

CONSTRUCTION OF BOSONS AND FERMIONS OUT OF QUONS

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Abstract

The quon algebra describes particles, “quons,” that are neither fermions nor bosons, using a label q that parametrizes a smooth interpolation between bosons ($q = 1$) and fermions ($q = -1$). Understanding the relation of quons on the one side and bosons or fermions on the other can shed light on the different properties of these two kinds of operators and the statistics which they carry. In particular, local bilinear observables can be constructed from bosons and fermions, but not from quons. In this paper we construct bosons and fermions from quon operators. For bosons, our construction works for $-1 \leq q \leq 1$. The case $q = -1$ is paradoxical, since that case makes a boson out of fermions, which would seem to be impossible. None the less, when the limit $q \rightarrow -1$ is taken from above, the construction works. For fermions, the analogous construction works for $-1 \leq q \leq 1$, which includes the paradoxical case $q = 1$.

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1 Introduction

Why are all identical particles in nature either bosons or fermions? Attempts to answer this question have gone in two directions: (1) to study other possible types of quantum statistics theoretically and to propose and analyze experimental searches for violations of bose or fermi statistics [1], (2) to find theoretical arguments that only bose or fermi statistics occur in nature and thereby understand better why only bose and fermi statistics can occur [2]. The book edited by R.C. Hilborn and G.M. Tino [3] has the proceedings of a recent conference on statistics.

A better understanding of the relation of bose and fermi statistics to other possible statistics should be a useful tool for analysis of both these issues. In this letter we study the construction of bose and of fermi operators using quons [4], a type of particle statistics that interpolates smoothly between bose and fermi statistics and can provide a description of particles that violate bose or fermi statistics by a small amount[5]. We give ansatzes for the construction of bosons from quons with q in the range $-1 \leq q \leq 1$ and for the construction of fermions from quons in the range $-1 \leq q \leq 1$. In each case the ansatzes represent bose or fermi creation or annihilation operators in terms of an infinite degree series of quon operators. Surprisingly, our constructions are valid for bosons represented by an infinite series of products of fermions as well as for fermions represented by an infinite series of bosons.

The bose and fermi algebras are

$$[b_k, b_l^\dagger]_- = \delta_{k,l} \quad (1)$$

$$[f_k, f_l^\dagger]_+ = \delta_{k,l}, \quad (2)$$

and the Fock representation that we will always consider is given by

$$b_k|0\rangle = 0, \quad \text{or} \quad f_k|0\rangle = 0. \quad (3)$$

The quon algebra is

$$a_k a_l^\dagger - q a_l^\dagger a_k = \delta_{kl}. \quad (4)$$

We consider the Fock-like representation with the vacuum condition

$$a_k|0\rangle = 0. \quad (5)$$

These two conditions determine all vacuum matrix element of polynomials in the creation and annihilation operators. In the case of free quons, all non-vanishing vacuum matrix elements have the same number of annihilators and creators. For such a matrix element with all annihilators to the left and creators to the right, the matrix element is a sum of products of “contractions” of the form $\langle 0|aa^\dagger|0\rangle$ just as in the case of bosons and fermions. The only difference is that the terms are multiplied by integer powers of q . The power can be given as a graphical rule: Put \circ 's for each annihilator and \times 's for each creator in the order in which they occur in the matrix element on the x-axis. Draw lines above the x-axis connecting the pairs that are contracted. The minimum number of times these lines cross is the power of q for that term in the matrix element [4]. For $q = \pm 1$ this agrees with the usual rule for bosons and fermions.

The physical significance of q for small violations of fermi statistics is that $q = 2v_f - 1$, where the parameter v_f gives the deviation of the two-particle density matrix from fermi statistics,

$$\rho_2 = (1 - v_f)\rho_a + v_f\rho_s, \tag{6}$$

where $\rho_{a(s)}$ is the antisymmetric (symmetric) two-particle density matrix. For small violations of bose statistics, the two-particle density matrix is

$$\rho_2 = (1 - v_b)\rho_s + v_b\rho_a. \tag{7}$$

For this case $q = 1 - 2v_b$.

For q in the open interval $(-1, 1)$ all representations of the symmetric group occur. As $q \rightarrow 1$, the symmetric representations are more heavily weighted and at $q = 1$ only the totally symmetric representation remains; correspondingly, as $q \rightarrow -1$, the antisymmetric representations are more heavily weighted and at $q = -1$ only the totally antisymmetric representation remains. Thus for a general n -quon state, there are $n!$ linearly independent states for $-1 < q < 1$, but there is only one state for $q = \pm 1$. We emphasize something that many people find very strange: *there is no commutation relation between two creation or between two annihilation operators*, except for $q = \pm 1$, which, of course, correspond to bose and fermi statistics. Indeed, the fact that the general n -particle state with different quantum numbers for all the particles has $n!$ linearly independent states proves

that there is no such commutation relation between any number of creation (or annihilation) operators. An even stronger statement holds: There is no two-sided ideal containing a term with only creation operators.

Quons are an operator realization of “infinite statistics” which were found as a possible statistics by Doplicher, Haag and Roberts [6] in their general classification of particle statistics. The simplest case, $q = 0$, [7] suggested to one of us (OWG) by Hegstrom, was discussed earlier in the context of operator algebras by Cuntz [8].

It is amusing that, even though quon fields are not local, the free quon field obeys the TCP theorem and Wick’s theorem holds for quon fields [4].

We consider three ways to construct bose or fermi creation and annihilation operators using quons. The first way, in Sec. 2, uses a general ansatz for the bose or fermi operators in terms of an infinite degree series of normal ordered products of quon operators with undetermined coefficients. Direct substitution of the ansatz in the bose or fermi commutation relation yields conditions that determine the coefficients. We show that the number of constraints equals the number of coefficients, but we do not give a formula for the general term using this method. The second way, in Sec. 3, uses the same ansatz, but determines the coefficients by requiring the validity of the bose or fermi permutation symmetry relation for states with a given number of quanta. As the number of quanta is increased, the higher coefficients are determined. Again we do not give a formula for the general term. The third approach, in Sec. 4, is to expand the bose operator in terms that act on only one sector with a given number of quanta; that is to expand using terms that have projection operators on the subspace with a given number of quanta of the quons, or, in other words, to isolate the terms that contribute for a given number of quanta. This third approach yields simple explicit formulas for an arbitrary term in the infinite series representation of the bose operators in terms of quon operators. The corresponding results for the representation of fermi operators in terms of quon operators are analogous to the results for bose operators.

We were surprised to find that our results hold for the expansion of a bose operator using fermi operators ($q = 1$) and for the expansion of a fermi operator using bose operators ($q = -1$). We discuss the relevant limiting cases in Sec. 5.

2 Direct calculation in the algebra

Choose the a_k and a_l^\dagger operators to obey the quon relations (4) in the Fock-like representation (5). Expand a bose operator b_k as

$$b_k = c_{kl}a_l + c_{klmn}a_l^\dagger a_m a_n + \dots \quad (8)$$

Unless stated otherwise, repeated indices are summed over. The expression for a_k^\dagger is always the adjoint of the expression for a_k . We want to find the simplest ansatz that will allow the construction. Translation and rotation invariance are compatible with the following much simpler ansatz,

$$b_k = a_k + xa_t^\dagger a_k a_t + ya_t^\dagger a_t a_k + \dots \quad (9)$$

The terms in (9) suffice to give all terms of the form $a^\dagger a$ in the commutator $[b_l^\dagger, b_k]_-$. There are two different types of terms bilinear in a^\dagger and a , $a_l^\dagger a_k$ and $\delta_{kl} a_s^\dagger a_s$; these lead to two equations that determine x and y ,

$$q(x^2 + y^2) + 2xy + 2(x + qy) + q - 1 = 0, \quad (10)$$

$$x^2 + y^2 + 2qxy + 2(qx + y) = 0. \quad (11)$$

The solutions are

$$x = \pm \frac{1}{\sqrt{2(1+q)}}, \quad y = \pm \frac{1}{\sqrt{2(1+q)}} - 1. \quad (12)$$

Inserting these solutions in (9), we find

$$b_l^\dagger = a_l^\dagger + \pm \frac{1}{\sqrt{2(1+q)}} a_t^\dagger a_l^\dagger a_t + (\pm \frac{1}{\sqrt{2(1+q)}} - 1) a_l^\dagger a_t^\dagger a_t + \dots, \quad (13)$$

where the result is valid up to terms in the commutator with four operators. For the upper choice of sign the result reduces to $b_l^\dagger = a_l^\dagger$ when $q = 1$, as we expect. We disfavor the lower choice of sign, which does not so reduce, because it gives an unwanted negative sign for the two-particle state even though it gives a symmetric two particle state for $q = 1$,

$$b_{l_2}^\dagger b_{l_1}^\dagger |0\rangle = -a_{l_2}^\dagger a_{l_1}^\dagger |0\rangle. \quad (14)$$

Similarly, for the fermi case, we can expand the fermi operator as:

$$f_k = c_{kl}a_l + c_{klmn}a_l^\dagger a_m a_n + \dots \quad (15)$$

which can be written more simply as:

$$f_k = a_k + x'a_t^\dagger a_k a_t + y'a_t^\dagger a_t a_k + \dots \quad (16)$$

Now we consider the anticommutator $[f_l^\dagger, f_k]_+$, which gives us the following set of equations:

$$q(x'^2 + y'^2) + 2x'y' + 2(x' + qy') + q + 1 = 0, \quad (17)$$

$$x'^2 + y'^2 + 2qx'y' + 2(qx' + y') = 0. \quad (18)$$

which yields

$$x' = \pm \frac{1}{\sqrt{2(1-q)}}, \quad y' = \mp \frac{1}{\sqrt{2(1-q)}} - 1. \quad (19)$$

Inserting the solutions the fermi operator becomes

$$f_l^\dagger = a_l^\dagger + \pm \frac{1}{\sqrt{2(1-q)}} a_t^\dagger a_l^\dagger a_t + (\mp \frac{1}{\sqrt{2(1-q)}} - 1) a_l^\dagger a_t^\dagger a_t + \dots \quad (20)$$

Here the lower choice of sign reduces the result to $f_l^\dagger = a_l^\dagger$ and the upper choice gives the two-particle state with an unwanted extra minus sign.

The general ansatz for b_k will be an infinite series with terms of the form $a^\dagger{}^{n-1}a^n$. Taking account of the possibility of renaming indices, there can be $n!$ different terms of this type, since the creation operators can be labeled by indices $t_1, t_2, \dots, t(n-1)$ in fixed order and the annihilation operators can be labeled by $k, t_1, t_2, \dots, t(n-1)$ in $n!$ different orders, each such term having an independent coefficient. The commutator $[b_k, b_l^\dagger]_-$ will have terms of the form $a^\dagger{}^{n-1}a^{n-1}$ of two types, those with a factor $\delta_{k,l}$ and those in which the indices k and l appear on operators in the product of annihilation and creation operators. There are $(n-1)!$ independent terms of the first kind, since the indices $k_1, k_2, \dots, k(n-1)$ can be fixed for—say—the creation operators and the indices can be permuted in $(n-1)!$ ways in the annihilation operators. To count the number of different terms of the second type, fix the order of $a_{t_1}^\dagger a_{t_2}^\dagger \dots a_{t(n-2)}^\dagger$. The creation operator a_l^\dagger can be put in $(n-1)$

places among the a^\dagger 's. The annihilation operators $a_{t_1}a_{t_2}\cdots a_{t_{(n-2)}}$ can be permuted in $(n-2)!$ ways. The annihilation operator a_k can be put in $(n-1)$ places. Thus the number of different terms of the second type is $(n-1)(n-2)!(n-1) = (n-1)(n-1)!$ and the total number of different terms is $(n-1)! + (n-1)(n-1)! = n!$ The equality of the number of independent coefficients and the number of independent constraints makes it plausible that a solution exists for an arbitrary term in the infinite series representation of the bose operators in normal-ordered products of the quon operators.

3 Direct calculation on the states

A representation of the bose operators in terms of quon operators will have the property that a product of n bose creation operators acting on the vacuum will be symmetric under permutations of the bose creation operators. When the bose creation operators are represented by quon operators, the n particle state will be symmetric under permutations of the quon creation operators. We can use this property to simplify the expansion of the bose operators in terms of the quon operators.

For the case of one quantum, we clearly need

$$b_k^\dagger = a_k^\dagger. \quad (21)$$

For the case of two quanta, we can take

$$b_k = a_k + xa_t^\dagger a_k a_t + ya_t^\dagger a_t a_k \quad (22)$$

as before. Direct calculation gives

$$b_{k_2}^\dagger b_{k_1}^\dagger |0\rangle = [(1+y)a_{k_2}^\dagger a_{k_1}^\dagger + xa_{k_1}^\dagger a_{k_2}^\dagger]|0\rangle. \quad (23)$$

We require

$$b_{k_2}^\dagger b_{k_1}^\dagger |0\rangle = \mathcal{N}(2; q)[a_{k_2}^\dagger a_{k_1}^\dagger + a_{k_1}^\dagger a_{k_2}^\dagger]|0\rangle \quad (24)$$

in order to enforce bose symmetry on the two-particle state. Using [4]

$$\mathcal{N}(n; q) = \frac{1}{\sqrt{n![n]_q!}}, \quad [n]_q = 1 + q + \cdots + q^{n-1}, \quad [n]_q! = [1]_q[2]_q \cdots [n]_q, \quad (25)$$

we find $\mathcal{N}(2; q) = 1/\sqrt{2(1+q)}$ which yields the same solution for x and y found in (12).

We can rewrite these solutions as

$$b_l^\dagger = a_l^\dagger(1 - a_t^\dagger a_t) \pm \frac{1}{\sqrt{2(1+q)}}(a_t^\dagger a_l^\dagger + a_l^\dagger a_t^\dagger)a_t. \quad (26)$$

When we act with $b_{l_2}^\dagger$ as just defined on the single-particle state $b_{l_1}^\dagger|0\rangle = a_{l_1}^\dagger|0\rangle$ we find that the first term in (26) annihilates the single-particle state and the second term, acting on the single-particle state, creates the correctly normalized symmetric two-particle state. This suggests a simple strategy to construct the general form of the bose creation operator: for the term in the creation operator that is to act on the $(n-1)$ -particle state and create the properly normalized symmetric n -particle state, divide the term into two parts, $b_l^{(n)\dagger} \equiv b_l^{1(n)\dagger} + b_l^{2(n)\dagger}$, make the first part annihilate the $(n-1)$ -particle state and the second part create the normalized symmetric n -particle state. We only require the bose commutation relation to be satisfied on the Fock space of bose operators. Thus we can assume that the states created by $n-1$ bose creation operators expressed in terms of the quon creation operators acting on the vacuum are totally symmetric and normalized; i.e., have the form

$$b_{l_1}^\dagger b_{l_2}^\dagger \cdots b_{l_{(n-1)}}^\dagger |0\rangle = \mathcal{N}(n-1; q) \sum_P a_{l_{P1}}^\dagger a_{l_{P2}}^\dagger \cdots a_{l_{P(n-1)}}^\dagger |0\rangle. \quad (27)$$

The second part, $b_l^{2(n)\dagger}$ can immediately be written down. The $n-1$ annihilation operators standing to the right can be written in a fixed order, since they will be symmetrized automatically when they contract with the $n-1$ symmetrized creation operators that act on the vacuum to create the $n-1$ -particle symmetric state. The summed over creation operators standing to the left can also be written in a fixed order; they will be symmetrized because they are summed over with the symmetrized annihilation operators. The only creation operator that must be moved around is the creation operator a_l^\dagger that carries the index of the bose operator b_l^\dagger that we are constructing. The result is

$$\begin{aligned} b_l^{2(n)\dagger} &= \mathcal{M}(n; q)[a_l^\dagger a_{t_1}^\dagger a_{t_2}^\dagger \cdots a_{t_{(n-1)}}^\dagger + a_{t_1}^\dagger a_l^\dagger a_{t_2}^\dagger \cdots a_{t_{(n-1)}}^\dagger \\ &\quad + \cdots + a_{t_1}^\dagger a_{t_2}^\dagger \cdots a_{t_{(n-1)}}^\dagger a_l^\dagger] a_{t(n-1)} \cdots a_{t_2} a_{t_1}, \end{aligned} \quad (28)$$

$$\mathcal{M}(n; q) = \frac{1}{n[n]_q! [n-1]_q!}. \quad (29)$$

(To construct $b_l^{2(n)\dagger}$ that can be used to represent the bose operators on an arbitrary quon state, replace the product of a 's in (28) by $\sum_P a_{tP1} a_{tP2} \cdots a_{tP(n-1)} / [n-1]_q!$) The form of the first part, $b_l^{1(n)\dagger}$ is more complicated, because the term that creates the n -particle state acting on the state with $n-1$ particles will do complicated things acting on states with more than n particles. We can avoid this problem and gain a simple result if we use projection operators so that the term in b^\dagger that makes an n -particle state contributes only when it acts on the $n-1$ -particle state.

4 Calculation using projection operators

Define the projection operator Λ_n to give one acting on an n -particle state and zero otherwise. We construct Λ_n by

$$\Lambda_n = \frac{\sin \pi(N-n)}{\pi(N-n)}, \quad (30)$$

where the explicit formula for N in terms of the quon operators has been given by S. Stanciu [9]. Using the results of the previous section the formula for b_l^\dagger is

$$b_l^\dagger = \sum_{n=1}^{\infty} \Lambda_n b_l^{2(n)\dagger} \Lambda_{n-1} \quad (31)$$

Equation (31) is the desired formula for the bose creation operator expressed in terms of the quon operators.

The analogous formula for constructing fermions from quons is

$$f_l^\dagger = \sum_{n=1}^{\infty} \Lambda_n f_l^{2(n)\dagger} \Lambda_{n-1}, \quad (32)$$

$$\begin{aligned} f_l^{2(n)\dagger} = & \mathcal{M}(n; -q) [a_l^\dagger a_{t1}^\dagger a_{t2}^\dagger \cdots a_{t(n-1)}^\dagger - a_{t1}^\dagger a_l^\dagger a_{t2}^\dagger \cdots a_{t(n-1)}^\dagger \\ & + \cdots \pm a_{t1}^\dagger a_{t2}^\dagger \cdots a_{t(n-1)}^\dagger a_l^\dagger] a_{t(n-1)} \cdots a_{t2} a_{t1}, \end{aligned} \quad (33)$$

$$\mathcal{M}(n; -q) = \frac{1}{n[n]_{-q}![n-1]_{-q}!}. \quad (34)$$

5 Bosons in terms of fermions and fermions in terms of bosons

Consider the solution (26) for the first nontrivial term in the expansion of the bose operator in terms of quons. The limiting case $q = -1$ seems to be ill-defined because of the factor $\sqrt{2(1+q)}$ in the denominator; however the numerator has the factor $(a_t^\dagger a_t^\dagger + a_t^\dagger a_t^\dagger)$ which also vanishes for $q \rightarrow -1$. To evaluate the limit for $q \rightarrow -1$ we calculate the matrix element

$$(b_{l_2}^\dagger b_{l_1}^\dagger |0\rangle, b_{k_2}^\dagger b_{k_1}^\dagger |0\rangle). \quad (35)$$

The result

$$\delta_{l_1, k_1} \delta_{l_2, k_2} + \delta_{l_1, k_2} \delta_{l_2, k_1} \quad (36)$$

is valid for all q and shows that for two-particle states the bose operator is correctly represented by the quon series in the limit $q \rightarrow -1$. Analogous results hold for the general case for both the bose and fermi operators.

It has been known for a long time [10] that fermions can be represented by bose operators in two spacetime dimensions. The present construction holds in any number of dimensions.

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References

- [1] O.W. Greenberg and R.N. Mohapatra, Phys. Rev. D 39 (1989) 2032.
- [2] C.-K. Chow and O.W. Greenberg, Phys. Lett. A 283 (2001) 20.

- [3] *Spin-Statistics Connection and Commutation Relations*, (American Institute of Physics, Melville, NY, 2000), ed. R.C. Hilborn and G.M. Tino (American Institute of Physics, New York, 2000), pp 113-127, hep/th.0007054.
- [4] O.W.Greenberg, Phys. Rev. D 43 (1991) 4111.
- [5] E. Ramberg and G.A. Snow, Phys. Lett. B238 (1990) 438; K. Deilamian, J.D. Gillaspay and D.E. Kelleher, Phys. Rev. Lett. 74 (1995) 4787; M. de Angelis, G. Gagliardi, L. Gianfrani, and G. Tino, Phys. Rev. Lett. 76 (1996) 2840; R. C. Hilborn and C. L. Yuca, Phys. Rev. Lett. 76 (1996) 2844; G. Modugno, M. Inguscio, and G. M. Tino, Phys. Rev. Lett. 81 (1998) 4790; D. DeMille, D. Budker, N. Derr and E. Devaney, Phys. Rev. Lett. 83 (1999) 3978.
- [6] S. Doplicher, R. Haag and J. Roberts, Commun. Math. Phys. 23 (1971) 199 and *ibid* 35 (1974) 49; R. Haag, *Local Quantum Physics* (Springer, Berlin, 1992).
- [7] O.W. Greenberg, Phys. Rev. Lett. 64 (1990) 705.
- [8] J. Cuntz, Commun. Math. Phys. 57 (1977) 173.
- [9] S. Stanciu, Commun. Math. Phys. 147 (1992) 211.
- [10] P. Jordan and E.P. Wigner, Zeitsch f. Physik 47 (1928) 631.