A tale of two skyrmions: the nucleon's strange quark content in different large N_c limits

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The nucleon's strange quark content comes from closed quark loops, and hence should vanish at leading order in the traditional large N_c (TLNC) limit. Quark loops are not suppressed in the recently proposed orientifold large N_c (OLNC) limit, and thus the strange quark content should be non-vanishing at leading order. The Skyrme model is supposed to encode the large N_c behavior of baryons, and can be formulated for both of these large N_c limits. There is an apparent paradox associated with the large N_c behavior of strange quark matrix elements in the Skyrme model. The model only distinguishes between the two large N_c limits via the N_c scaling of the couplings and the Witten-Wess-Zumino term, so that a vanishing leading order strange matrix element in the TLNC limit implies that it also vanishes at leading order in the OLNC limit, contrary to the expectations based on the suppression/non-suppression of quark loops. The resolution of this paradox is that the Skyrme model does not include the most general type of meson-meson interaction and, in fact, contains no meson-meson interactions which vanish for the TLNC limit but not the OLNC. The inclusion of such terms in the model yields the expected scaling for strange quark matrix elements.

During the past two decades there has been an extensive experimental program to study strange quark matrix elements of the nucleon [1]. They are of interest in large measure because they are sensitive to physics clearly beyond the naive quark model — they are nonzero only due to closed strange quark loops. Thus they are an ideal way to explore an important theoretical issue: the distinction between two variants of the large N_c limit of QCD. In this paper we focus on strange matrix elements in Skyrme models[2], which are chiral soliton models often justified by appeals to large N_c QCD[3, 4]. Attempting to understand the N_c scaling of strange matrix elements in the context of Skyrme models raises an apparent paradox which this paper resolves.

The traditional method for generalizing QCD to many colors[3, 5] treats the quark as being in the fundamental representation of $SU(N)$. We will refer to this approach as the 't Hooft (or "traditional") large N_c (TLNC). Recently, an alternative method — dubbed the "orientifold large N_c" (OLNC) limit[6, 7, 8] — for generalizing to large N_c has been proposed, where quarks are taken to be in a two-index representation of color. The principal theoretical motivation for studying this limit was the connection of one flavor QCD in this limit to large N_c supersymmetric Yang-Mills theory; this allows one to exploit powerful mathematical tools in the analysis of one-flavor QCD. However, there is an important connection to phenomenology: for $N_c = 3$ the anti-symmetric representation is isomorphic to the fundamental representation.

The fundamental difference between the two approaches is that the TLNC limit suppresses quark loop effects while the OLNC does not. Quarks are double-color-indexed objects in the OLNC limit and scale in essentially the same was as gluons; all planar diagrams are leading order. Thus, mesons in the OLNC limit scale with N_c in the same way as glueballs[11] which is distinct from the scaling in the TLNC limit:

$$\Gamma_n \sim N_c^{2-n} \quad \text{(OLNC)},$$
$$\Gamma_n \sim N_c^{1-n/2} \quad \text{(TLNC)}, \qquad (1)$$

where Γ_n is a generic n-meson vertex. In effect, there is a rule to convert the generic scaling from the TLNC limit to the OLNC limit, namely, the substitution $N_c^k \to N_c^{2k}$.

An obvious consequence of the scaling of Eq. (1) is on Skyrme models. The N_c scaling in such models is the result of the N_c scaling of the parameters in a model. If one alters the scaling of the parameters of a Skyrmion in the TLNC limit through the generic replacement $N_c^k \to N_c^{2k}$ one finds that the mass of the Skyrmons in the OLNC limit scales as $M \sim N_c^2$. As shown in refs. [12, 13] the N_c scaling of *all* generic properties of the baryon (mass, couplings, cross-sections, etc.) in the OLNC limit is consistent with the nucleon behaving as a Skyrmion. The consistency of this description is made even stronger due to Bolognesi's observation[12] that the coefficient of the Witten-Wess-Zumino term in the OLNC limit is $N_c(N_c-1)/2 \sim N_c^2$, while in the TLNC limit it is N_c[19].

The consistency of the Skyrme model with large N_c QCD is deeper than merely showing that all of the generic N_c scaling rules apply; spin and flavor play an essential role. The hedgehog structure of the classical solution to the Skyrme model imposes correlations between spatial directions and isospin. These correlations impose relations between certain observables computed at leading order in the collectively quantized Skyrmions which are independent of the details of the Skyrme Lagrangian [14]. These relations in all Skyrme-type models encode an emergent symmetry of QCD — a contracted SU(2 N_f) symmetry where N_f is the number of flavors. These rules follow solely from the fact that the pion-nucleon coupling constant diverges at large N_c while the pion-nucleon scattering amplitude is finite due to unitarity[15, 16, 17]. Since this condition holds for both the TLNC limit ($g_{\pi NN} \sim N_c^{1/2}$) and OLNC limit ($g_{\pi NN} \sim N_c$) the contracted SU(2 N_f) spin-flavor symmetry must emerge in both variants of the large N_c limit of QCD.

To begin, let us focus on *the* Skyrme model, *i.e.* Skyrme's original model[2], but generalized to three flavors so that the question of strangeness is relevant. The action for the model is

$$S = \int d^4x \left(\frac{f_\pi^2}{4} \mathrm{Tr}(L_\mu L^\mu) + \frac{\epsilon^2}{4} \mathrm{Tr}([L_\mu, L_\nu]^2) \right) + S_{WWZ} \tag{2}$$

where the left chiral current L_μ is given by $L_\mu \equiv U^\dagger \partial_\mu U$, with $U \in SU(3)_f$ [2, 4]; S_{WWZ} is the well-known Witten-Wess-Zumino (WWZ) term, the addition of which is necessary for the Skyrme model to respect the symmetries of QCD[18, 19]. The U field can be written as $U = \exp(i\vec{\tau} \cdot \vec{\pi}/f_\pi)$ where $\vec{\pi}$ is the pseudoscalar meson field, and $\vec{\tau}$ is a vector composed of the first three Gell-Mann matrices, $\vec{\tau} \equiv (\lambda_1, \lambda_2, \lambda_3)$. From the scaling rules in Eq. (1), it is apparent that $f_\pi \sim \epsilon \sim N_c^{1/2}$ for the TLNC limit, while for the OLNC limit the scaling is $f_\pi \sim \epsilon \sim N_c$. The only way that N_c enters is through the parameters f_π and ϵ, and through the Witten-Wess-Zumino term[12, 13]. To show the N_c dependence of the parameters in an explicit form, we can write

$$f_\pi = \sqrt{N_c}\, \overline{f_\pi} \qquad \epsilon = \sqrt{N_c}\, \overline{\epsilon} \qquad \text{(TLNC)}$$
$$f_\pi = \sqrt{\frac{N_c(N_c-1)}{2}}\, \overline{f_\pi} \qquad \epsilon = \sqrt{\frac{N_c(N_c-1)}{2}}\, \overline{\epsilon} \quad \text{(OLNC)}$$

where the barred quantities do not depend on N_c. This implies that the action can be written as

$$S = N_c\, \overline{S} \;\; \text{(TLNC)} \qquad S = \frac{N_c(N_c-1)}{2}\, \overline{S} \;\; \text{(OLNC)} \tag{3}$$

with $\overline{S}$ independent of N_c and of the same form for both the TLNC limit and OLNC limit. The choice of the form $\sqrt{N_c(N_c-1)/2}$ rather than N_c for the scaling of the parameters ensures that the Witten-Wess-Zumino term scales in the same way as the rest of the system, and is related to the fact that the baryon consists of $N_c(N_c-1)/2$ quarks in the OLNC limit[12].

This leads to an apparent paradox. In general, when a system is in the semi-classical regime, the size of a prefactor multiplying the action plays two roles: i) It controls the convergence of the semi-classical expansion, and ii) specific powers of the prefactor act as multiplicative factors for particular observables. Thus, when N_c is large enough to justify the neglect of subleading effects in both $1/N_c$ expansions, the only effect of going from the TLNC limit to the OLNC limit for the Skyrmion is to make the replacement $N_c \to N_c(N_c-1)/2$ in multiplicative factors for the various observables.

This is a surprising result, because it appears to leave no room for the effects of the different behaviors of quark loops in the two large N_c limits. At leading order, quark loops are suppressed in the TLNC limit, while not being suppressed in the OLNC limit. Thus, one would generically expect that strange quark matrix elements should scale as N_c^2 in the OLNC limit (that is, with leading order scaling), while in the TLNC limit they should be zero at leading order (that is, they should scale as N_c^0, one order below leading). However, given the simple replacement rule above, it appears that the Skyrme model must predict strange quark matrix of the same scale, N_c^0, for both large N_c limits. The paradox is how to reconcile the expectations for the scaling of strange quark matrix elements from *a priori* quark loop effect considerations with the apparent Skyrme model results.

The resolution would be trivial if the TLNC limit of the Skyrme model had a leading order contribution to strange quark matrix elements. While this is counter to our expectations, calculations of strange quark matrix elements of the nucleon in assorted variants of Skyrme models have larger typical values than for other models on the market[1]. Since the calculations do not include *any* explicit $1/N_c$ corrections, the very fact that the results are non-zero seems to suggest that the leading order term does survive. However, a careful analysis shows that the strange quark matrix elements of the nucleon in the Skyrme model *are* zero at leading order in a systematic expansion around the TLNC limit. While there are no *explicit* $1/N_c$ corrections in the existing calculations based on collective quantization, there are *implicit* effects which are subleading in $1/N_c$ and which account for the entire result.

To illustrate this, consider the nucleon's strange scalar matrix element at zero momentum transfer for a Skyrmion in the exact SU(3) flavor limit. It is convenient to analyze this matrix element as a fraction, denoted X_s, of the total scalar matrix elements of the three light flavors:

$$X_s \equiv \frac{\langle N \,|\overline{s}s - \langle \overline{s}s\rangle_{\rm vac}|\, N\rangle}{\left\langle N \left|\overline{u}u + \overline{d}d + \overline{s}s - \langle \overline{u}u + \overline{d}d + \overline{s}s\rangle_{\rm vac}\right| N\right\rangle}\,, \tag{4}$$

where $|N\rangle$ represents the nucleon state and the quantities with subscript "vac" indicating a vacuum subtraction. For the exact SU(3) limit, X_s can be computed via collective quantization, with collective quantum variables specified by an SU(3) rotation A on the standard classical static hedgehog. That is, U is given by $U = A^\dagger U_h A$ with the hedgehog

Skyrmion defined as $U_h \equiv \exp\left(i\hat{r}\cdot\vec{\tau}f(r)\right)$; the profile function $f(r)$ is determined by minimizing the energy subject to the condition that the system has unit winding number. A standard calculation for X_s in the Skyrme model [20] gives

$$X_s = \frac{1}{3}\langle N\,|1 - D_{88}|\,N\rangle = \frac{1}{3}\int dA\;\psi_N^*(A)\,(1 - D_{88})\,\psi_N(A) \tag{5}$$

where dA stands for the Haar measure for SU(3), $D_{88} = \frac{1}{2}\mathrm{Tr}\left[\lambda_8 A \lambda_8 A^\dagger\right]$ (which is an SU(3) Wigner D-matrix), and $\psi_N(A)$ is the collective wave function for the nucleon—*i.e.,* an appropriately normalized SU(3) Wigner D-matrix.

As was discussed in another context, tracing the N_c dependence cleanly requires that the calculation be done with the coefficient of the WWZ term having an arbitrary explicit N_c dependence[21]. This implies that the nucleon lies in a representation that is the generalization[22] of the octet for arbitrary N_c. The generalized representation "8", which at $N_c = 3$ corresponds to the familiar octet, is specified by $(p,q) = \left(1, \frac{N_c-1}{2}\right)$ for the TLNC limit. The evaluation of Eq. (5) for arbitrary N_c can be done straightforwardly with the aid of the SU(3) Clebsch-Gordan coefficients appropriate for the "8" representation[21, 22]. The result is

$$X_s = \frac{2(N_c+4)}{N_c^2 + 10N_c + 21} = \frac{2}{N_c} + \mathcal{O}\left(1/N_c^2\right)\ . \tag{6}$$

Thus, X_s goes to zero as N_c^{-1} as $N_c \to \infty$. X_s is subleading in a formal $1/N_c$ expansion around the TLNC limit — exactly as expected on general grounds.

We note in passing that phenomenological calculations of strange quark matrix elements are typically done with $N_c = 3$ at the outset in the WWZ term. This builds in some subleading effects in $1/N_c$. For example, calculations of X_s for the exact $SU(3)$ limit [20] gave 7/30, which is in agreement with Eq. (6) for $N_c = 3$.

Thus, it appears that despite our expectations that strange quark matrix elements of the nucleon in the OLNC limit are of leading order (*i.e.,* $\mathcal{O}(N_c^2)$), the Skyrme model of Eq. (2) has strange quark matrix elements which are of order N_c^0 regardless of whether one is in the OLNC limit or the TLNC limit. As it happens, this conclusion is correct, but fortunately it is not the entire story. The fault lies not in our expectations for the OLNC limit but with the model: while the direct SU(3) generalization of Skyrme's original model given in Eq. (2) does indeed have strange matrix elements of order N_c^0 in the OLNC limit, more general Skyrme-type models have matrix elements of order N_c^2.

We must recall that the general arguments from large N_c QCD do not justify *the* Skyrme model in the sense of Skyrme's original model. *The* Skyrme model does manage to capture the generic large N_c scaling rules for all observables, as well as the model-independent relations of the contracted SU($2N_f$) symmetry required from large N_c consistency rules. Beyond this, however, values of various couplings predicted by the Skyrme model should be viewed as model-dependent and thus essentially arbitrary from the point of view of large N_c QCD. Indeed, we know *a priori* that the model does *not* capture all of the physics at leading order in $1/N_c$. For example, *the* Skyrme model only has one kind of meson field, the light pseudoscalar meson field, whereas large N_c in fact has an infinite number of meson fields. Even for the light pseudoscalar meson interactions, terms which are allowed in large N_c QCD are set to zero to make the calculations tractable; indeed, an infinite number of such terms are neglected.

Thus, while all terms in *the* Skyrme model correctly encode the leading order large N_c scaling laws for both large N_c limits (provided the coefficients are scaled properly), the converse is not true: all terms with the correct leading order scaling behavior are not included in *the* Skyrme model. The paradox we are considering is resolved provided there exist terms in Skyrme-type models which, while absent in *the* Skyrme model, are allowed at leading-order in the OLNC limit and which give rise to strange quark matrix elements. Such terms cannot contribute at leading order in the TLNC limit, as they represent quark loop effects.

It is not hard to see how this can happen. Recall that the principal difference between the two limits was the suppression of quark loop effects in the TLNC limit and not in the OLNC limit. One consequence of this at the level of meson-meson interactions is that all terms for an underlying SU(3) symmetric theory which require more than one summation over flavor indices are suppressed in the TLNC limit by one factor of $1/N_c$ for each summation beyond the first. The reason for this is simple: each distinct sum over flavors for mesons corresponds to distinct quark loops at the quark level. Each additional quark loop is suppressed by a factor of $1/N_c$ in the TLNC limit, but not in the OLNC limit. Terms with more than one flavor trace that are suppressed in the TLNC limit are known to exist in chiral perturbation theory[23]. If terms of this sort contribute to strange quark matrix elements of the nucleon when included in a Skyrme-type model, the paradox would be resolved.

To illustrate this idea, consider the effect of the inclusion of one such term from chiral perturbation theory,

$$\mathcal{S}' = \int d^4x\, L_4\, \mathrm{Tr}\left(L_\mu L^\mu\right) \mathrm{Tr}\left(\chi^\dagger U + \chi U^\dagger\right). \tag{7}$$

The coefficient L_4 is one of the standard constants in chiral perturbation theory at order p^4, and the scalar source χ is taken to be proportional to the quark masses:

$$\chi = 2B_0 \begin{pmatrix} m_u & 0 & 0 \\ 0 & m_d & 0 \\ 0 & 0 & m_s \end{pmatrix}, \tag{8}$$

where B_0 is a constant of proportionality which is order N_c^0. From the discussion above, it is evident that

$$L_4 \sim N_c^0 \;\; \text{(TLNC)} \qquad L_4 \sim N_c^2 \;\; \text{(OLNC)} \tag{9}$$

Consider a model with an action given by $S_{SK} + S'$. We can probe the strangeness content of this model by considering the strange scalar matrix element at zero momentum transfer, *i.e.,* the strange sigma term:

$$\sigma_s \equiv \langle N \,|\, m_s \, (\langle \overline{s}s \rangle - \langle \overline{s}s \rangle_{\rm vac}) |\, N \rangle = m_s \frac{\partial M_N}{\partial m_s} \, . \tag{10}$$

The second form for σ_s is obtained via the Feynman-Hellmann theorem[24]. Note that this quantity is intimately related to X_s of Eq. (4) and contains the same information.

As with the usual Skyrmion, the mass of the nucleon is dominated by the mass of the classical hedgehog Skyrmion. The profile function $f(r)$ is obtained by varying the action subject to the hedgehog ansatz and imposing a unit winding number.

By standard large N_c rules, the profile function is independent of N_c at large N_c, regardless of whether one studies the TLNC limit or the OLNC limit. However, the detailed form of $f(r)$ is different in the two limits: S' contributes to the leading order action and hence to the variational equations at leading order in the OLNC limit, but not in the TLNC limit.

The contribution of the S' term to the mass of the nucleon may be computed straightforwardly, and from this the Feynman-Hellmann theorem can be used to compute its contribution to σ_s, which we denote σ_s':

$$\begin{aligned} \sigma_s' \;&=\; L_4 \, (32\pi \, m_s \, B_0) \int_0^\infty dr \, r^2 \left(f'^2 + \frac{2\sin^2(f)}{r^2} \right) \\ \sigma_s' \;&\sim\; N_c^0 \;\; \text{(TLNC)} \qquad \sigma_s' \sim N_c^2 \;\; \text{(OLNC)} \end{aligned} \tag{11}$$

where the scaling with N_c follows since everything on the righthand side of Eq. (11) scales as N_c^0 except for L_4; the scaling of L_4 with N_c is N_c^2, as given in Eq. (9).

The scaling of σ_s' in Eq. (11) is precisely as one would have expected from general arguments involving quark loops in the two limits. Moreover, this behavior is generic. In Skyrme-type models, the inclusion of meson-meson interaction terms the coefficients of which vanish at leading order in the TLNC limit, but not in the OLNC limit, can give rise to strange quark matrix elements of order N_c^2 in the OLNC limit. In the TLNC limit, however, such terms make only subleading (order N_c^0) contributions by construction. This resolves the paradox.

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