Novel Critical Behavior in Inhomogeneous Systems

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(Received 8 June 1993)

The effect of inhomogeneous couplings on the scaling behavior in critical phenomena is considered using renormalization group analysis, Monte Carlo simulations, and other numerical tests. Novel scaling relationships result from nonuniform perturbations of a uniform fixed point at criticality. An effective dimension entering scaling equalities is elucidated. Nonuniversal behavior is found in fully connected systems with exchange couplings arrayed in a hierarchical manner.

PACS numbers: 64.60.Ak

In recent years, new forms of scaling, richer than those occurring in standard critical phenomena, have been proposed to explain many physical problems ranging from growth processes to turbulence [1]. A basic issue is to identify which dimensions or other properties associated with fractal, multifractal, or inhomogeneous structures influence or determine the basic scaling laws of critical phenomena. In this Letter we address this question in two different contexts and find new results for the scaling properties of critical phenomena when multifractal features are incorporated in the models. First, we consider inhomogeneous perturbations about a uniform critical fixed point and deduce the effective dimensionality that enters scaling relationships. Second, we consider fully connected structures with inhomogeneous exchange couplings arranged in a hierarchical manner and argue that nonuniversal critical behavior, with unusual corrections to scaling, should be obtained.

Our study was stimulated, in part, by the critical behavior of superfluid He in porous media [2,3]. In the experiments, the critical exponents were found to be sensitive to the structure of the host medium. While bulk values were seen for He in Vycor, gels possessing long range correlated structure [4] yielded variable exponents. It is not clear whether these results are consequences of the geometry *alone* or of the correlated energetics of the adsorption potential resulting from the morphology.

Let us consider d-dimensional Ising models with inhomogeneous coupling constant arrangements as sketched in Figs. 1(b)-1(c). An iterative scheme underlies the construction [Fig. 1(a)]. We first consider these structures to define small perturbations of an otherwise uniform critical Hamiltonian. For simplicity we imagine an Ising model on a hypercubic lattice with the reduced Hamiltonian,

$$-\beta H(S)=\sum_{\langle ij\rangle}(K_c+\delta K_{n(i,j)})S_iS_j\,, \tag{1}$$

where $S_i=\pm 1$, the sum is over nearest neighbor (n.n.) pairs of sites, and $n(i,j)$ is the order of the bond $\langle ij\rangle$. If we choose $\delta K_m=\delta K_0 R^m$, with $0\le R\le 1$, $m=0,1,2,\ldots$, and δK_0 close to zero, the perturbation in Eq. (1) acquires a multifractal character [1]. Indeed, if we consider, for example, a d-dimensional structure of Nth order in a lattice of linear size $L=2^N$, we get

$$M_N(q)\equiv\sum_{\langle ij\rangle}[\delta K_n(i,j)]^q=d\delta K_0^q\sum_{m=0}^{N}\binom{N}{m}(2^d-1)^{N-m}R^{mq}=d\delta K_0^q(2^d-1+R^q)^N\equiv d\delta K_0^{q_2 N\tau(q)} \tag{2}$$

with $\tau(q)=\log_2(2^d-1+R^q)$. Thus, the qth moment M_N scales with an exponent $\tau(q)$ typical of a multiscaling distribution. The perturbation in Eq. (1) is a sum over this lattice of the usual local energy density operator, modulated by a position-dependent source factor with multifractal properties. We want to determine the scaling dimensions of this operator and see whether it is connected to any of the exponents characterizing the multifractal distribution. This goal can be addressed by renormalization group (RG) methods and achieved exactly for $d=1$. By putting $t_n=\tanh(K_c+\delta K_n)$, a standard decimation with rescaling $b=2$ yields: $t_n'=t_nt_{n+1}$, $n=0,1,\ldots$. These recursions have the $t_n^*=1$, uniform $T=0$ fixed point; linearization of the RG transformation around it leads to the following simple eigenvalue problem:

$$(b^y-1)\delta t_n\equiv\delta t_{n+1},\quad n=0,1,\ldots, \tag{3}$$

determining the acceptable ($\lim_{n\to\infty}\delta t_n=0$) eigenperturbations and their dimensions y. The relevant eigenvectors correspond to $0<y\le 1$ and have $\delta t_n=(b^y-1)^n\delta t_0$. The eigenperturbation satisfying Eq. (3), in terms of t (the

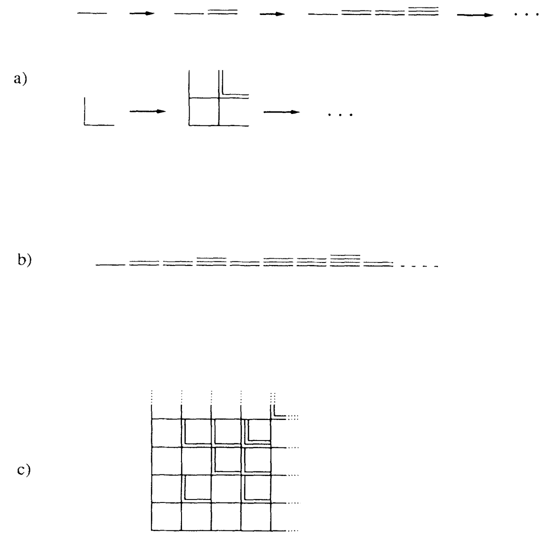

FIG. 1. Inhomogeneous coupling structures in $d=1$ (b) and $d=2$ (c), obtained using the recursive algorithms in (a). n-tuple bonds correspond to couplings, or perturbations, of $(n-1)$th order.

appropriate parameter in $d=1$), has exactly the multifractal structure postulated in Eq. (1), with $R=b^y-1$. In particular, $y=\tau(1)$. Thus, in $d=1$, our multifractal energylike perturbations have a continuous range of associated relevant dimensions. For $d>1$ an exact evaluation of these dimensions is not possible. We therefore apply to the problem an approximate Migdal-Kadanoff (MK) RG scheme [5].

We consider explicitly the $d=2$ case. With the bond moving sketched in Fig. 2, we find (with $b\equiv 2$)

$$t_n'=\left(\frac{t_n+t_{n+1}}{1+t_nt_{n+1}}\right)\left(\frac{2t_n}{1+t_n^2}\right),\quad n=0,1,\ldots. \tag{4}$$

By linearizing around the uniform fixed point, we get

$$\delta t_n'=\frac{b^{y_u}}{4}\delta t_{n+1}+\frac{3b^{y_u}}{4}\delta t_n=b^y\delta t_n\,, \tag{5}$$

where y_u indicates the thermal exponent associated with uniform perturbations and the last equality again specifies the eigenvalue problem determining y. Acceptable relevant perturbations are multifractal also in this case with $\delta t_n=(4b^{y-y_u}-3)^n\delta t_0$ and $0.325\ldots\le y\le y_u$. This means that if we choose a perturbation with $\delta K_n\propto R^n\delta K_0$ and $R=4b^{y-y_u}-3$, the corresponding exponent will be y. Moreover, using the expression for τ we get $\tau(1)=\log_b(4b^{y-y_u})=2+y-y_u$. In general d this becomes the scaling relation

$$y-\tau(1)=y_u-d\,. \tag{6}$$

According to Eq. (2), $\tau(1)$ can be interpreted as a fractal dimension associated with the perturbation. In Eq. (6) the relation between y and y_u depends only on the dimensions of the corresponding perturbations. Another interpretation of Eq. (6), substantiating the role of dimension for $\tau(1)$, involves the "defect" free energy, i.e., the difference Δf between the free energy corresponding to the Hamiltonian (1) and that obtained with all δK_m equal to zero. For a finite system of linear size b^N, Eq. (6) is shown below to be consistent with this free energy, made intensive by a normalization factor $b^{N\tau(1)}$ for $N\to\infty$, and scaling as

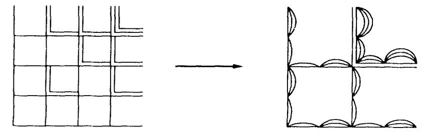

FIG. 2. Migdal-Hadanoff bond moving scheme for the $d=2$ structure. Decimation leads to an RG transformation with $b=2$.

$$\Delta f(K_c+\delta K,\{\delta K_n\})=l^{-\tau(1)}\Delta f(K_c+l^{y_u}\delta K,\{l^y\delta K_n\}) \tag{7}$$

near the bulk fixed point. For nonfractal defects [6], scaling relations like (6) hold with $\tau(1)$ replaced by the geometrical dimension of the defect (e.g., 1 for a line defect). Let us indicate by ω the dimension of the energy density operator; at criticality the two-point energy-density correlation function behaves like $r^{-2\omega}$ at large distance r. Since y_u is the dimension of the conjugate parameter δK [Eq. (7)], the static relation between fluctuations and correlations implies $\omega=d-y_u$. We now differentiate Δf twice with respect to δK_0. Since δK_0 is the parameter conjugate to the multifractal part of the Hamiltonian (1), we expect that, for a system of linear size b^N at $K=K_c$ and $\delta K_0=0$:

$$\frac{\partial^2\Delta f}{\partial(\delta K_0)^2}=\frac{1}{b^{N\tau(1)}}\sum_{\substack{\langle ij\rangle\\ \langle kl\rangle}}R^{n(i,j)+n(k,l)}\times(\langle S_iS_jS_kS_l\rangle-\langle S_iS_j\rangle\langle S_kS_l\rangle)\,, \tag{8}$$

where the correlation function in brackets decays like $r^{-2\omega}$, if r is the distance between $\langle ij\rangle$ and $\langle kl\rangle$. On the other hand, standard scaling considerations applied to the finite size version of Eq. (7) imply that the quantity in Eq. (7) should scale like $b^{N[2y-\tau(1)]}$. Since $\omega=d-y_u$, Eq. (6) entails $2y-\tau(1)=-2\omega+\tau(1)$. Thus the second derivative in Eq. (8) should scale as $b^{N[\tau(1)-2\omega]}$. Such behavior implies a simple, but nontrivial, way in which the energy correlation in Eq. (8) couples to the correlations implicit in the multifractal structure [7]. We have tested the scaling behavior of the quantity in Eq. (8) numerically on *Euclidean* lattices for arbitrary values of ω and R and have obtained clear evidence that our assumption, and thus Eq. (6), are satisfied generally.

The above results can be generalized in various ways. One can, of course, consider either systems different from

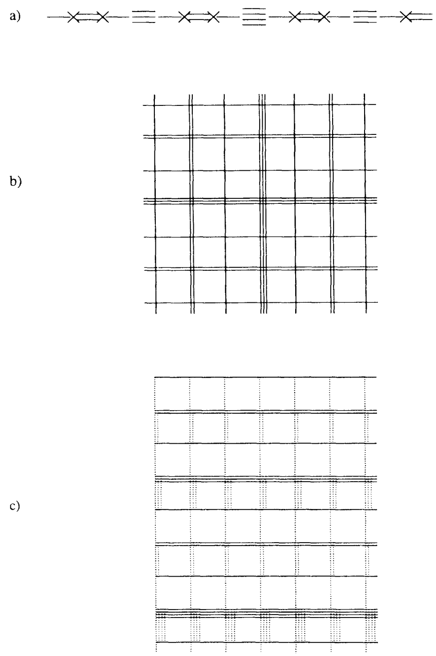

FIG. 3. (a) Ultrametric structure in $d=1$. Crosses indicate surviving spins under a $b=2$ decimation. (b) Ultrametric structure in $d=2$. (c) Self-dual, anisotropic, hierarchical pattern used in Monte Carlo simulations. Ultrametric structures may be constructed by superposing lattices of periodicity $a, 2a, 4a, 16a, \ldots$ with bonds of order $0,1,2,3,\ldots$, respectively, with the higher order bonds superceding the lower order ones.

the Ising model or multifractal perturbations of magnetic rather than thermal character and conjecture the validity of a scaling relation like (6). We have checked numerically that Eq. (6) holds when the perturbation in Eq. (1) has a quenched random character. In the construction of the multifractal structure [Fig. 1(c)], the upgrading of bond orders is not regularly applied to the right upper quarter of the structure but instead to a quarter chosen at random. Such a scheme gives rise to a random structure having the same τ's as the deterministic one. A numerical study of the behavior of the quenched average of Eq. (8) confirms that the scaling law (6) holds.

We now turn to a case of hierarchical inhomogeneities, sketched in Figs. 3(a)–3(c). Hierarchical, ultrametric structures of this kind have often been introduced in connection with dynamical problems inspired by spin glass physics [8,9]. They can be considered as trivial limiting cases in the multifractal family; in the example of Fig. 3, $\tau(q) \equiv \tau(0) = 2$, the dimension of the lattice, independent of the coupling's reduction ratio, R, for $R \leq 2$. Such a feature suggests the absence of a band of subdominant relevant perturbations, according to Eq. (6).

In $d=1$, a $b=2$, decimation of the structure can be performed as indicated in Figs. 3(a), leading to $t_0' = t_0^2 t_1$, $t_n' = t_{n+1}$, $n \geq 1$. One finds a structure $\delta t_n = b^{ny}(b^y - 2) \times \delta t_0$ for an eigenperturbation with exponent y. This leaves the choice $y = y_u = 1$, with periodic eigenperturbation $(\delta t_0, 0, 0, \ldots)$ as the only possibility. For the $d=2$ example of Fig. 3(b) a MK decimation with $b=2$ leads to

$$t_n' = \left(\frac{t_0 + t_{n+1}}{1 + t_0 t_{n+1}} \right)^2 . \tag{9}$$

Linearization around the uniform critical fixed point yields $\delta t_n = [2^n b^{n(y-y_u)} - (1 - 2^n b^{n(y-y_u)})/(1 - 2b^{y-y_u})] \times \delta t_0$. This again leaves $y = y_u$ and the fully uniform perturbation $\delta t_n \equiv \delta t_0$ as the only acceptable solution. We have verified that this behavior, consistent with Eq. (6), is indeed implied by the scaling of the corresponding quantity in Eq. (8).

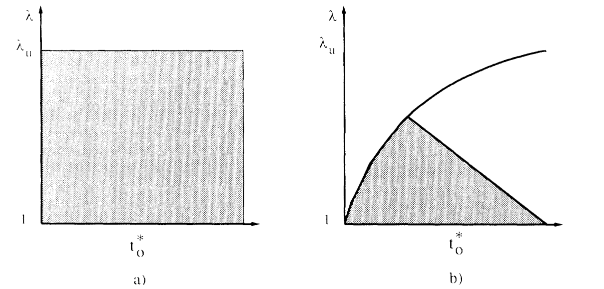

FIG. 4. Eigenvalue spectrum for the multifractal (a) and hierarchical (b) distributions of coupling strengths. The shaded regions denote a continuum of subdominant eigenvalues. λ_u is the eigenvalue of the uniform system. t^* is the fixed point parameter for the uniform system. In (a), $0 < t_0^* < 1$, with t^* being between 0 and 1. In (b) $0 < t_0^* < t^*$.

So far, we have discussed infinitesimal nonuniform perturbations away from a uniform system at criticality. We now consider the situation where the intrinsic couplings are inhomogeneous. Within the MK scheme the drastic difference found between multifractal and hierarchical perturbations in the linear regime remains when we investigate the fixed point structure of recursions like (4) or (9) in an infinite dimensional parameter space.

Consider first the multifractal interaction pattern of Fig. 1(b), with the recursion relation given by Eq. (4). The fixed point is characterized by a set of nonuniform coupling constants determined uniquely by $t_0^* \equiv \tanh K_0^*$. This set of fixed point parameters $\{t_n^*\}$ has the property that $\lim_{n\to\infty} t_n^* = t^*$ for all t_0^*. t^* is the fixed point parameter of the uniform system; at long length scales the lattice becomes trivially uniform. Perturbations about the fixed point can be studied numerically and result in a continuous spectrum of eigenvalues—the dominant eigenvalue, $\lambda_{\max} = 1.917\ldots$, being that of the uniform system [Fig. 4(a)]. Generally, multifractal interaction patterns lead to the generation of a continuous, gapless spectrum of eigenvalues, while preserving the universality of the dominant eigenvalue. This spectrum of subdominant relevant eigenvalues may lead to unusual corrections to scaling.

A new feature is found, however, for the case of hierarchical interactions, Fig. 3(b). The recursion relation, Eq. (9), again has a nonuniform fixed point structure of $\{t_n^*\}$. Here, in the asymptotic limit, the corresponding distribution of K_n^* increases linearly with n and meaningful solutions exist only for $0 \le t_0^* \le t^*$. The eigenvalue spectrum obtained by linearizing about these new uniform fixed points is shown in Fig. 4(b). The dominant eigenvalue no longer retains the value of the uniform system but shows nonuniversal behavior dependent on t_0^*. Furthermore, as $t_0^* \to 0$, $\lambda_{\max}$ crosses over from the uniform 2D value to that obtained within the MK approximation for a 1D system. (The MK scheme in $d=1+\varepsilon$ yields a discontinuity in the limit $\varepsilon \to 0$.) This behavior can be understood in terms of the long-length-scale nature of the fixed point distribution: As $t_0^* \to 0$, K_n^* increases with n and never recovers the uniform system limit. In addition, the corresponding eigenvectors are no longer simply related to the distribution $\{K_n^*\}$. Thus, any perturbation, even a uniform change in K_n, is controlled by $\lambda_{\max}$ and exhibits nonuniversal behavior. Similar results were obtained in 3D for the hierarchical interaction pattern: $\lambda_{\max}$ now crosses over from its 3D value in the uniform limit to the 1D value as $t_0^* \to 0$. As in the multifractal case, a continuum of subdominant, relevant eigenvalues is found [Fig. 4(b)]. The Widom scaling relation, for the vanishing of the surface tension at criticality, $\mu = (d-1)\nu$, is found to hold with the *Euclidean d* value independent of the value of t_0^*. The same holds in the multifractal case, as well.

Is this nonuniversality an artifact of the MK approximation? We have carried out Monte Carlo (MC) simulations in an attempt to answer this question. We considered finite Ising systems in $d=2$ with the anisotropic self-dual interaction pattern in Fig. 3(c). The horizontal bonds in the ith row have strength $K_{n_i}^h$, while the vertical bonds between rows i and $i+1$ have strength $K_{n_i}^v = -\frac{1}{2} \times \ln(\tanh K_{n_i}^h)$. By construction this $2^N \times 2^N$ lattice is self-dual, eliminating the need to search for a critical point. For this system we computed the specific heat and the susceptibility to obtain, from a finite-size scaling analysis, α/ν and $\gamma/\nu - 2 = -\eta$, respectively. We chose couplings from a two-parameter family of the form $K_n = K_0 + n\Delta$. We used $N=3$ to 8; the 256×256 lattice was the largest for which we could obtain adequate statistics, even using a standard acceleration algorithm [10]. This sort of problem poses a major challenge to MC simulation because of the broad range of characteristic energies. Specifically for $K_0=0.1$, $\Delta=0.4$ we found $\alpha/\nu = 0.08 \pm 0.02$ and $\eta = 0.45 \pm 0.03$, whereas for $K_0 = 0.2$ and $\Delta = 0.3$, $\alpha/\nu = 0.11 \pm 0.01$ and $\eta = 0.62 \pm 0.02$.

In summary, we have investigated the effect of heterogeneous couplings on the critical behavior of an Ising spin system. In the perturbative limit (around a uniform system at criticality), the coupling of multifractal correlations with those of a critical spin system leads to an enlargement of the standard algebra via a continuum of new relevant operators [11]. An RG analysis combined with MC simulations is suggestive of nonuniversal critical behavior of spin systems with a hierarchical distribution of exchange couplings. It is an intriguing possibility that a similar mechanism may be the cause of nonuniversality in the experiments of Ref. [2] and [3].

This work was supported by grants from NASA, NATO, NSF, ONR, the Petroleum Research Fund administered by the American Chemical Society, and the Center for Academic Computing at Penn State. T.L.E. is a guest worker at NIST and A.L.S. acknowledges partial support from INFM. We are grateful to E. Cheng and R. Schmolke for assistance at an early stage of this project.

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[10] Our runs for $N=3,4,5$, and 6 were carried out using an efficient vectorized code. Values of K_0 and Δ studied ranged between 0.1 and 0.4. The simulation entailed averaging times of the order of 10^5–10^6 Monte Carlo passes. Fewer values of K_0 and Δ were investigated for $N=7$ and 8 using a parallel implementation of the Swendsen-Wang cluster algorithm [R. H. Swendsen and J-S. Wang, Phys. Rev. Lett. **58**, 86 (1987)] on the Connection Machine. These results were in accord with those for smaller values of N.

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