\times 10^6\, km$$

$$M_E \cong 6\times 10^{24}\, kg$$



$$\underset{\to}{F}_G = \frac{-Gm\cdot M_E}{R_E^2}\hat{r}$$

$$\simeq -9.8mN\hat{r}$$

Similarly, force on Earth du to m is by Newton's 3rd law $+9.8mN\hat{r}$ acting at Ctr. Of Earth.

II. KEPLER'S LAW $\qquad T_P^2 \;\alpha\; R_P^3$

Since the planetary orbits are nearly circular, we assume uniform circular motion of a planet of mass M_p on a circle of radius R_p. If the angular velocity is w_p the planet requires a centripetal force

$$\underset{\longrightarrow}{F_C} = -M_p R_p w_p^{\,2}\hat{r}$$

and for the orbit to be stable the sun must provide $\underset{\longrightarrow}{F_G} = \underset{\longrightarrow}{F_C}$. That is

$$\underset{\longrightarrow}{F_G} = \frac{-GM_p M_S}{R_p^{\,2}}\hat{r}$$

$$= \underset{\longrightarrow}{F_C}$$

where M_S is the mass of the sun.

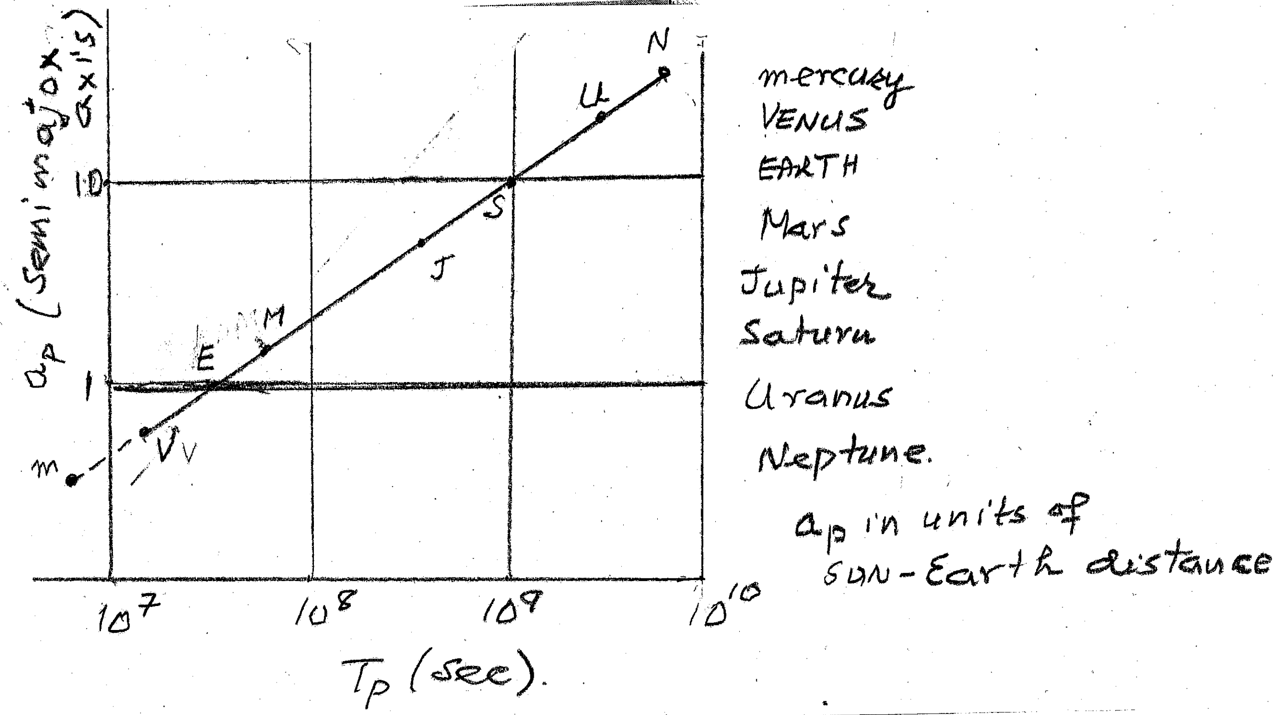
$$\frac{-GM_S}{R_p^{\,2}} = R_p w_p^{\,2}$$

But the period $T_p = \dfrac{2\pi}{w_p}$

So $\quad \dfrac{-GM_S}{R_p^{\,2}} = R_p \dfrac{4\pi^2}{T_p^{\,2}}$

So $$T_p^2 = \frac{4\pi^2}{GM_S} R_p^3$$

We will prove the next Kepler Law after we have discussed Angular Momentum.



III. NOTE THAT FOR SATELLITES* IN CIRCULAR ORBITS AROUND THE EARTH THE KEPLERIAN LAW BECOMES

$$T_S^2 = \frac{4\pi^2}{GM_E} R_S^3$$

Where T_S = Period of Satellite

R_S = Radius of Orbit

*INCLUDE THE MOON

To prove this we note that a satellite of mass m_S going around the Earth on a circular orbit of radius R_S, measured from the center of the Earth, requires a centripetal force

$$\underline{F_C} = -m_S R_S w_S^2 \hat{r}$$ and the Earth provides $\vec{F}_G$

Equating the two and setting $T_S = \frac{2\pi}{w_S}$

Gives

$$T_S^2 = \frac{4\pi^2}{GM_E} R_S^3$$

Thus, once you pick the period of a Satellite the radius of its orbit is fixed

Facts

→ Moon, being a Satellite of Earth continuously falling toward the Earth at all times!

→ Astronauts in stable orbit become "weightless" ($N_R \to 0$)!