ABSTRACT

Title of Document: **AN EPISTEMIC FRAMING ANALYSIS OF UPPER LEVEL PHYSICS STUDENTS' USE OF MATHEMATICS**

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Mathematics is central to a professional physicist's work and, by extension, to a physics student's studies. It provides a language for abstraction, definition, computation, and connection to physical reality. This power of mathematics in physics is also the source of many of the difficulties it presents students. Simply put, many different activities could all be described as "using math in physics". Expertise entails a complicated coordination of these various activities.

This work examines the many different kinds of thinking that are all facets of the use of mathematics in physics. It uses an epistemological lens, one that looks at the type of explanation a student presently sees as appropriate, to analyze the mathematical thinking of upper level physics undergraduates. Sometimes a student will turn to a detailed calculation to produce or justify an answer. Other times a physical argument is explicitly connected to the mathematics at hand. Still other times quoting a definition is seen as sufficient, and so on. Local coherencies evolve in students' thought around these various types of mathematical justifications. We use the cognitive process of framing to model students' navigation of these various facets of math use in physics.

We first demonstrate several common framings observed in our students' mathematical thought and give several examples of each. Armed with this analysis tool, we then give several examples of how this framing analysis can be used to address a research question. We consider what effects, if any, a powerful symbolic calculator has on students' thinking. We also consider how to characterize growing expertise among physics students. Framing offers a lens for analysis that is a natural fit for these sample research questions.

To active physics education researchers, the framing analysis presented in this dissertation can provide a useful tool for addressing other research questions. To physics teachers, we present this analysis so that it may make them more explicitly aware of the various types of reasoning, and the dynamics among them, that students employ in our physics classes. This awareness will help us better hear students' arguments and respond appropriately.

AN EPISTEMIC FRAMING ANALYSIS OF UPPER LEVEL PHYSICS STUDENTS' USE OF MATHEMATICS

By

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Dedication

It takes a little bravery to write a dissertation.
This one is dedicated to Marilyn Bing,
the bravest person I know.

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Great thanks also goes out to my family, starting with my wife, Kristin. Her straightforward encouragement has been clear, direct, and invaluable. A final word of thanks is for my parents, Jeff and Joan. If I have reached new heights, it is because they have let me stand on their shoulders.

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Chapter 1: Introduction

Physics education research, as a field, has two main research goals. Some projects focus on designing new curricula for physics classes. Researchers work to identify particular difficulties physics students have in traditionally taught physics courses and then build new laboratory activities, lecture presentations, instructional worksheets, and so forth in an attempt to address these specific student difficulties. The other main strand of physics education research concerns modeling how physics students think. Researchers in these projects focus on finding the best frameworks and vocabulary to describe what goes on internally in our students' minds. Ideally, research on how physics students think informs the curriculum design studies.

This dissertation is primarily concerned with modeling the thinking of physics students. Its findings certainly have implications for curriculum design, and these implications will be addressed at various points throughout this document. Still, the main goal of this study is to provide both researchers and teachers with a simple, natural, and meaningful framework for hearing and interpreting how our students think about the physics we work to teach them.

Describing a system for analyzing all physics students' thinking in all possible physics areas is much too broad a goal for one study. A few broad cuts will be made right away. First, we will focus on students' use of mathematics in physics. Physics is, after all, about building mathematical models for describing and predicting nature's behavior. Learning this mathematical language of nature is a very important part of a physics student's education. Focus will also be given to upper level undergraduate physics students. These sophomore, junior, and senior level classes comprise an important step up in mathematical rigor and complexity compared to the simpler mathematics required in introductory physics courses.

This dissertation will thus focus on modeling the mathematical thinking of upper level undergraduate physics students. Further cuts that will further focus this broad goal will be detailed in Chapter Two. For now, consider the following story that will motivate this work.

1.1 A Story: Lots of Things Count as "Using Math in Physics"

While this study will focus on upper level physics classes, a simpler example is in order for this introduction. Let's take an example of math use from a more basic, introductory physics course. Perhaps using math in such a low level course will be a straightforward thing to describe.

Consider four university professors who are all assigned to teach sections of their department's large introductory physics course. This physics course is the standard semester-long, calculus-based mechanics course that is a general education requirement for many science and engineering majors at our professors' university.

Our professors are a dedicated lot. They genuinely care about their students' success in physics, even those students like the engineering and chemistry majors who are not planning on continuing as professional physicists. Indeed, one might even say our professors care most about making a good impression on these non-physics major students. They realize their class might be the only university physics class these students take, so they want them to at least leave with a healthy appreciation for the subject.

These four professors decide to meet a week before classes start for the semester. After laying out the logistics of the lab schedules, exam times, and TA assignments, talk turns to their goals for their students. They decide that each of their four lecture sections should explicitly share a common goal. That way, even students or teaching assistants from different sections could work and study together and be assured of a reasonable overlap among the different lecture sections. All of our professors decide that using math fluently is very important in physics, so they agree to put special emphasis on mathematical thinking during the coming semester.

The story now turns to each professor sitting in his or her office preparing the first lecture of the semester. Each quickly comes across the familiar $x_f = x_o + v_o\Delta t$ kinematic equation. Remembering the pledge to emphasize math use in this semester's lectures, each thinks about what they're going to talk about when this first important equation of the semester comes up.

Professor Alpha looks at $x_f = x_o + v_o\Delta t$ and thinks to himself, "All right, that equation encodes a calculation scheme. If v_o is 4, Δt is 2, and x_o is 3, then that equation tells us how to calculate x_f. It's just 4*2 + 3." He plans on working a few sample calculations for his class and refreshing them on some simple algebra techniques. If you wanted to solve $x_f = x_o + v_o\Delta t$ for Δt, for example, there is a certain algebraic order of operations that must be observed. First subtract the x_o from each side and only then divide by v_o.

Professor Beta has a different reaction when $x_f = x_o + v_o\Delta t$ appears in her lecture plan. She sees that equation and is reminded of how appropriate uses of math in physics correctly model whatever physical system is at hand. Dr. Beta plans on talking with her class how $x_f = x_o + v_o\Delta t$ encodes a physical idea. Velocity is how far an object moves for every, say, one second. The quantity $v_o\Delta t$ is how many seconds' worth of motion you're dealing with. Tack that distance traveled onto x_o, which is where you started from, and you'll have where the object must end up, x_f.

Professor Gamma thinks something still different when he's sitting at his desk. "Oh, the point of $x_f = x_o + v_o\Delta t$ is that it's a convenient rule for kinematics," he muses. "There are several other rules too, like $x_f = x_o + v_o t + \frac{1}{2}at^2$ and $v_f^2 = v_o^2 + 2a\Delta x$. I'll present these various rules and talk with my students about how important it is to make sure you're quoting a rule that is applicable to your current problem. $x_f = x_o + v_o\Delta t$, for example, is only true if your acceleration is zero." Professor Gamma also plans on talking about how math, in general, provides a convenient and time-saving system for physicists. No one, practically speaking, starts every physics problem from absolute first principles every single time. Physicists sometimes take shortcuts, quoting previously packaged mathematical results. Mathematics is powerful, in part, because it allows such packaging.

Professor Delta's mind goes in yet another direction when she realizes that $x_f = x_o + v_o\Delta t$ is going to come up in her lecture. "The great thing about using math in physics," she thinks to herself, "is that you get this whole big web of interconnected math ideas. Math gives a formal, logical structure that connects superficially different applications. I'm going to emphasize to my students how $x_f = x_o + v_o\Delta t$ fits in with a web of other math ideas." Dr. Delta plans on talking about how $x_f = x_o + v_o\Delta t$ can be derived from the definition of average velocity: $\langle v\rangle = \frac{\Delta x}{\Delta t}$. She also wants to note how $x_f = x_o + v_o\Delta t$ has a base-plus-change structure to it just like, for example, $v_f = v_o + a\Delta t$. Stepping way back, $x_f = x_o + v_o\Delta t$ is a solution to a general class of differential equations: $\frac{d^2x}{dt^2} = k$, for the case $k=0$.

Our four hypothetical physics teachers illustrate an important point about using mathematics in physics. Many different lines of thought can all be considered to be "using math in physics". Professors Alpha through Delta are all correct. All of them were focusing on a type of mathematical thinking that is sometimes useful to professional physicists.

That mathematics is used in so many different ways in physics is not only a great source of its power but also likely a significant source of the difficultly for many students. Sorting through all these different mathematical options to find a presently appropriate one is not necessarily a natural or easy task for students. There's not even a guarantee that a student won't be thinking about, say, a calculation scheme when his professor is trying to demonstrate how a physical idea maps to an equation.

It is especially striking that so many mathematical options exist for such a basic example as $x_f = x_o + v_o\Delta t$. Indeed, this equation is often the very first one encountered by a physics student. This dissertation will go much further in mathematical complexity, focusing on upper level physics students' uses of mathematics. We will see that the mathematics they encounter will be at least as rich in multiple interpretations as this introductory example.

1.2 Physics Problem Solving as Navigation Among Different Uses of Math

The story about Professors Alpha through Delta may have seemed somewhat constructed and artificial. In reading about each professor's thoughts regarding $x_f = x_o + v_o\Delta t$, you likely thought, "Well, sure, I agree with what that Professor Alpha is thinking, but he's obviously forgetting this and that." Professor Beta then went on to think exclusively about "this" (and only "this") while Professor Gamma conveniently thought about "that" (and only "that").

The compartmentalization of our four professors' thinking is what makes the story seem a bit unnatural to an experienced physicist. Real examples of math use in physics tend to reflect several of the four professors' approaches. Finding an appropriate use of math for whatever specific situation is at hand can be a nontrivial matter. Perhaps expert physicists have developed a good instinct for making this choice in real time, but students may struggle.

This focus on the various mathematical options that exist leads to a certain characterization of problem solving in physics. Solving physics problems comes to be seen as a real-time navigation among a wide variety of options. A physics student's thinking "evolves" along some trajectory during a problem solving episode. Primitive humans did not consider all possible biological options before deciding to develop a larger cranial cavity. A series of random mutations led to a larger cranial cavity which turned out to be useful and advantageous. Likewise, a physics student does not consider all possible uses of mathematics before making a selection. A student may choose to try a calculation scheme for a while. If that calculation approach isn't advantageous, it could die out and the student may happen to try examining the physical referents for his mathematics. If this physical approach appears to be helping, it is that much more likely to survive. Biological evolution is not guaranteed to produce the best possible outcome. The present human physiology is by no means the most efficient or hardy possibility imaginable. Likewise, a student's path of thought during a physics problem will not necessarily lead to the most elegant or successful outcome.

This dissertation will be concerned with finding a natural, useful language for describing physics students' thinking during their real-time navigation through physics problems. Compare this approach with a more knowledge-bank approach to describing problem solving. Experts are good problem solvers, in part, because they have both more and better organized knowledge about math and physics. One could imagine a study of physics students' problem solving that focuses on describing the breadth of their available knowledge and its efficiency of organization. This dissertation, however, will be much more concerned with how students use, in real-time, whatever knowledge happens to be familiar to them.

1.3 Two-Part Organization of this Dissertation

There are two main parts to this dissertation. The first part develops an analytical tool for modeling the mathematical thinking of upper level physics students. This tool relies on a cognitive process called epistemic framing. Briefly, epistemic framing is the process by which a student pares down the set of all his available knowledge, selecting (often subconsciously) a subset of his knowledge for the purpose of solving problems, constructing new knowledge, and evaluating what he knows. The second part of this dissertation applies this epistemic framing tool towards various research questions.

1.3.1 Chapters 2 Through 5: Developing an Epistemic Framing Analysis Tool

This first section describes this dissertation's analysis tool for studying upper level physics students' use of mathematics. It develops a method, based on the cognitive process of epistemic framing, for analyzing how these students navigate the myriad mathematical options available to them. Evidence for how a student is currently framing their math use comes primarily from looking at the justification they offer in their mathematical arguments. What, in this moment, counts as sufficient justification to the student? Answering that question, this dissertation argues, provides excellent evidence for how the student is framing his use of mathematics.

Chapter Two situates this dissertation's work within the larger body of physics education, math education, and the cognitive science literature. This chapter describes how this project's analysis of student thinking focuses on modeling the internal mental processes of individual students, as opposed to focusing on the broader social and cultural systems in which these students interact. This chapter also details the knowledge-in-pieces framework for analyzing students' thinking. Physics students' thought is seen as emerging from in-the-moment associations of many small-grained knowledge pieces (as opposed to wholesale activations of large, coherent blocks of reasoning). After setting this background, Chapter Two discusses the epistemic framing process that will be central to the coming analysis of physics students' problem solving. Evidence for how a physics student is framing his math use will come from examining the justifications he offers in his mathematical arguments. Connections to argumentation theory research are thus explored as well.

Chapter Three describes the methodological details of this project's analysis. All of this dissertation's data comes from video recordings of physics students at work. Most episodes are either from groups of students working on their homework outside of class or from individual problem-solving interviews with students. This chapter describes the relevant data collection procedures. Most importantly, Chapter Three details how over one hundred hours of video data were condensed into the analysis presented in this dissertation. A first pass involved identifying episodes containing either an argument or miscommunication between students. A framing issue was often at the root of such disagreements, which usually amount to one student saying (implicitly), "Look at the math in this physics problem this way," while another student would argue, "no, this other way of using math is more appropriate." By closely analyzing many such episodes, commonalities began to emerge. The various ways the students in each individual episode were framing their math use were seen to gather into a smaller number of common framing clusters.

Chapter Four detailed the four framing clusters that emerged from this dissertation's data set: Calculation, Physical Mapping, Invoking Authority, and Math Consistency. Each of these four framings corresponds to a different cluster of mathematical justifications that students were seen to offer. Again, these four clusters (which match the musings of Professors Alpha through Delta in this chapter) emerged from this projects' analysis of physics students' thinking. They were not initially assumed to exist by the researcher. Several short examples of each framing cluster are provided.

With the epistemic framing analysis thus developed, Chapter Five applies this analysis tool to two extended episodes of physics students at work. Each of these extended episodes features a disagreement among students. The disagreement, however, runs much deeper than some factual issue, like whether the work done by gravity in some specific setting is independent of path. The disagreements in these episodes are over what would count as a sufficient reason for deciding whether gravitational work was independent of path. Chapter Five demonstrates how an epistemic framing analysis can naturally bring out this often implicit dynamic in students' thinking.

1.3.2 Chapters 6 and 7: Applying an Epistemic Framing Analysis Tool

The second section of this dissertation turns toward applying this framing analysis tool towards a variety of research questions. Chapter Five illustrates how this framing analysis offers a natural way to meaningfully parse an extended episode of physics students at work on a problem. A stronger statement of this framing analysis's value, however, would come from its success in addressing other research questions.

Chapter Six addresses the effect of a powerful symbolic calculator, like Mathematica, on physics students' thinking. The conventional wisdom is that such a powerful calculator shuts down physics students' mathematical abilities. This dissertation's framing analysis allows a more refined description of a symbolic calculator's effect. The calculator does not simply shut down students' math abilities. It does, however, preferentially select a certain subset of their mathematical knowledge: that corresponding to a Calculation framing. The physics students in Chapter Six's episodes show evidence of powerful mathematical ability when using a symbolic calculator, although this tool does tend to project their mathematical thought along a certain axis.

Chapter Seven turns to a different research question. What does an epistemic framing analysis say about expertise in physics? Several examples are given illustrating how experts are especially adept at looking for coherency among different epistemic framings. They might perform a long calculation and then see if that result matches a commonly quoted rule, for example.

Chapter 2: Literature Review and Connections to This Study

In this chapter, I orient this study within the existing body of work of the physics education research community and several other larger fields including cognitive science, linguistics, and education. This task consists primarily of three tasks:

→ Define the relevant unit for our analysis
→ Select an appropriate grain size upon which to focus
→ Detail the ramifications of such choices

Defining the relevant unit for analysis entails choosing between an individual or social group focus. Focusing on individuals views thinking as a process that primarily occurs in a given person's mind. There are certainly external influences on such a process, such as social setting, available tools, and time constraints, but these are viewed as (sometimes quite strong) perturbations to what is ultimately an individualistic process of meaning-making. Alternatively, one can conceptualize thinking as a fundamentally social activity. Such a viewpoint views meaning as being constructed in the interaction between individuals or between an individual and some aspect of his social and cultural environment. The first section of this chapter elaborates on such a socio-cultural view of cognition so as to better situate the more individual-focused cognitive viewpoint that will later be detailed and adopted for the bulk of this dissertation.

One must also select an appropriate grain size. Given that we will situate ideas in an individual's mind, what is the relevant "size" of these ideas? When a physics student voices a claim such as "the box sitting on the table isn't moving," are they likely thinking in terms of a big, largely coherent framework of forces, vector addition, and Newton's Second Law or are they likely thinking in terms of much smaller, more primitive units like "blocking"? The box isn't moving because the table is in its way, preventing it from falling. After detailing these two grain size choices, labeled "unitary" and "manifold" respectively, a principled choice of the manifold view is made and the relevant implications of this choice are discussed.

These choices of analysis unit and grain size have ramifications. If we are to analyze episodes of physics students' cognition through a lens that views thinking as being made up of activations of many small elements, the natural question is how any kind of coherent product comes from such a wide expanse of possibilities. The next section of this chapter details framing, the process by which a mind interprets what the present activity is about and activates a corresponding subset of responses. In a nutshell, framing allows an individual to pare down the list of all possible actions into a much smaller, more manageable subset of responses. The history of framing research across a wide variety of academic disciplines is briefly reviewed before detailing its past use in physics education research, to which this dissertation contributes.

The data set used in this dissertation comes from video of physics students at work solving physics problems. It is thus appropriate to review the wide body of research that has been published on students' problem solving, both in physics classes and, since this dissertation focuses on physics students' use of mathematics, math classes. Two main foci exist in this problem solving research: some studies focus on the content or organization of the problem solvers' knowledge while others focus more on the activation and usage patterns of whatever knowledge students happen to have. This dissertation more addresses the latter subset of existing research on physics and math problem solving. Indeed, a manifold stance towards cognition naturally leads one to focus on knowledge activation issues. This work offers a new lens through which to view and analyze their thinking, one that looks at the framing of mathematics these physics students use during their debates and arguments.

2.1 Cognitive Ontology: What Things are Candidates for Analysis?

Before attempting to model something as complicated as physics students' thinking, one must carefully define the system at hand. Many factors influence human thought, including past experiences, emotions, social relationships, confidence, available tools, comfort with prerequisite knowledge, power dynamics, and so forth. Without some up-front attempt to narrow our focus to a subset of these possibilities, our account of physics students' thinking could meander along an unfocused path.

The following subsection details the first, coarsest analysis cut made in this dissertation. There is a choice to be made between focusing on communities of learners or on individual learners.

2.1.1 Defining the Unit of Analysis: Why We Choose a Cognitive Rather Than a Socio-Cultural Focus

Two broad approaches exist for researchers hoping to analyze people's thinking. The cognitive approach, the one ultimately adopted for this dissertation, views thinking and learning as primarily individualistic enterprises. Any account of a student's thought will ultimately center on what sort of processes his brain is carrying out. A socio-cultural approach, in contrast, conceptualizes thinking and learning as fundamentally communal activities. The aim is to understand the social setting, which includes the student, as a whole. This socio-culture viewpoint is now reviewed in an effort to ultimately define this dissertation's cognitive stance via comparison.

Briefly, a socio-cultural viewpoint sees knowledge as strongly dependent upon the learner's situation and culture. Investigating whether a person knows "X" will inevitably include watching her do something that closely resembles "X". If knowing and doing are so closely intertwined (Brown, Collins, and Duguid, 1989), one should not ignore the real-world setting in which the person does "X". This choice to include the thinker's surroundings then leads to defining large units for analysis. An "activity system" (Engestrom and Miettinen, 1999), for example, includes the collection of objects, community members, tools, and social rules that surrounds the thinker. This activity system is not seen as a mere backdrop for the thinker's efforts. Rather, the activity system is itself the relevant unit of analysis.

To illustrate, consider how an available tool (Engestrom, 1987) can alter a student's thought. A student with a calculator handy might be especially likely to think about “ten divided by one-half” as a number-churning exercise as they punch calculator buttons, while a student without one may be more likely to interpret division by one-half as a simple doubling operation. The system of calculator plus student behaves quite differently than the system without the calculator. An activity system approach would seek to understand how the student-calculator interaction develops. What social environment brought this student-tool system together? What has society defined the uses of this calculator tool to be? How did the student become aware of these socially defined uses of a calculator?

Such a socio-cultural research paradigm leads to relatively expansive definitions of thinking, learning, and so forth. Even something as apparently modular as a “concept” becomes fundamentally linked with the learner’s setting when it’s argued that any instantiation of a concept necessarily involves some sort of non-inert background situation (Barsalou, 2002). For example, can you conjure up an example of a “chair” without, at least fleetingly, considering where that type of chair would be found and what you’d be doing in it? Wider issues, such as personal identity, also come to be seen through a social lens. Identity becomes cast as meaningful participation within a community. You feel like you are part of a community via your participation within it (Wenger, 1998). Learning is cast as the gradual trajectory from the fringes of a community towards eventual legitimate participation within its core (Lave and Wenger, 1991).

While these conceptualizations of thinking and learning are quite broad, there are many practical applications of this socio-cultural research lens to people’s thinking. Navigation on a Navy ship is a good example (Hutchins, 1996). The navigational task on a large ship is spread out among many people: observers, charters, pilots, etc. No one person can steer the vessel. The main goal is to understand how the ship (i.e. the whole system) thinks. Understanding any individual’s actions can only be done by considering whom he’s taking information from and whom he’s passing information to. Only as a system, with various safety redundancies and error checks built in, does the crew navigate. Smaller scale examples of individuals’ thinking being fundamentally tied to their environment also exist (Greeno, 1989). Studies have been done on dairy workers sorting product via a complex manipulation of full and fractional pre-packaged crates without explicit reference to the underlying mental mathematics (Scribner, 1984). Other research has described a person in a dietary class measuring two-thirds of three-fourths a cup of cottage cheese not by calculation but by measuring out three-fourths a cup of cottage cheese, dumping it out on a table, separating it into four equal parts, and selecting three (Lave, 1988) and young coconut street vendors using very coconut-packaging-oriented math to figure prices (Carraher, Carraher, and Schliemann, 1985). All these case studies make the argument that the subjects couldn’t have done the mathematics behind their actions in the abstract. Their success depended on some part of their environment, again suggesting a focus on a system larger than a mere individual thinker.

The distinction between a socio-cultural view and an individual cognitive view is ultimately one of foregrounding and backgrounding. A socio-cultural view assumes the existence of an actively thinking individual while it focuses on social-level interactions. A cognitive view assumes the existence of perturbative environmental influences while it focuses on the details of the actively thinking individual's thought. These two research viewpoints are best seen as complimentary, not mutually exclusive (Cobb, 1994). A broad, continuum picture is most appropriate, with cognitive viewpoints on one end flowing into socio-cultural viewpoints on the other (Greeno, 1997; Hedegaard, 1999).

Given such a continuum of research viewpoints, it is then the researcher's responsibility to make a principled choice for his work. This dissertation adopts a more individual cognitive view for two main practical reasons. First, it is the primary goal of this work to better understand the thinking of physics students in a university setting. The goal of a university, or any school for that matter, is to ultimately have the students leave the institution and carry with them, as individuals, some newfound knowledge that they can then apply in whatever career situation they find themselves in. Granted, an important element of modern day career success is working well in groups, but the individual doesn't bring these career teammates with her from college. She does bring her knowledge base with her from college.

The second motivation for adopting a cognitive viewpoint is more pedagogical. It is hoped that this study will help physics teachers be more cognizant of their students' thinking. Such awareness of a student's thinking is especially important in one-on-one and small group discussions. The situation, of course, becomes more complex in a twenty-five student classroom. Still, the twenty-five minds in a given physics classroom tend not to approximate the synchronicity and order of, say, a naval navigation crew. This work will thus focus primarily on describing the dynamics of individual physics students' thought within their own minds. Certainly external factors affect this thought, and several will be explicitly discussed in later chapters, but the bulk of this work concerns individual minds.

2.1.2 Selecting an Appropriate Grain Size for Modeling Students' Thoughts

Given that we have decided to make the individual student the primary unit of analysis, another research design question exists. This dissertation will be concerned with modeling the thoughts of individuals, but what will be the most useful grain size to employ? Loosely, what are the relevant "sizes" of these physics students' thoughts for our analysis?

A common default assumption for modeling students' thinking is to speak in terms of relatively large, coherent mental units. Successful physics students think productively because they have a well-integrated mass of knowledge, say about Newton's Second Law, that they are able to apply systematically to many different situations. Novice students struggle because they either lack such a coherent unit or they have some equally big naïve theory, say an Aristotelian model of motion, that gets applied to many situations and produces incorrect physical reasoning. A large grain size model such as this is referred to as a unitary account of thought.

An alternative grain size is a manifold account of student thought (Hammer and Elby, 2002). Such a view sees students' thinking as being made up of much smaller pieces that activate and chain together in especially context-dependent ways. It sees local coherencies in thought as evolving in primarily a bottom-up way as opposed to a top-down account of a broad "concept" being applied to a situation and giving it order.

This dissertation adopts a manifold view of physics students' thinking. The following subsections detail these unitary and manifold stances in an effort to motivate a principled choice for adopting a manifold stance. Implications of this choice, such as why the analysis in this dissertation is not couched in terms of transfer of knowledge from mathematics classes to physics, are detailed in later sections.

2.1.2.1 The Large Grain Size (Unitary) Account

A unitary account of students' thinking emphasizes the role of large-scale systems in their reasoning. Coherencies that are observed in students' work are attributed to these relatively organized blocks of ideas. Disagreement of students' answers with the physics canon is attributed to either a lack of an organized block of knowledge or to a relatively robust, organized misconception.

As an example of a unitary viewpoint, consider an early study on physics students' understanding of projectile motion (Caramazza, McClosky, and Green, 1981). These authors gave a paper and pencil quiz asking fifty students to draw the trajectory of a pendulum bob after the pendulum string was cut at various points in its swing. They proceed to observe patterns in many of the students' responses. Some students used the same response across all release points. About ten percent always drew the bob falling straight down, regardless of when the pendulum string was cut. About a quarter of the students drew identical parabolic shapes for all release points along the pendulum's path, irrespective of the particular size and direction of the bob's initial velocity at the particular release points. Other systematicities were observed as well.

Caramazza et al. proceed to discuss their quiz results in terms of "beliefs" and "gross misconceptions". They argue that more research should be done to better understand both the nature and the source of these students' beliefs about projectile motion. Implicit in all their discussion is the assumption that there exists a relatively stable "belief" in these students' minds, a reasonably stationary target for future investigations to probe.

A variety of studies have either adopted or implicitly defaulted to a unitary view of students' thought. Many topics have been investigated, including motion (Caramazza, McClosky, Green, 1981; McClosky, 1983), slopes of graphs (Woolnough, 2000), and geography (Vosniadou and Brewer, 1992). Unitary viewpoints are also evident in many studies that don't deal directly with specific science topics. For example, researchers have investigated students' understanding of the distinction between theory and evidence in a scientific argument (Kuhn, 1989), using localized results to infer a widespread conflation of the roles of theory and evidence. Work has also been done on students' ontological classifications of physics concepts (Chi, Slotta, and deLeeuw, 1994; Slotta, Chi, and Joram, 1995). This work looks at whether students tend to view something like electric current in an object-like way or a process-like way. That is, do they think of current as a thing in and of itself that can travel, split, and recombine, or do they think of current as a collection of smaller particles that behave in a certain way to produce an emergent phenomenon, i.e. current? These ontological studies assume that students classify physics concepts in an organized, stable way that is relatively independent of context.

A researcher taking a manifold view would not be surprised at coherences like those in Caramazza et al.'s pendulum quiz but would view these patterns of responses more locally. This researcher would see the particulars of this problem, the pendulum, the point mass in motion, even the physics quiz setting, as happening to cue a variety of smaller knowledge elements that fired together and built these locally coherent responses on the spot. One would not necessarily assume the same assembly of cues would occur on a different problem (or even the same problem asked in a different context). Coherency in students' responses is seen as an emergent phenomenon rather than a belief-directed output. A more detailed look at the manifold viewpoint is in the following subsection.

To summarize, unitary views see coherency as evidence of application of a stable mental structure whereas manifold views interpret consistency of student responses as an on-the-spot emergent phenomenon. A unitary viewpoint must then see learning as a relatively dramatic process. If, as basic constructivism believes, any input to a student's mind is filtered through past experiences and beliefs, then a student with a stable misconception will tend to see any incoming data through that tainted lens. How to get students to see evidence contrary to their misconceptions if they have such a strong filter in place? There is an inconsistency present: if a student has a robust misconception that is always applied, how can a student put such a misconception on hold so as to impartially evaluate counter evidence (Carey, 1986; Smith, diSessa, and Roschelle, 1993)?

2.1.2.2 The Small Grain Size (Manifold) Account

A researcher with a manifold view would not assume the existence of relatively stable belief structures. It holds that the on-the-spot dynamics of small knowledge elements can produce local coherencies which should not be assumed to indicate stable, reliable thought patterns (Minstrell, 1992; Strike and Posner, 1992). Relatively stable belief structures can certainly evolve, as with an expert's conception of Newton's Second Law, but a researcher with a manifold view would require evidence of consistent responses from a wide variety of contexts before making such an attribution.

Such a view of students' thinking addresses the inconsistency referred to at the end of the previous section. If an incorrect answer is not seen as indicative of a robust misconception, then the novice to expert transition starts to be seen as a much more gradual and gentle process. Non-canon responses from students may contain productive reasoning seeds, little bits of knowledge or intuition that are present in expert answers as well (Hammer, 1996). The novice-expert gap comes to be seen as much more of a continuum rather than a discontinuity (Warren et al., 2001). Even expert thought, in many situations, is best seen as a context-dependent activation of particular knowledge bits rather than a repeated application of the same, stable knowledge structure (Smith, diSessa, and Roschelle, 1993).

What is the nature of these small knowledge elements that are so central to a manifold stance? This dissertation will use the general term "resources" to refer to any element of students' thinking that contributes to the real-time evolution of an idea. Resources are compiled bits of knowledge that an individual treats as irreducible and devoid of substructure. There is no set size of a resource since different individuals (and even the same individual in different situations) can compile their knowledge differently (Redish, 2004). The primary definition of a resource concerns its function in a person's thinking: it is a small (relative to whatever idea that is currently at hand) chunk of knowledge that participates in the active, on-the-spot construction of an idea.

As a general rule, a resource, in and of itself, is neither correct nor incorrect. Its correctness depends on the specific context in which it is activated. Contrast that point about resources with a unitary belief, such as an Aristotelian model of motion, that can be declared incorrect essentially in vacuum, independent of context. The next subsections offer common examples of various types of resources that have been identified in the physics education literature.

2.1.2.2.1 Examples of Resources: Phenomenological Primitives

A classic example of a type of resource would be a phenomenological primitive (diSessa, 1993). A phenomenological primitive, p-prim for short, is a knowledge element abstracted from one's everyday experience. An example is *overcoming*. That is, a result follows because a cause has a greater effect on the system than other identified causes that would prevent that effect. Such a long-hand definition of a given p-prim is somewhat misleading. It is certainly not meant to imply an individual consciously thinks through a sentence like "a cause has a greater effect on the system than other identified causes that would prevent this effect" whenever he uses the p-prim overcoming. An earlier paragraph defined a resource as "a compiled bit of knowledge" that is "small, relative to whatever idea is at hand…that participates in the active, on-the-spot construction of an idea." The overcoming p-prim can thus be used in an almost self-evident way to explain an observation. Why did the box accelerate down the ramp? Someone could answer, "Because gravity overcame the friction force."

Such an answer does not necessarily imply that the student is thinking in terms of an elaborate Newton's Second Law framework, complete with vector addition of forces. Nor does such an answer imply the existence of any other similarly large, organized theory or belief in the students' mind. Why does the box move down the ramp? Gravity just wins, that's all there is to it. A request for further explanation might trigger an attempt to detail Newton's Second Law, but that forthcoming explanation emphatically does not imply the student was consciously aware of it back when he gave his original "gravity wins" answer.

If the *overcoming* p-prim is active in the student's thinking, it's quite possible a request for more detail on the original "gravity wins" answer will be met with an awkward confusion. "What do you mean? Gravity just overpowers the friction force and the box slides down the ramp." As the earlier definition of a resource stated, a resource is a compiled bit of knowledge that (at least in a given context) tends to be treated as irreducible. A student using the *overcoming* p-prim would tend to pause at such a request to explain such an evidently obvious (at least in this context) bit of knowledge as *overcoming*.

Overcoming is neither correct nor incorrect in the abstract. The previous example was an appropriate context for using that p-prim. Explaining that a rope pulling a bucket out of a well at a constant speed is overcoming gravity would be an inappropriate use of *overcoming.* A p-prim like *balancing* would be more appropriate. All of these preceding p-prim uses can happen outside of any conscious thought of a formal issue like Newton's Second Law. Such is even the case in some instances of expert thought. Experts, diSessa argues (diSessa, 1993), are experts in part because they have learned to coordinate p-prims with overarching physical laws that allow for activation of appropriate p-prims in a given context.

2.1.2.2.2 Example Resources: Symbolic Forms

A second type of resource is a symbolic form, a tightly bound physical interpretation and mathematical template (Sherin, 2001; Sherin, 2006). An example would be *balancing*, symbolized as □ = □ (although other symbolic forms could use the same math template). It is only a partial coincidence that the previous paragraph mentioned a p-prim of the same name. Symbolic forms, like p-prim, include some sort of physical knowledge element that has been abstracted from everyday experience. Symbolic forms, however, differ from p-prims in that they also include some sort of mathematical notation. The physical notion of balancing and the mathematical template □ = □ are bound together to create the symbolic form *balancing*.

As with p-prims, indeed as with most resources, the *balancing* symbolic form tends to be treated as a self-evident chunk of knowledge as a student builds up a larger idea. Sherin describes several such instances of the *balancing* symbolic form (Sherin, 2001). Groups of students were asked to find an expression for the terminal velocity of a ball that is dropped from a great height. One group wrote down the equation $R = mg$, where mg is the familiar form for the force of gravity and R stood for the air resistance force. Another group began by writing down $F_g = Cv^2$ where F_g is the gravitational force and Cv^2 is a common expression for the velocity-dependent force of air resistance. Both groups simultaneously talked about how the two forces equaled each other at terminal velocity, evidence that their physical notions of balancing were (at least at this moment) tightly bound with their use of the □ = □ template.

Just as a student's use of the *overcoming* p-prim did not imply the use of a robust conception of Newton's Second Law, neither did these students' use of the *balancing* symbolic form. Sherin proceeded to question these students about the origins of their written equations. Many were unable to articulate how an expression like $F_g = Cv^2$ could come from $\sum \vec{F} = m\vec{a}$. Furthermore, many of the students were actually surprised when presented with such a derivation. A manifold view of cognition nicely fits this data. These students were using small chunks of reasoning, like the *balancing* symbolic form, in a real-time construction of an idea. Yes, their thinking produced an expression that was consistent with a big, self-consistent conception (i.e. Newton's Second Law), but that consistency does not imply they were actively conscious of anything beyond a nearly self-evident mathematical expression of balancing.

Another example of a symbolic form is *base plus change*, symbolized as □ + □. As with p-prims and most other resources, *base plus change* is neither correct nor incorrect in and of itself. Sherin includes an example of students applying such a resource towards, correctly, trying to understand $v_f = v_o + at$ and, incorrectly, trying to justify $v_f = v_o + \frac{1}{2}at^2$. In both examples, the students were piecing together their explanations on the spot, basing their work on the *base plus change* resource that was treated as essentially self-evident.

2.1.2.2.3 Resources Activate in Context-Dependent Ways

Describing thought in terms of a manifold model entails not only identifying the resources in play but also discussing the dynamics of their activation and interaction. This dissertation's main goal is to provide a useful lens for identifying and mapping the dynamics among resources in one particular area of thought: upper level physics students' use of mathematics. In this investigation of dynamics among resources, we will be less concerned with identifying specific types of resources like an *overcoming* p-prim or a *balancing* symbolic form.

The zeroth-order issue regarding resources' dynamics is that they activate in context dependent ways. "Context" can refer to many things: the previous problem the student worked, the lecture he just came from, the tendencies of the fellow students around him, even his current mental state. Consider again the terminal velocity problem from the previous section. Many contextual factors could effect whether a student activates, say, an *overcoming* p-prim or a *balancing* symbolic form. If the last problem the student did had gravity overcoming friction as a box slid down a ramp, maybe the student will (incorrectly) try to think of the terminal velocity problem in terms of gravity overcoming air resistance. If a fellow student gives an argument about equal forces at terminal velocity, perhaps he will apply a "balancing" resource.

Compare this context dependent resource activation with the dynamics of a more unitary belief. Ascribing a unitary theory like an Aristotelian model of motion to a student means you would expect that student to apply this incorrect model across a wide variety of cases. A unitary belief is taken to be relatively context independent.

2.1.2.2.4 Resources Form Networks in the Mind

Since this dissertation will be mostly concerned with describing the dynamics among different resources, it is important to take time to describe the stage upon which resources act. The topology of this stage will constrain the subsequent dynamics of the resources. The wider stage of resources' interactions and activations that forms the background for this dissertation is primarily that of a large, weighted network.

We now motivate this weighted network picture of resources in two ways. The first is by analogy to the brain's neurology. The gap between the neuronal level and what a student actually says (as modeled by resources) is too great to admit anything more rigorous than this broad analogy. The second motivation for a network picture of resources comes from a review of resource-level studies of students' thinking. Readers who wish can skip over the following neurological section.

2.1.2.2.4.1 Neurological Analogy for Networks of Resources

Claiming that resources activate and associate in network-like ways should have a basis in the brain's neurology. Any product of the thought process must, after all, ultimately arise from the physical brain itself. While the leap from neuronal level processes to an *overcoming* p-prim is far too great for an explicit connection, there are certainly microscopic brain processes from neuroscience and cognitive science that parallel the network behavior I am claiming for macroscopic resources. Of particular importance are the neuroscience descriptions of three things:

→ neurons
→ association
→ executive function

For a more detailed description of the links between neuroscience and physics education research, see Redish's chapter on cognitive models (Redish, 2004).

Neurons themselves exist in networks. Each neuron cell can link to one or many other neurons. Information, i.e electrical impulses, can travel via these links from neuron to neuron. These impulses regularly feed back upstream to neurons that have already fired once. Multiple levels of activation can exist in any one neuron.

Association is well known trait of basic memory (Fuster, 1999; Baddeley, 1998). A particular smell, for example, may be especially likely to make you think about happily sitting at your family's dinner table when you were a child. Since neurons can exhibit many different levels of activation, it is perhaps not so surprising that thinking of one particular thing may prime a related piece of information, making it significantly more likely that you'll become consciously aware of this related information as well. When the students above saw the terminal velocity problem, their minds associated (in that particular moment) such a problem with a *balancing* symbolic form. This association then led to others, such as the awareness of an air resistant force and then the Cv^2 expression for it. Association helps a student navigate a network of resources by much the same neural mechanisms as a certain smell might remind you of your mother's pot roast.

Neuroscience also speaks of an executive function within the brain. The brain is made up of a great many neurons that encode a nearly incalculable amount of information. It is reasonable that some control mechanism exists that helps regulate what knowledge is brought into conscious consideration. Executive function is this control mechanism. Without such a gatekeeper, of sorts, human thought would tend towards a jumbled stream of unrelated information. While the exact workings of this "executive function" are unknown, its absence in brain damaged patients can be striking. Neurological fMRI studies and brain lesion patient studies have helped associate this executive function characteristic with the brain's pre-frontal cortex area (Goldberg, 2001).

There is thus a neurological basis for considering student thinking in terms of networks. Neurons themselves are physically interconnected in a network way and can exist in various stages of excitation. This variable excitation can lead to associations. Certain information is more likely to cue certain other information. A neurological executive function helps govern what part of this huge network's information is selected for conscious consideration.

2.1.2.2.4.2 Resource-Level Evidence of Networks

We now turn to the more macroscopic network behavior observed in physics students' use of various resources. We will talk of the different weights a resource can have in a network (plasticity) and of studies that have more explicitly adopted a network model of resources. Framing is the process by which a student decides what is the nature of his current activity and hence activates the appropriate area of his huge network of resources. Framing, the main focus of this dissertation, is examined in much greater depth in a later section of this chapter.

Resources themselves can have different weights, different degrees of commitment an individual associates with them. Some resources are relatively new and unfamiliar to an individual. These more plastic resources can be especially sensitive to context in their activation and use. Other resources are more established and familiar to a given individual. These more solid resources tend to be applied more readily across a wider variety of contexts.

Sayre et al. provide an example illustrating the relative "plasticity" of different resources in a given student's thinking (Sayre, Wittmann, and Donovan, 2006). Their example has a physics student trying to understand the inclusion of the negative sign in the standard expression for an air resistance force: $\vec{F}_R = -c\vec{v}$. The student quickly sets up a right-is-positive, left-is-negative coordinate system and then spends considerable time carefully checking the relative signs of the object's velocity and the resulting air resistance force. If the velocity is to the right (positive), then the air resistance force must be to the left. Let's see, a positive $\vec{v}$ combines with the external negative in $\vec{F}_R = -c\vec{v}$ to give $\vec{F}_R$ to the left.

The authors go on to illustrate how the student's coordinate system resources are less plastic than those he is associating with the sign of the resistive force. He comfortably sets up his coordinate system at the start without further explanation. He constantly refers back to this right-positive, left-negative coordinate system which anchors his thinking. In contrast, his reasoning about the expression for the resistive force is more plastic, more uncertain. He carefully, explicitly steps through his argument, even pausing at the end to ask for verification from the interviewer.

The relative value an individual places on the resources in play at a given time thus influence how these resources are combined to produce claims and inferences. Resources combine and combinations of resources combine as evermore complex networks of ideas are formed by students. These networks are, in general, fluid and dynamic (Wittmann, 2006). A mental unit like a "concept" comes to be interpreted by researchers as a large net of resources that help a person identify and manipulate relevant features of the concept across a wide variety of situations. A concept can appear and be used in very different ways in different situations via an individual focusing a different subset of the relevant resource network (diSessa and Sherin, 1998; Barsalou, 2005). diSessa and Sherin, for example, detail some of the different ways a student can read out information about "force" from a variety of situations.

Sometimes the student focuses on the presence or absence of accelerations, sometimes the same student focuses on weights and gravity, sometimes she focuses on action/reaction pairs, etc. The student's specific approach to thinking about "force" varies with respect to the question asked and the interviewer's prods. A good metaphor is to see a concept's instantiation evolving, rather than being applied, in a given situation as a person travels the relevant network of resources.

The many examples of student thinking detailed in this dissertation are best fit by a networked, manifold model of cognition. This work will focus on the type of justifications students offer for the mathematics they use in their physics problem solving. Tracking their use of various justifications will offer a window to the dynamics of their thinking. Data obtained through this window will align much more naturally with a manifold, as opposed to a unitary, view. Many specific examples validating this choice will be presented in later chapters. Suffice it to say, for now, that the mathematical arguments these physics students make can be said to "evolve" in a much stronger sense than they can be said to be "systematically applied". The word "evolving" is not meant to imply wandering or inexperience. If the mind's relevant topology is more manifold and networked, that's simply how an individual's thought must behave.

2.1.3 A Ramification of Choosing a Manifold Grain Size: Why Not Transfer?

There is not necessarily a huge dichotomy between the unitary and manifold views of cognition. Smaller knowledge pieces that are routinely activated and chained together in similar contexts become more and more tightly bound by the brain. They begin to act more as a unit, with their step-by-step details increasingly hidden, like a piece of compiled computer code (Redish, 2004). Unitary cognitive chunks can result. It is ultimately, of course, an empirical task to find which theoretical stance is most appropriate for the particular student data at hand.

Nonetheless, a researcher's theoretical stance, even if it's completely implicit, will alter the types of research questions that are asked. This dissertation investigates upper level physics students' use of mathematics in their physics work. The manifold stance implies, for example, that this study will <u>not</u> be framed as a transfer of knowledge from these students' math classes to their physics classes.

Much work has been done in both education and cognitive science on the issue of transfer. Transfer is traditionally seen as the application of an idea or concept learned in one context to a new, different situation. The transferred knowledge gives the learner structure and new insight in the new situation (Gick and Holyoak, 1980; Brown and Kane, 1988; Bassok, 1990). One possible way to approach this dissertation's subject matter would be looking to see how, or if, these upper level physics students transferred concepts from their math classes to their physics classes.

Such a framing of this study would presuppose a unitary view. In order for there to be transfer, in a classical sense, of a concept from math class to physics class, there would have to be a relatively packaged, exportable math concept in the first place. The thinking of the physics students in this study is more manifold. They are certainly using math ideas in their physics work, but these ideas are activated and flow into their conversation in a much more networked, evolving manner.

To motivate this complex view of transfer, we will turn to a detailed description of a well-known transfer study (Gick and Holyoak, 1980). These authors set up an experiment with a relatively straightforward transfer situation. Subjects were told a story about an army that wanted to invade a castle. The castle had several roads leading to it, but each of the roads was mined. If too great a weight passed over any of the roads, the explosives would detonate. The army's general solved this problem by dividing his army into several smaller units, sending each unit down a separate road leading to the castle. The relatively light weight of each piece of the army kept the mines from exploding. Subjects were then posed an analogous medical problem. Suppose a patient had a tumor in his stomach. An operation is out of the question, but there luckily exists an instrument that can send an energy beam into the body to irradiate the tumor. The catch is that any single beam strong enough to kill the tumor will also destroy whatever healthy tissue it passes through on its way in. How to safely irradiate the tumor?

Gick and Holyoak found that very few subjects used the castle situation to construct a solution to the tumor problem without a hint to do so. When a sufficient prompt was included, many more subjects "transferred" information from the army story to their tumor solution. The most relevant observation, however, is that Gick and Holyoak had trouble defining exactly what their subjects were noticing about the army story and importing to the tumor problem. In their modeling of this transfer situation, they acknowledge that their subjects could be doing a wide range of things. Perhaps they were detailing a very elaborate abstract structure in the army story and carefully mapping that structure onto the tumor problem. Both problems deal with a large deadly force (army or beam) that must be split to avoid disaster (mines or killing healthy tissue) and sent along different paths (roads or angles to the stomach) so that it can destroy (raze or irradiate) a target (castle or tumor). Were the subjects consciously mapping all this structure? Perhaps they were doing something much simpler, being consciously aware of only a vague notion that these were both division-type problems. Perhaps the average subject was doing something in between, beginning his work with only a rough notion of "division-type problems" and filling in some undetermined number of details one by one (in some unknown order) over time.

The point for us is that Gick and Holyoak's clinical transfer study was quite a straightforward "transfer" task, at least compared to watching physics students in their natural setting (at work on their homework) and nebulously hoping they will find a nugget of math knowledge to "transfer" to their physics problem. If Gick and Holyoak couldn't find a clean description of what exactly is transferred in their experiment, there isn't much hope for doing so here.

Pinning down the moment when the "transfer" of the math ideas relevant to this study happens is thus difficult for two reasons. It's very difficult to draw boundary lines where the relevant "idea" begins and ends, and even if that could be done, not every facet of the idea would necessarily enter the conversation at the same moment in time. Classical transfer (and the unitary view of packaged ideas it presupposes) is not a good empirical fit to either Gick and Holyoak's castle/tumor situation or to the math/physics data presented later in this dissertation.

In response to these types of issues, transfer has begun to be seen in a more manifold manner (Hammer, Elby, Scherr, and Redish, 2005) in the literature as well. Several authors are explicitly discussing a network-based view of the transfer process (Dufresne st al., 2005; diSessa and Wagner, 2005). The across-context activation of knowledge is cast as a "chaotic" process of networked bits of knowledge. "Chaotic" is not meant to mean unfocused or naïve but rather reflects how the specific knowledge that gets activated tends to be quite sensitive to the initial conditions within the learner's mind. Ideas from math class tend to be accessed physics class in relatively small pieces that are integrated in real time as these students work.

Many modern "transfer" studies view transfer on an individual basis (Lobato, 2003; Cui, 2006). Assessing the degree to which transfer occurs must be done student by student on the basis of what that individual sees as similar about the two domains at hand, as opposed to starting from the researcher's normative judgment about what knowledge should be relevant. There are many axes to consider, including situation similarity, metacognitive tools, and long-term memory content (Royer, Mestre, and Dufresne, 2005; Rebello et al., 2005). The exact way that the "transferred" knowledge helps the individual in the situation at hand tends to be nebulous and difficult to exactly define (Schwartz, Bransford, and Sears, 2005; Ford, 2005).

Hence, our manifold perspective affects how we frame this study. A cleanly definable transfer of knowledge from mathematics classes to these students' physics classes is neither expected by a manifold stance nor, as it will turn out, found. These students certainly use math knowledge in their physics work, but not in a uniform, regular way. Their math use occurs in a much more open-ended, evolving manner. It is sensitive to context and very subject to how the student is currently framing the situation.

This framing process is detailed in the next section and is the main focus of this dissertation's analysis of the dynamics of physics students' thinking. It will be our foundation for describing how these students use math in their physics work, a situation far too complicated to be described by a unitary classical transfer model.

2.2 Framing: How Does One Navigate a Myriad of Cognitive Resources?

If a manifold view of cognition does not default to attributing large coherent chunks of knowledge to students' thinking, then an important question arises. With the myriad of resources that could be called upon by a student at any given time, how does any coherent thought occur at all? A manifold view must include some process by which the set of all possible resources is pared down to a manageable size. Framing is such a paring-down process.

2.2.1 A Framing Story: Sarah Shifts Her Reasoning

We first turn to a published example from a physics student's reasoning (Wittmann and Scherr, 2002). This student's thinking displays a marked shift midway through the episode.

"Sarah" is an undergraduate who was willing to sit for an interview. The researcher has been asking her questions aimed at her understanding of electrical conductors and insulators. Sarah has just explained how insulators are so dense that current cannot flow through them. Wanting to further explore this insulator/density connection Sarah has mentioned, the interviewer brings up the case of Styrofoam. When the interviewer asks her whether Styrofoam is an insulator, Sarah responds that it is. Her response to "Why?" is that she "memorized it". The conversation continues, and when the next opportunity arises for Sarah to justify a claim she makes a blanket statement citing "organic chemistry". She is relying on authority in her explanations, quoting rules and facts.

After the interviewer prods her to give "any explanation you find," Sarah's reasoning undergoes a shift. She gives a more detailed, more conceptual account of conductance. Sarah puts together a little story about electrons getting torn away from their parent atoms and then being free to move. She explains how a battery could perhaps cause this electron-tearing and how a higher temperature wire would also have more energy available to tear electrons off the atoms.

The shift we care about in Sarah's reasoning concerns the types of explanations she gives. She began by quoting facts. Implicit was Sarah's interpretation of her situation and the interviewer's intentions. Oh, OK, this interviewer wants to find out what facts I know about conductors and insulators. I'll give him some facts I know.

The interviewer's apparent dissatisfaction with her quoted facts and subsequent "any explanation you find" prompt caused Sarah to reinterpret her activity. She came to see the interviewer's questions as prompts to tell a little story about conduction. Sarah is less sure of her story about tearing off electrons than she was about her quoted facts, but she sees this uncertainty as permissible now. Now we're constructing stories, not quoting facts.

Briefly, Sarah has framed her activity differently in the two parts of this episode. The different framings, different implicit answers to "what kind of activity is going on here", led Sarah to bring different subsets of her knowledge stores to bear on the interviewer's questions. We now turn to a more detailed account of this framing process.

2.2.2 Framing in Other Fields: Controlling Access to Knowledge

Framing is the, usually subconscious, choice the mind makes regarding "What kind of activity is going on here?" Within a manifold viewpoint, the mind's answer to this question will prime a subset of an individual's available resources while inhibiting others. The individual is hence most likely to respond to the present situation in a way that is locally coherent and consistent with his social setting. The list of all possible resources is pared down to a much more manageable set for the person to consciously consider. Framing is at least functionally analogous to the brain's previously discussed "executive function" (Redish, 2004), the neurological ability to control what information coherently bubbles to conscious consideration.

As a quick example, consider entering a library. Even if you have never been in that particular library building before, you will immediately have a general idea how to proceed. You would expect there to be computers with easy access to the library's home catalog search page, stacks of books organized in a particular way, and copy machines. You would plan on doing certain types of work in this building like quiet reading, writing, and note taking. You would also have social expectations. You would not plan on shouting across a room or sprinting down an aisle.

Framing should not be equated to activating a large, stable instruction list. It's not as if you immediately run down a checklist upon entering a library. Where's the check-out desk? There it is. Where are the computers with the search engines? There they are. Where's the periodical section? There it is. Large data structures like this library list are like a set empty slots ready to be filled in with the particulars of a situation. Several early studies in artificial intelligence (from which modern framing studies partially evolved) were concerned with identifying (and then programming) such data structure "frames" (Minsky, 1975; Rumelhart and Ortony, 1977; Schank and Abelson, 1977).

This dissertation does not equate framing with the recall and activation of organized, rigid data structures. Rather, this dissertation sees framing as the cuing of fuzzy, adaptable networks of cognitive resources. It may be statistically likely you'll find an electronic copy of the journal you want, but finding only a microfilm edition is easy to adapt to. Finding only an old fashioned card catalog would merely push your thought into a different part of your network of library strategies.

Framing has been studied in a wide array of academic disciplines including linguistics, sociology, art, psychology, and anthropology (Bateson, 1972; Goffman, 1974; Tannen, 1993; MacLachlan and Reid, 1994). All of these studies implicitly agree on the existence of what has been called "Felicity's Condition" (Goffman, 1997a). Felicity's Condition is the unspoken premise naturally adopted by an individual that incoming information, whether it be spoken, read, observed, etc, comes from a rational source, and it is thus up to the individual to attempt to contextualize and hence interpret that incoming information. Framing is the process by which the mind attempts this contextualization and interpretation.

Different individuals can certainly frame the same incoming information in different ways (Tannen, 1993). A quick example is to note that what is play to the golfer is work to the caddy (Goffman, 1997b). Miscommunications arise when two individuals frame their interaction differently, each bringing a different subset of their available resources to bear on the situation. Framing should not be thought of as something that happens only once at the start of a new activity. People continually recheck their framing of a situation and may alter it accordingly, bringing new resources into conscious consideration while temporarily disregarding other ones (Frake, 1977; Tannen and Wallat, 1993). Sarah is one such example.

Framing can lead people to subconsciously disregard some strands of input information that are not seen as currently relevant. A latecomer taking his seat at a theater can be ignored, possibly not even noticed, by other audience members (Goffman, 1974).

Psychologists have long been aware of selective attention effects where a person's concentration on a particular task can cause them to miss other dramatic occurrences. We briefly turn to a particularly striking (and amusing) demonstration of selective attention (Simons and Chabris, 1999).

The Simons and Chabris study began by sitting a subject down in front of a video screen. The subjects were told that they would be watching a video of people passing basketballs back and forth. It would be their job to silently count the number of passes between, say, people wearing white shirts.

When the video starts, the subject is greeted with a complicated scene. There are two teams of three people each. One team has white shirts and one team has black shirts. Each team has a basketball. White shirts only pass to other white shirts and black shirts only pass to other black shirts. The catch is that they are constantly dribbling about between passes, each individual weaving in and out all of the others. The subject must pay careful attention and try to focus only on the people with the white shirts.

About halfway through the 75-second video, a person in a gorilla costume slowly strolls in from the right. She even stops in the middle of the scene to look at the camera and beat her chest before slowly strolling out of view to the left. The six basketball players simply weave around her just as if she were another player.

One third to one half of the subjects never noticed the gorilla. I have personally been in several seminars where this video was shown to a roomful of people. About a third of the room will proudly announce they counted seventeen passes among the white shirts and then indignantly swear someone must have switched videos when they're told to watch a second time and forget about counting the passes.

The point is that the human mind is capable of selective attention, sometimes very markedly so. This gorilla example is mostly perceptual. Subjects were told to concentrate on the white shirt passes, and they missed seeing the gorilla in their midst.

2.2.3 Framing in Physics Education Research, with Epistemic Resources as a Window to Framing

Physics students, as any teacher can attest, display signs of selective attention as well. Sarah certainly did during her interview about electrical conductors and insulators. Her case wasn't a directly perceptual case like the gorilla example, but she displayed an analogous compartmentalization. She focused on fact-quoting, which caused her miss alternate types of explanation. Her interpretation of the purpose of an activity led her to disregard other information.

This dissertation examines upper level physics students' use of mathematics. Their thought processes show signs of selective attention as they juggle which components of their available math and physics knowledge to bring to bear at a given time. They can seem temporarily oblivious to a course of action that is obvious, at least to the researcher. These observations will be interpreted through the framing analysis scheme for math use in physics classes that is put forward in this work.

Framing is a process by which resources are selected and primed for conscious consideration. The examples of resources cited up to this point, such as p-prims and symbolic forms, are conceptual resources. They refer to what could broadly be called the subject matter knowledge of physics. Another class of resources is epistemic resources. Epistemic resources are tightly coupled to the framing process as well.

Epistemic resources deal with how students perceive the nature of the knowledge under current consideration. Do they see scientific knowledge as fixed and absolute or as being relative to one's point of view? Do they view scientific knowledge as something they can construct for themselves or as something that must be handed down from an authority figure (Hammer and Elby, 2002; Hammer, 2004a)?

Just as much of students' conceptual reasoning, including that which is presented in this work, is more appropriately viewed from a manifold perspective, so too is students' epistemic reasoning. Broad, decontextualized questions such as "Do you see science knowledge as being handed down from authority," at least by themselves, are unlikely to elicit meaningful information on students' epistemologies. Such a question assumes that students have relatively stable, context-independent beliefs about the nature of science. Much like the case with conceptual knowledge, authors have argued that students' epistemic stances are manifold and highly sensitive to context (Hammer, 1994; Elby and Hammer, 2001). Sarah, for example, displayed a shift from "knowledge as authority driven" to "knowledge as constructed by oneself" in her brief electric conduction interview. This shift happened in response to an interviewer's prod. It was an in-the-moment reaction to the natural flow of the conversation. One would certainly not expect that this isolated shift signals a large-scale change in Sarah's approach towards physics. It is unreasonable to think she never saw physics as being about telling conceptual stories before, nor is it reasonable to think she will never quote authority again. There are many similar published examples of in-the-moment shifts in students' reasoning (Redish, 2004; Hammer, 2004b; Hammer, Elby, Scherr, and Redish, 2005).

As further evidence to the manifold nature of students' epistemologies, there also tends to be a disconnect between how students view the nature of formal science and how they proceed to interpret their own work in science class (Sandoval, 2005). Epistemic stances evolve, in a time averaged sense, in complex ways as a student progresses through his education (Bromme, Kienhues, and Stahl, 2008).

Tracking what epistemic resources are in play at a given time is a vital piece of evidence for how a student is framing her current activity. Sarah reframed her activity in the interview from seeing her activity as a knowledge quoting process to seeing her activity as a personal explanation generating process. An epistemic lens, one that looked at how Sarah was viewing the nature of the scientific knowledge at hand at a given point in time, is vital in identifying this framing shift. Different conceptual resources were activated by each framing, producing different responses.

This dissertation will detail many examples of frame shifts in how upper level physics students use mathematics in their physics work. It will develop a system for identifying and analyzing these frame shifts that is based on what epistemic resources these students have activated at a particular time. What do they see as the nature of the math knowledge at hand?

2.2.4 This Study's Theoretical Stance

It is now possible to summarize the theoretical stance adopted for this dissertation. This study will look at the use of mathematics in the work of upper level physics majors. It aims to develop a modeling framework that can be used to describe their mathematical thinking.

While almost all the student data used in this study is from groupwork episodes, this modeling will be done from an individual cognitive perspective. We will focus on what is inferred to be happening in a given student's mind. There will certainly be communication between students as they influence what goes on in each other's minds, but such influences are cast as, often fairly strong, perturbations to the individuals' thought. Common ways of thinking about an issue may certainly emerge in a group, but these will be described as the result of individuals influencing the evolution of each other's thought through their natural discussion. The individual, not the group as a whole, is the primary unit of analysis.

A manifold perspective is used to describe a student's mathematical thought. Small bits of knowledge activate and cluster in context-dependent ways. Framing is the process by which the set of all a students' mathematical knowledge is pared down to a manageable subset for conscious consideration. This study uses an epistemic lens to identify and interpret how these upper level physics students frame their mathematical activity. What do they see as the nature of the math they are currently using? Is it about calculation, rule quoting, etc.? Using this epistemic lens, various episodes of student thinking are analyzed and common framings emerge. Upper level physics students' mathematical thought is modeled as a complex navigation among these common framings.

2.3 Previous Work in Problem Solving: Two Foci

Having established the theoretical grounding for this study, this chapter now turns to the more practically oriented connections for this work. All of the data analyzed in the later chapters will be of physics students working on physics problems. There is a large collection of physics and mathematics education research literature that focuses on students' problem solving, addressing questions such as: What do experts do when they solve problems? How are students different? How can we help students to do it more like experts?

This section reviews this body of existing work. While the main goal of this dissertation is to provide an analysis tool for understanding physics students' mathematical thought, it should be noted how the insights this analytical lens offers researchers meshes with existing work done in physics and math problem solving.

Two main threads of research have developed in the math and physics problem solving literature. Each focuses on a different aspect of what accounts for experts' problem solving success and, by extension, what instructors should focus upon with their students. One block of work emphasizes the amount, structure, and organization of knowledge. Experts are expert problem solvers because they have more knowledge that is indexed in highly efficient ways. A second block of work focuses on people's real-time use of their knowledge. Experts are expert problem solvers because they are more adaptive, expecting to draw on several different types of knowledge and checking for coherence among them as the need arises.

The following subsections outline these two threads within the published work on mathematics and physics problem solving. Reasons are given as to why and how this dissertation more closely addresses the latter part of the problem solving literature.

2.3.1 The Structure and Amount of Knowledge Influence Problem Solving

A good deal of the problem solving literature has focused on relatively static traits of expert problem solvers. They have both more and better-organized knowledge, which translates to more successful problem solving. Several review articles attest to the prominence of this view on problem solving (Maloney, 1994; Hsu, Brewe, Foster, and Harper, 2004).

Larkin et al. provided an often-cited, clear articulation of this knowledge bank description of problem solving (Larkin, McDermott, Simon, and Simon, 1980). Their paper begins by noting how experts display a certain difficult-to-define quality that helps them solve problems quickly and efficiently. People commonly call this quality "intuition" or "talent" or "imagination", but Larkin et al. quickly point out that simply naming this quality offers little insight into what actually comprises it or how it may be acquired. Their paper argues that an expert's breadth and organization of knowledge are the most important components of this "intuition" or "talent" in problem solving.

Larkin et al. offer three sources of evidence for this assertion. They cite studies of expert chess players, the existence of computer programs that solve problems by careful parsing and alignment with memory banks, and observations of human problem solvers.

Larkin et al. begin to make their case for the importance of experts' knowledge banks by citing a body of chess expertise literature. The most striking is a study where chess novices and grandmasters were shown positions from the middle of a game between strong chess players. The grandmasters could quickly memorize and reproduce these board configurations even though they sometimes contained upwards of twenty-five pieces. Novices only managed to remember the positions of a few pieces. When shown a random arrangement of twenty-five pieces, however, the grandmasters faired no better at memorizing the arrangement than the novices. The grandmasters' feats of memory (and, by extension, their "talent" or expertise in chess) were closely tied to their ability to break a game position into familiar chunks of pieces. They could quickly parse the position from an expert game because it contained these familiar pieces. The random arrangement had no such familiar chunks, rendering the grandmasters' considerable knowledge banks useless.

Larkin et al. continue with a description of two computer programs, STUDENT and ISAAC, that can solve standard textbook math and physics problems. STUDENT can take an input math problem and parse the problem statement into its grammatical components like noun phrases, comparative clauses, etc. It can then assign algebraic symbols and operators, translating the text-based problem statement into an algebraic equation, which it then solves. ISAAC is a more advanced program that can find the answers to statics problems in physics. ISAAC includes an English-parsing routine similar to STUDENT's that helps it find the relevant quantities and relations from an input problem. ISAAC also includes an explicit memory bank of schemata, carefully organized data structures, for common components like levers, fulcrums, frictionless surfaces, etc. When ISAAC realizes a problem is, in essence, a lever problem, it can access its "lever" data structure and allow it to organize its subsequent "thinking". Larkin et al. make no claim that STUDENT and ISAAC accurately reflect humans' problem solving. They do, however, point to these programs' existence as a proof-of-concept of sorts. These programs are using their programs, their carefully organized knowledge banks (and, obviously, only their carefully organized knowledge banks) to allow them to solve problems. Is it such a stretch, Larkin et al. argue, to assume human experts rely on their carefully organized knowledge banks as well?

Larkin et al.'s third source of evidence for the importance of an experts' knowledge breadth and organization comes from observations of human problem solvers. They detail the mapping process a human must carry out to translate a written problem into a form that can be solved to produce a correct answer. Experts are especially adept at using their well-organized knowledge banks to quickly represent a problem in a powerful, general way (McDermott and Larkin, 1978). They can, for example, take a problem about a block sliding down an incline from its literal representation on the page to a symbolic representation of forces and vectors organized according to over-arching mathematical principles. This ease of translation, they argue, is very important to an expert's problem solving success.

Closely correlated to experts' organized knowledge structures is their ability to productively categorize problems upon first reading them. Experts have been shown to be able to categorize physics and math problems into categories more efficiently according to general principles than do novices (Chi, Feltovich, and Glaser, 1981; Schoenfeld, 1985a; Snyder, 2000).

Chi et al.'s landmark study is often cited as the first in physics education to carefully examine how experts can categorize physics problems. Their experiment gave subjects (physics novices and experts) a stack of several dozen index cards, each with a standard textbook physics problem on it. Subjects were asked to sort the cards according to "similarity of solution". They demonstrated how experts were much more adept at sorting the problems according to "deep structure". For example, an expert will tend to look at a box sliding down an inclined plane and a planet in a circular orbit around the sun as two instances of a Newton's Second Law. Novices tend to focus on "surface structure" and see a problem about a ramp and a problem about an orbit.

Some teaching reforms, such as incorporating a computer program to help students make deep categorizations (Mestre et al., 1993), focus on helping students build such categorization abilities. Even when their categorization attempts fail, experts are still able to work in a more systematic way. They make substantial effort to relate their present work to more familiar examples, and they more regularly perform consistency checks on their work. These traits are due, at least in part, to a better organized knowledge base (Singh, 2002).

The Larkin et al. and Chi et al. studies are two of the foundational problem solving studies focusing on the breadth and organization of experts' knowledge. A large amount of knowledge is required for solving even the most straightforward problems in upper level physics classes (Manogue, Browne, Dray, and Edwards, 2006). Helping physics students become better problem solvers is complicated by the fact that experts' knowledge tends to become quite compiled over time. When shown a map of the electric field in a region of space, for example, the expert tends to comprehend a great deal of information at a glance. It is usually not necessary for him to dwell on such matters as what the electric field would mean for a test charge, what kind of mathematical structure a vector field is, and so on. Experts' conscious awareness of all these important details tends to decay over time as their knowledge of the subject becomes increasingly automatic.

Some research confronts this issue of experts' knowledge compilation in problem solving in a very direct way. These studies attempt to carefully dissect the relevant expert knowledge structure and then use that newly learned anatomy to design better teaching methods. A few examples include detailed studies of electromagnetic plane waves (Podolefsky and Finkelstein, 2007), Cartesian vectors (Poynter and Tall, 2005), and electrostatics (Redish, Scherr, and Tuminaro, 2006).

A good deal of evidence, then, exists in the literature attesting to the importance of experts' broad, well organized knowledge banks. This dissertation does not deny this importance. It does, however, argue that a complete description of problem solving expertise (and hence a complete description of physics students' thinking) must also include a focus on in-the-moment navigation. It's not enough that an expert has a static, huge store of knowledge, he must also be able to access the relevant bits at the relevant times. This dissertation provides a system for analyzing how physics students access various parts of their mathematical knowledge as the moment-to-moment demands of their problem solving change. We now turn to a review of the second part of the problem solving literature, that part which focuses explicitly on these in-the-moment navigation issues.

2.3.2 The Use of Knowledge (Framing) Influences Problem Solving

The last section explained how some of the problem solving literature focuses on the quantity and organization of a person's knowledge. Another large thread in the physics and math problem solving literature is more focused on real-time, in-the-moment issues. Experts tend to be better in-the-moment navigators as they solve problems. They become aware of potential dead-end attempts and consciously navigate away from them, whereas novices tend to drift along in whatever problem solving current they happened to enter (Schoenfeld, 1992; Redish 1999; Sabella and Redish, 2007).

We first turn to a detailed description of a well known study on these in-the-moment navigation issues (Schoenfeld, 1985b). Schoenfeld's study was conducted around a semester-long math course he taught. The course was meant as a general problem solving class. A typical classroom session lasted two and a half hours. Much of the students' time was spent in small problem solving groups working on various problems posed by Schoenfeld. As the students worked, Schoenfeld would circulate around the room to check on their progress and offer guidance as needed.

Schoenfeld's primary goal for this course was a focus on helping his students become more flexible problem solvers, better at navigating the solution process as different demands popped up. He worked towards this goal by making as many explicit references to heuristic problem solving techniques as possible. First and foremost was the sign he conspicuously posted at the front of the room for every class. The sign had three questions on it: What exactly are you doing? Why are you doing it? How does it help you? His standing rule was that he could stop any student in any group at any time and ask one or more of these questions, which he frequently did. Embarrassed silences often resulted, especially at the start of the semester. Schoenfeld also reported how he focused on problem solving navigation during the class discussions he led. He would often allow the students to carry him along a solution path he knew wouldn't work out so that he could explicitly demonstrate these monitoring strategies.

Schoenfeld gauged his success at teaching these navigation strategies with pretest and posttest problem solving interviews. His best piece of evidence concerns how often students "plunged into" a particular solution method and stuck with it, for better or worse. His students self-reported that they felt they had done less plunging during the posttest. This encouraging bit of self-reporting was supported by Schoenfeld's careful analysis of their problem solving transcripts. His students had attempted a significantly greater variety of solution methods during the posttest, even though the questions were of similar difficulty to the pretest (as measured with a control group). A marked increase in the number of correct solutions accompanied this increase in the variety of attempts. This increase in the variety of solution methods occurred even though his students also reported having a better idea how to start the problems in the posttest. They were more confident with their initial ideas, but they still managed to try a greater variety of solution methods.

Schoenfeld's study thus illustrates how a full description of problem solving must include something besides a static description of a students' knowledge bank. In-the-moment navigation during problem solving also deserves serious, explicit consideration. The particulars of exactly what experts are doing when they navigate from moment to moment are difficult to define precisely. Mapping out the details of a problem solving prescription, including calls to "focus the problem", "describe the physics", "plan the solution," and so on tends to be overly linear compared to expert thought (Heller and Reif, 1984) and not especially helpful for novices becoming better problem solvers (Huffman, 1997).

This dissertation aligns best with this second thread of the problem solving literature, being much more concerned with students' real-time thought dynamics rather than with their knowledge bank's breath or organization. The breadth and organization of a student's knowledge is sometimes an important consideration, but there are two main reasons for this focus on real-time navigation.

First, the knowledge bank approach has a more unitary underpinning. Searching for the structure of students' knowledge assumes there's a relatively stable structure there to be found. This assumption is significant and not necessarily valid in all situations, especially if students are working with relatively new material. The examples given later in this dissertation will demonstrate more flexibility and context dependence in students' thinking than would be expected in a unitary model of thinking.

Second, and more importantly, the unitary assumption that underlies much of the knowledge-bank problem-solving literature often leads to a biased experimental design. Much of the knowledge bank literature deals with students working on questions that are certainly "exercises" to experts and often are perceived as such by the students themselves as well. By "exercises", I mean the question is of a type that is very familiar to the solver, often a canonical textbook question. Something like "How long does it take a ball dropped from ten meters to hit the ground, neglecting air resistance?" would be an exercise, at least to an expert. Giving the students "exercises" is a very practical decision for these knowledge bank studies. If one assumes (explicitly or implicitly) a unitary model of students' thinking, then the goal of a problem solving study will be to identify and describe the order and consistency in students' thought. It is best to probe that order with as clear-cut, straightforward a tool as possible (i.e. asking "exercise" questions). Larkin et al.'s motivating description of problem solving computer programs certainly only dealt with straightforward exercise problems. In drawing their analogy to human problem solvers, all their humans are working on similar textbook exercises that had a clear-cut structure. Chi et al.'s question sort was also done with standard exercise problems lifted from a popular introductory textbook.

Of course, there is always the danger that such an experimental design will overemphasize the role of consistency and context independence in students' (and even experts') thinking. One wouldn't expect such knowledge organizations and categorizations that Larkin, Chi, at el. examined to apply nearly as readily to broader, more ill-defined problems. Schoenfeld's pretest and posttest questions were significantly more complicated than the typical "exercise" problem, and this design feature opened up a different aspect of students' problem solving to his investigation. He could focus on their real-time navigation, which turned out to be vital in his case. The upper level students in this dissertation are similarly working to solve problems that they see as more complex than a mere exercise, making our focus on students' in-the-moment navigation appropriate.

There is certainly bleed-over between these two lines of problem solving research. Perkins and Salomon provide an especially useful overview that integrates many of the popular studies reviewed here (Perkins and Salomon, 1989). They point out, for example, that even Schoenfeld's successful heuristic-teaching class was successful, in part, because his students were learning to ask themselves "What exactly are you doing? How will it help you?" in a very specific context. They worked on problems during their class time that dealt with topics similar to his pretest and posttest questions. His students must have been good at in-the-moment navigation during their posttests due, at least in part, to their newfound familiarity with the material. It is, after all, considerably easier to ask yourself "How will this help me?" if you know a little something about the type of problem at hand. Will changing variables help me solve this differential equation? It's hard to tell if I don't have a particular solution method for the relevant canonical differential equation form stored in my knowledge banks.

Still, this dissertation is primarily focused on providing a lens through which to view and understand upper level physics students' thinking as they navigate physics problems. It characterizes the local coherencies that tend to emerge in the minute-to-minute ways these students frame their math use.

This framing research tool allows a characterization of what "expertise" means in physics problem solving. Briefly, it highlights the importance of coherence in expert thought. There are likely several ways to frame one's use of math in a physics problem at any one given time, and experts tend to devote serious effort to aligning the implications of these various lines of thought. This characterization is detailed later in Chapter Seven.

2.4 Argumentation as a Path to Framing Analysis

An earlier part of this chapter describes this study's theoretical stance. It will interprets physics students' use of mathematics through a framing analysis lens, using "a fundamentally epistemic lens…What do they see as the nature of the math they are currently using?"

But how, practically speaking, can one infer what a given student sees as "the nature of the math" he is currently using? Briefly put, a researcher must look at the type of proof the student is currently offering for his mathematical statements. Argumentation theory is a field of research that has long been concerned with how people build justifications and communicate them effectively to each other. Thus, this chapter briefly turns to an overview of several threads of argumentation research.

2.4.1 An Overview of Argumentation Research

There are several subfields that are sometimes colloquially lumped under "argumentation theory" (van Rees, 2007). On one end of the continuum is what is best called formal logic. Studies in formal logic deal with relatively clean and straightforward methods of proof-making that can easily be decontextualized from whatever given situation is at hand. The formal logic chains that result from such analysis, chains like "If A then B, if B then C but not D, etc," lend themselves readily to computational modeling (Atkinson and Bench-Capon, 2007). Although even such apparently straightforward applications of classical logic rely on fuzzy mental processes that are very difficult to describe in detail analytically (Carroll, 1895).

A second branch of research, the one that is most often actually called "argumentation theory", includes what is often called rhetoric. This field of research focuses most on presenting, as opposed to having, an argument. It attempts to parse the content of a given argument into some kind of structure and often carries some sort of evaluative tone with regard to that structure. A central pillar of this field, and an important basis for this dissertation's analysis, is the work of Stephen Toulmin. He devised an often-cited system for parsing an argument into such parts as *claims*, *data*, and *warrants* (Toulmin, 1958). A person will make a statement, the claim, that requires proof. They will then offer one or more relevant facts, the data. The warrant is the bridge, sometimes unspoken, that explains how the given data relates to the claim at hand. For example, I might state that Jack Nicklaus is the greatest golfer alive (claim) because he won the Masters six times (data). The relevant warrant that would link this data to that claim would be that the Masters is a very prestigious tournament that is played every year on Augusta National, one of the most difficult courses on the planet.

Because argumentation theory deals more with real-world arguments than formal logic, analysis schemes like Toulmin structures are best thought of as heuristic guides, not formal organizers, for parsing arguments. For example, attempts to carefully map out even the structure of a published, formal legal argument according to Toulmin's scheme resulted in an explosion of complexity (Newman and Marshall, 1992). The authors found it increasingly necessary to add sublevel after sublevel to the basic claim-data-warrant scheme as they encountered more and more interwoven lines of reasoning. Even with all these complicated sublevels in the argument's diagram, they had trouble accounting for large chunks of implicit information that the writers of the legal document simply assumed the reader would know. Another study trained a group of corporate professionals in Toulmin structures and then had them try to apply this tool to diagram an argument relevant to their profession. Their success was limited, and many participants noted that the resulting argument diagrams were less convincing than the original arguments themselves (Adelman, Lehner, Cheikes, and Taylor, 2007).

Naturally occurring arguments are more nebulous in structure than an argument fitting a clean Toulmin structure. Justifications that are logically unsound are often treated as acceptable in informal, real-time situations according to complicated, probabilistic mental processes (Hahn and Oaksford, 2007). A third branch of research, often gathered under the label "discourse analysis" (van Rees, 2007), concerns itself primarily with the in-the-moment patterns people employ in their speech and thought as they construct and communicate arguments.

These in-the-moment argument constructions are often verbally incomplete. They often refer to a body of knowledge that the speaker (correctly or incorrectly) assumes he shares with the listener. These flow-of-conversation arguments sometimes have holes in them that are consciously or unconsciously overlooked. For example, suppose I'm trying to convince my wife that we should spend our next vacation in Rome, not Amsterdam. I might say, "Well, at least Rome doesn't have bedbugs".

That single statement could well win the argument for me, but understanding its full meaning requires much more than carefully analyzing the seven words I actually spoke. For instance, that statement refers to a chunk of information implicit to my wife and me. A few years ago, we were on a trip to Europe and a few of our friends had a bad case of bedbugs in an Amsterdam hotel. Rome's lodgings were much less dramatic. My seven-word statement could also lead my wife to frame our discussion in a certain way, perhaps "We're remembering events from a previous vacation". Once her mind interprets our present discussion in this way, she might quickly remember other highlights from our previous trip like the Colosseum, St. Peter's, and the Forum. Those recollections, never spoken aloud, might convince her to book a plane to Rome. Note also that my in-the-moment argument has an obvious fallacy in it. Rome most likely has a few bedbugs of its own somewhere.

Real-world arguments can thus be very complicated to analyze. It would be practically impossible for a researcher to infer most of the last paragraph from merely analyzing the seven-word transcript "Well, at least Rome doesn't have bedbugs".

This dissertation's work will most closely align with this discourse analysis research approach. It takes a detailed look at physics students' mathematical arguments, but it does not attempt to analyze these arguments according to a regular, repeatable structure of logic. As later examples demonstrate, these students' thinking is too dynamic to allow such a structural interpretation. It is expected that significant parts of these students' mathematical reasoning will be unspoken, just as with the Rome argument above. The goal of this dissertation is to provide the best possible window onto understanding these students' arguments, even if important parts must be inferred.

2.4.2 Argumentation in Math and Physics Also Leaves Much Implicit

Just like the Amsterdam/Rome argument, mathematics use in a physics class is too laden with context dependence, too reliant on usually tacit ancillary knowledge for its interpretation, to allow for something like a formal Toulmin analysis. As a quick example, consider telling a physicist that $A(x, y) = k(x^2 + y^2)$ and then asking her what $A(r, \theta)$ must then be (Redish, 2005). She would probably quickly respond, like most physicists, that $A(r, \theta) = kr^2$. Implicit would be the way she contextualized r and θ to be polar coordinates instead of mere variable names. She would likely dismiss the statement $A(r, \theta) = k(r^2 + \theta^2)$, that most mathematicians would consider to be the correct answer, as nonsense for the same reason. You can't add a squared distance to a squared angle on dimensional grounds. In this example, ancillary information is used by physicists to re-interpret the meaning of mathematical expression.

Such examples of ancillary information being vital to the contextual use of math in physics are much more the rule rather than the exception. Consider a physicist looking at the equation $x_f = v_o t$. She would likely think about an object in motion and use that picture to note that $x_f = v_o t$ is only true if the object starts at the $x = 0$ position, starts at the time $t = 0$, and has no acceleration. Now consider the same physicist looking at $i = 12f$, where i was a number of inches and f a number of feet. She would interpret that expression as sensible by noting the "12" had a hidden unit on it, specifically inches per foot. The point is that our hypothetical physicist relied on appropriately cuing two different collections of ancillary information to interpret two expressions that were of the same mathematical form: a = bc.

Since ancillary knowledge is so important to understanding mathematical statements in a physics setting, any argument analysis must be done with respect to this largely tacit body of information. What Toulmin would call the warrants of a mathematical argument must be interpreted with respect to whatever ancillary knowledge the individual is calling upon at a given time. These warrants can shift from moment to moment, in accordance with the manifold view of mind of this dissertation.

2.4.3 This Dissertation Tracks How Physics Students Frame Their Math Use by Focusing on Their Mathematical Warrants

Practically speaking, this shifting of warrants results in physics students giving different kinds of proof at different times during a mathematical argument. This dissertation proposes a system of classifying the various types of warrants observed in physics students' mathematical argumentation. This system emerged from commonalities observed across a wide variety of physics problem solving episodes. The analysis of the students' mathematical warrants offers a powerful window to describing how they are currently framing their activity. It is this dissertation's solution to the problem of accessing some important implicit components of real flow-of-conversation arguments.

The idea of having different kinds of proof be accepted in an argument is not, in general, a new one. On a grander scale, researchers have noted that what counts as valid proof does not necessarily remain the same as one crosses social or cultural boundaries. One needs look no further than the Creationist/Evolution debate for an example (Lemke, 2001). On a smaller classroom scale, this phenomenon of shifting justification has also been noted with biology students and has been interpreted with respect to a manifold picture of the mind (Southerland, Abrams, Cummins, and Anzelmo, 2001). This idea of different kinds of reasoning counting as sufficient proof has also been noted in mathematics education research. Researchers have discussed, for example, the "embodied", "proceptual", and "formal" reasons 13 + 24 equals 24 + 13 (Watson, Spirou, and Tall, 2003; Tall, 2004). The embodied explanation is that adding twenty-four objects to a collection of thirteen objects gives you the same total number as if you started with thirteen objects and added twenty-four. A proceptual explanation focuses on how you can manipulate the meaning-laden symbols in the problem in a prescribed manner, i.e. you can do the column-addition you learned in elementary school, and get the same result either way. The formal reason 13 + 24 equals 24 + 13 is that it's assumed true by axiom. It's the commutative property.

The contribution of this dissertation is to provide a system for analyzing physics students' mathematical thinking that was built from observing loose commonalities across a wide variety of examples. This system focuses on the different types of proof these students give as they use math to argue during their physics problem solving. That the types of justifications they give shift throughout the course of an argument is not surprising given the manifold picture of the mind adopted in this dissertation. It will be demonstrated how the flow of a problem solving conversation can be parsed by viewing it as two or more individuals trying to juggle and coordinate various types of mathematical justifications in their reasoning.

2.5 Chapter Summary

This chapter set out to address three main issues, with respect to both this dissertation itself and the wider body of physics education literature:

→ Define the relevant unit for our analysis
→ Select an appropriate grain size upon which to focus
→ Detail the ramifications of such choices

The individual student, rather than the larger social setting, is the relevant unit for analysis in this dissertation. A large body of work exists that describes how a student's social environment, cultural background, available tools, sense of identity, and so on affects their thinking. The analysis in this dissertation does not ignore these factors, but it does treat them as (sometimes strong) perturbations to what is ultimately an attempt to describe what goes on in an individual's head. Most of this dissertation concerns cognitive processes that operate in a student's brain and hence give rise to various behaviors.

Given the choice to focus on the cognitive processes within individual students, there is still a grain size choice to make. What are the relevant "sizes" of students' ideas? Two broad stances on this issue exist in the physics education literature: the unitary view and the manifold view. These two views differ in how much coherence and context independence they implicitly attribute to students' ideas. A unitary view sees a student's responses as indicative of some relatively coherent mental model or theory that the student will tend to apply in a variety of different contexts. A manifold view sees students' ideas as being in-the-moment constructions. Students use a variety of smaller grained knowledge elements, called resources, to assemble their ideas in real time. Which resources get activated and how they assemble depends on the context, the particulars of the situation. This dissertation adopts a manifold view of students' cognition because it will be seen to best fit the examples of students' thinking in later chapters. These students' thinking will be seen to be more mercurial than a unitary view would suggest.

Choosing a manifold view has ramifications. With the myriad of resources that could be called upon by a student at any given time, how does any coherent thought occur at all? A manifold view must include some process by which the set of all possible resources is pared down to a manageable size. Framing is that process. Framing has been studied in a wide variety of academic disciplines.

This dissertation looks to build a tool for analyzing physics students' framing of their math use. It borrows some relevant language from argumentation theory to do so. Examining the warrants that physics students use in their mathematical arguments will be the primary method of determining what subset of math resources the students are currently considering (i.e. how the students are framing their math use). Adopting different warrants corresponds to activating different epistemic resources.

Identifying how physics students are framing their use of math while they work on physics problems aligns this dissertation with one of two main subsets of the large body of problem solving literature. Many problem solving studies focus on the breadth and organization of students' knowledge banks. The more a student knows and the better organized their knowledge is, the better problem solvers they tend to be. This dissertation better aligns with a second branch of the problem solving literature—that which focuses on students' in-the-moment navigation during problem solving. Experts are good problem solvers, in part, because they make better real-time decisions regarding their focus.

This dissertation thus uses a manifold view of cognition to describe physics students' mathematical thinking as they solve physics problems. Tracking the warrants for their mathematical arguments will be a powerful lens for tracking how they focus their attention on some ever-evolving subset of their total math knowledge (i.e. for tracking how they frame their math use).

Chapter 3: Data Sources and Methodology

This chapter begins with a description of the student population and classes from which this study's data set was drawn. It continues with a general account of how this dissertation's mathematical framing analysis was developed. This chapter leads in to chapter four, which details each of the four common framings that have emerged from this study's analysis.

3.1 Data Sources

This dissertation's aim is to study the dynamics of physics students' thinking as they use mathematics in their work. It presents a system for analyzing how these students are framing their math use based on examining the warrants of their mathematical arguments. This framing analysis tool can be applied towards other research goals, such as examining the effects of powerful symbolic calculators on physics students' thinking and characterizing expertise with regard to math use in physics.

We choose to develop this framing analysis tool from examples of student thinking from upper level undergraduate physics courses. "Upper level" refers to a physics class that is not intended as an introductory course fulfilling a general education requirement. An upper level course would not be meant as a student's first or only university physics class. This level of physics class is especially important to a physics student's development as it usually marks the first exposure to the more rigorous math use common in advanced physics. This dissertation examines these students' thinking during this critical time on their path towards physics expertise.

Most of the students in this dissertation's examples of "upper level" physics are thus physics majors. This blanket statement is especially true of the examples drawn from the junior and senior year courses like Advanced Mechanics, Electricity and Magnetism, Quantum Mechanics I, and Quantum Mechanics II. Any non-physics major in these courses tends to be an astronomy major for whom the typical line of study has heavy overlap with the physics department. We now turn to a brief description of the various courses referenced in this dissertation. Most of the students analyzed were from the University of Maryland, so that university's course numbers and syllabi are used below.

3.1.1 PHYS 401: Quantum Mechanics I

Quantum Mechanics I comprises a physics students' formal introduction to quantum theory and is usually taken in the third or fourth year at the University of Maryland. While most of these students have had a passing exposure to quantum mechanical ideas in their earlier physics classes, this class usually marks their first encounter with the full mathematical machinery involved with solving Schrödinger's Equation.

Introduction to Quantum Mechanics (Griffiths, 2005) is a commonly used textbook for this class. Its first four or five chapters make up the usual course of study. They address the usual one-dimensional potentials that admit exact solutions to Schrödinger's Equation, including the infinite well, harmonic oscillator, delta function, and square well potentials. The formal linear algebra structure of quantum mechanics is discussed as well. Quantum Mechanics I usually includes a detailed treatment of simple three-dimensional problems as well, with special focus given to the hydrogen atom problem.

Quantum I involves a formidable amount of mathematics. Schrödinger's Equation is a complex differential equation containing derivatives with respect to both length and time. The sheer mechanics of solving this equation in various situations often involve infinite series solutions and special functions. Most importantly, a formal vector space structure underpins all solutions to Schrödinger's Equation. Understanding how the eigenfunction solutions to Schrödinger's Equation comprise a basis set from which any other wavefunction can be built is a key mathematical insight.

3.1.2 PHYS 402: Quantum Mechanics II

Quantum Mechanics II is the typical follow-up to the previous course. Whereas Quantum I focuses on exactly solvable problems, Quantum II is mostly concerned with problems for which a closed form solution to Schrödinger's Equation does not exist, as is the case with most authentic quantum problems. This class focuses on several common approximation methods for handling these unsolvable cases. Time-independent perturbation theory, and sometimes the time-dependent theory as well, are discussed. The variational method and WKB approximations are commonly included.

All of these approximation methods depend on the vector space technology that underlies the solutions to Schrödinger's Equation. Grasping this vector space idea, how any wavefunction can be built from eigenfunctions, is critical to students in understanding both the development and application of these approximation methods.

3.1.3 PHYS 411: Electricity and Magnetism

PHYS 411 is meant as a first upper level course on electricity and magnetism, usually taken in a student's third or fourth year. A common textbook is <u>Introduction to Electrodynamics</u> (Griffiths, 1999). A typical semester covers basic electro- and magnetostatics, using the full machinery of the relevant Maxwell Equations. Time independent fields are usually treated both in vacuum and in media. Some basic material on time dependent fields is often covered near the end of such a course.

Such a course centered on the Maxwell Equations will have a heavy mathematical emphasis on three-dimensional vector calculus. Visualization and computation with vector fields are important components of a student's mathematical understanding in this course. Volume, surface, and line integrals are common as well. A student's first upper level course in electricity and magnetism is sometimes the first time they encounter the full mathematical treatment of such integrals outside of an abstract treatment in a third semester calculus class. PHYS 411 also typically focuses on solutions to Laplace's Equation for simple situations. The eigenfunction solutions to Laplace's equation can be pieced together to form an expression for the electric potential using the same vector space mathematics as used in piecing together eigenfunctions of Schrödinger's Equation to produce an arbitrary wavefunction.

3.1.4 PHYS 374: Intermediate Theoretical Methods

A significant number of episodes for this study have also been drawn from a physics class is mathematical methods that was taught through a physics department. This math methods class is perhaps more of an intermediate level undergraduate class. It is required of sophomores or juniors and is meant to prepare them for the more complicated mathematics required in their later undergraduate physics courses. A significant number of the students in this class are not physics majors, as this class also fulfills the requirements of some applied mathematics and computer science degree programs as well. Still, this math methods course is far from the first physics course for any of the students. Even the sophomores in that class have gone through an introductory physics sequence at the university and, often, during their high school years as well.

No standard textbook is in use for this course at the University of Maryland. As a result, the topics covered tend to shift from semester to semester. During the two semesters of data collection for this dissertation, the course was taught by the same professor and hence relatively consistent from year to year. The major topics were chosen to align with the important mathematical ideas in the other upper level physics classes. Vector spaces were a primary focus, both in the abstract and in the context of coupled oscillators and waves on a string. Three-dimensional vector calculus, in the context of fluid flow, was also discussed with an eye towards its eventual application to electricity and magnetism.

3.2 Data Collection

All of the analysis in this dissertation begins with video recordings of physics students at work. Video is a rich source of student data, recording not just what is said but also its tone, volume, accompanying gestures, facial expressions, and corresponding writing (at least when the students write in some shared space, like on a blackboard). Our video data comes from several sources:

→ Students working in homework groups
→ Interviews
→ Classroom work

Video was recorded in each of these three categories, although the homework group and interview data turned out to be the most useful by far. This dissertation's analysis depends on students talking as much as possible. Evidence of their thinking comes primarily from what they say. Most classroom recordings simply contained too little student talk.

3.2.1 Homework Group Video

The majority of the episodes in this dissertation were taken from recordings of students at work in homework groups outside of the class time itself. It is common for these students to meet outside of class to work on their assignments together. A researcher would simply visit several of these upper level classes at the start of a semester and ask for volunteer groups of friends who wouldn't mind having their homework session taped. Those groups would then meet in a prearranged room with the video camera placed in the corner as discreetly as possible and proceed to work on that week's homework assignment. This data gathering method was the most practical way to capture these upper level physics students' natural work on authentic physics tasks.

A researcher, usually the author, would normally stay in the room's corner with the video camera while the students worked in their homework group. This method had a downside in that it made the video equipment somewhat more noticeable, which could make the students more self conscious and change their behavior, thinking, and speech. The effect tended to be reasonably small, though. After a quick "hello" at the start of a session, most student groups would routinely go for thirty minutes or more between explicit acknowledgements of either the researcher or the camera.

Staying in the room while recording had benefits. It allowed the researcher to pan the camera from person to person. Some homework groups had six or more students, a number too great to always fit into the camera's frame. The researcher could also zoom the camera into the blackboard to record what the students wrote. When the students would simply write on the paper in front of them, the researcher could still listen to what they were saying and jot down reasonable notes on what they must be writing. From time to time, the researcher would sacrifice his unobtrusive corner seat to actually look over the students' shoulders if he thought their writing was especially important to record.

About 80 hours of homework group video was collected using this method. Most major physics topics, including quantum mechanics, E & M, and math methods, are represented.

3.2.2 Student Interview Video

Another 25 hours of video for this study came from one-on-one interviews of a student with a researcher. Almost all of these interviewed students came from the Math Methods course described earlier. As with the homework groups, the researcher would simply take a minute during class to ask for volunteers. The interviewed students, unlike the homework group students, were paid a small amount in compensation for their time.

These interview sessions were primarily problem solving interviews where the student was asked to solve a particular problem at a blackboard while vocalizing his thoughts for the researcher. The main problems themselves were preplanned, but the interviews were not scripted at any finer level of detail. Depending on what the students brought up as their work on the problems progressed, the researcher would ask various subquestions. The goal was rarely to get the student to arrive at the correct answer of a problem during the interview. An interview would usually begin with the researcher explaining that he simply wanted to hear what the student had to say about these various problems. The researcher would note that he'd be happy to answer whatever physics questions the student still had at the end of the interview.

The researcher made an effort to keep these interviewed students talking as continuously as possible, vocalizing their ideas as they tried to resolve whatever questions came up in real time. Effective prodding was very student dependent. Some students were simply more talkative and comfortable discussing the physics. If such a student was quiet for ten seconds, a quick "What are you thinking about?" prod would typically make them start talking again. Other students were either less comfortable with the physics or merely liked to quietly form their sentences more completely before they started to talk. Constantly prodding these students after every ten silent seconds would be, at best, distracting and, at worst, belittling. Each individual interview's dynamics, then, depended a lot on the researcher's impression of that particular student's talkativeness and comfort level.

3.2.3 Classroom Video

Another 40 hours of video was recorded directly in upper level physics classrooms. Most of these 40 hours came from the Math Methods class described earlier. These recordings were made in much the same way as the homework group recordings. A researcher would sit with a camera in a corner as unobtrusively as possible, zooming from instructor to student and to the board when appropriate.

As noted above, these recordings tended to be less useful for the detailed analysis presented in this dissertation. Most upper level physics classes are dominated by the instructor's talk. These episodes usually contained too little extended student conversation to make a detailed analysis of their thinking. It is hoped that future work can take the framing analysis developed from the homework group and interview data and apply it to the instructors' speech in these classroom examples. Looking at framing mismatches between physics instructors' thinking and students' questions would likely be fertile ground for an in-classroom application of this dissertation's work.

3.3 Developing Our Analysis Tools

The previous chapter discussed the main goal of this dissertation: to analyze how upper level physics students think about the mathematics they use in their physics work using an epistemic lens. What do these students currently see as the nature of the mathematics at hand? Of all their mathematical knowledge, what part is appropriate to use right now? Closely looking at the different types of justifications these students offer for their mathematics will offer the researcher a window onto how they are framing their math use.

3.3.1 Students' Framings Are Easiest to Identify Via Contrasts and Shifts

Chapter Two details how a student's current epistemic framing corresponds to how they interpret the math at hand, leading them to consider certain subsets of their mathematical knowledge. Evidence for a student's current framing appears in the type of warrants they use in their mathematical reasoning.

Evidence for how students are framing their math use, or any other part of their activity for that matter, is easiest to pick out when there is some sort of contrast or misunderstanding present. Such contrasts make framing differences stand out. If two people are talking past each other, neither one seeming to comprehend what the other is trying to say, there is often a framing issue as the root cause. Such framing confusions are common sources of disagreements, even in non-physics settings (Tannen 1992).

Arguments are thus convenient places to look for evidence of students' framing. Many mathematical disagreements physics students have with each other reduce to the first student essentially saying "Look at this math issue this way" while the second student is claiming "No, you should be looking at it this other way". Such disagreements are fundamentally framing disagreements. The students are debating which parts of their mathematical knowledge are currently relevant. Examining the warrants (Toulmin, 1958) physics students use in their mathematical arguments offers a good window to how they are currently framing their math use.

Occasionally, at least in individual interviews, a student will explicitly signal that he is playing both sides of an argument himself. He will verbally signal that he's adopting another way of interpreting the math at hand. More often, a miscommunication or a disagreement is the only explicit marker of a frame clash or shift.

3.3.2 Data Selection Process

As described earlier, about 150 hours of raw video of upper level physics students at work were collected for this study. There needed to be some sort of selection process that could pare down this 150-hour set to a collection appropriate for a close, careful analysis.

The author was present during 95% of the tapings themselves and took detailed notes of the students' activity. These notes allowed the video databank to be quickly searched for the best debates, arguments, and misunderstandings alluded to earlier. At this early point in the pare-down process, "best" simply meant the debates and arguments that most likely had a lot of material available for possible analysis. Sometimes "best" translated to a simple clock reading. If students spent five minutes arguing about a certain point, there was a good chance a closer analysis might find a relatively large amount of speech that clearly annunciates their ideas. Other times the "best" arguments were selected for the novelty of their content. An argument about whether an expression simplifies to $x^2 - 3x + 2$ or $x^2 + 3x + 2$ is likely to be routine. The students likely won't engage very excitedly, and even if they did, they are likely to quickly agree on a useful way to resolve the argument. That is, they will likely share a common framing, which means there won't be much explicit evidence for that framing. However, an argument about a novel issue is much more likely to bring about a variety of approaches, a variety of framings. A richer student discussion for analysis would be expected around an issue like whether one can differentiate with respect to Planck's constant. (This episode appears in Chapter Five.)

This first pass through the 150-hour data set yielded about 50 snippets containing the arguments, debates, and misunderstandings most likely to be explicit and long enough to offer good evidence (i.e., clearly identifiable mathematical warrants) for how the students were framing their math use. Eventually, a framing analysis was carried out on other episodes that didn't contain such obvious arguments. Such an extension was necessary to help assure the generality of the framing analysis developed in this dissertation. It was important to check that the analysis framework developed from this 50-episode subset could be applied to more episodes of student thinking besides those containing explicit debates or arguments.

Several of the episodes quoted in this dissertation come from this later selection round and do not contain explicit disagreements, attesting to this generality.

Nonetheless, it was important to first closely analyze the 50-episode subset of explicit arguments. Are there common framings that physics students tend to use to interpret their mathematics? If so, what is the most natural number of common framings to analyze students' thinking in terms of? Would just two common framings usually be sufficient? Maybe we need a set of ten framings, ten types of mathematical warrants that tend to enter physics students' work. This 50-episode subset was meant to offer the best evidence for deciding whether a set of common framings exist and, if they do, what they specifically are. The next section describes the methodology used (referred to as knowledge analysis) to help these common framings emerge.

3.3.3 Knowledge Analysis: Common Framings Emerge from the Data Set

What was then done with these 50 episodes fits a common analysis pattern in physics education research. The basic idea was to use some sort of common thread through all the episodes to condense the fifty individual episodes according to a common analysis scheme. One doesn't aim to condense all of the information in each of the fifty episodes. Rather, one chooses a particular axis to look along (i.e. the "common thread") and focuses the analysis along that axis. Some information will of course be lost, but a good choice of axis will allow for meaningful comparisons to be made across what are superficially different episodes of student thinking. In this way, a large area of interest is projected down to a more manageable size for analysis, and a practical (but still meaningful) set of generalizations may emerge.

This dissertation fits this general pattern of taking a large area of interest to physics education, selecting a certain axis along which to analyze it, and producing a set of generalizations that relate to the large area. Specifically, this work looks at upper level physics students' use of mathematics (the large area of interest) through an epistemic lens (the selected axis). It will identify four common clusters of epistemic framing (the smaller generalizations) that can be used to parse these students' thinking.

But how to best identify these four common clusters of epistemic framing? At the start, there was no reason to expect there to be four instead of, say, ten. A methodology called "knowledge analysis" was used to make these four common epistemic framing clusters emerge from the 50-episode set.

Knowledge analysis is essentially an iterative methodology. One begins by taking a small number of individual episodes and first simply trying to describe the students' thinking each one individually. Look for commonalities across each of these episodes. Perhaps you notice themes A, B, C, D, and E that seem to come up several times each. Then select a new set of episodes, describe each individually, and see how well A, B, C, D, and E span this new space. Maybe you'll notice F and add it to your list. Maybe you'll notice that B and C are actually difficult to distinguish in the majority of these new cases, so you'll combine them. Then you see how well the new set of A, B, D, E, and F span a new selection of episodes. After several iterations, the clustering scheme should stop evolving significantly.

Specifically, this project began with identifying a small sample of about 50 episodes by looking for arguments and miscommunications, as described above. These episodes, again, were most likely to contain relatively easy-to-spot frame shifts. A subset of these 50 sample episodes were analyzed individually at first, the goal being to simply describe how that episode's students were interpreting the mathematics they were using. What type of warrants were they using in their mathematical arguments?

Once a small collection of these individual analyses were collected, it became possible to look for consistencies across episodes. Do the various framings observed cluster in any way? Are there common epistemic threads running across episodes dealing with very different mathematical topics? In looking for such consistencies, the goal was to identify general classes, not mutually exclusive categories, of student reasoning. A manifold theory of students' minds would not hope to identify anything more rigid than general classes of students' framings of math use.

Several clusters incorporating similar individual math framing examples were identified from this original small data selection. The next step was to do a similar framing analysis on a new set drawn from those 50 episodes and see if these original clusters could incorporate these new examples of students' mathematical thinking as well. Appropriate changes were made to the clusterings in light of this new data set, and then a third set of episodes were considered. After several iterations, the clustering scheme stopped evolving significantly. Eventually the whole 50 episode subset was used, with each individual episode cycled through more than once.

Chapter Four illustrates the four main clusters that emerged from this data set's examples of physics students' framing of their math use. They capture four general types of justification these students offer for their mathematics. These four clusters are labeled "Calculation", "Physical Mapping", "Invoking Authority", and "Math Consistency". Again, these clusterings are meant to be neither mutually exclusive nor sufficient to span all possibilities. They are merely presented as the most convenient way found of structuring comparisons across many different episodes in our data set. Without such an attempt at generalization, this dissertation would simply be a collection of individual case studies with limited promise for application to either other student data sets or curriculum design.

With the clustering scheme in place, it became much easier to analyze episodes where there was no obvious contrast or misunderstanding among students that clearly demarcated a frame clash. There now existed a language, a trustable system to use for attempting an epistemic framing analysis of any upper level physics student episode.

This knowledge analysis methodology has been explicitly used and described in other physics education studies as well. As stated before, this methodology is especially useful in identifying a set of small, general elements that can be used to describe students' thinking along a selected axis in some large area of research interest. Sherin's development of symbolic forms (see the extended description of symbolic forms in Chapter Two) is one such example. A knowledge analysis methodology was used to identify these small, tightly bound packages of math and physical intuition (the smaller general elements) that comprise physics students' conceptual understanding (the selected axis) of intermediate physics (the large area of interest) (Sherin, 2001). A knowledge analysis methodology has also been used to identify the core ideas (the smaller generalizations) that comprise students' data analysis (the selected axis) in introductory physics labs (the large area of interest) (Lippmann, 2003).

3.4 Chapter Summary

This chapter first described the various sources of data used in this dissertation's analysis of physics students' thinking. Video recordings of students came from several upper level physics classes including Quantum Mechanics I and II, Electricity and Magnetism, and Math Methods. Much of this dissertation's episodes came from recordings of homework group sessions held outside of formal class time. Some data comes from one-on-one interviews as well. Recordings of formal class time were made as well, although these were difficult to analyze because of the low levels of student discussion.

This chapter also concerned the more important methodological design of this dissertation's study. Chapter Four will introduce the four common framing clusters that emerged from the video data set. Later chapters will then analyze many episodes in terms of these framing clusters. How did these framing clusters initially emerge from the data set?

These clusters emerged from an iterative cycle of analysis and generalization, a methodology called knowledge analysis. Early attempts were simply aimed at describing how the students in a small selection of episodes were framing their math use. What kinds of warrants did they currently see as appropriate for their math arguments? Preliminary clusters were identified from these few examples of students' framing. This clustering scheme was modified after its attempted application to a new set of episodes. After several iterations, the framing clustering scheme stopped evolving significantly.

Chapter 4: Framing Clusters That Emerged from Upper Level Physics Students' Use of Mathematics

This section introduces the four epistemic framing clusters that emerged from this dissertation's analysis. Details of the iterative analysis process that led to these four clusters were given in Chapter Three. Examples are given illustrating how these various framings can be identified in students' work. The fundamental question that all these episodes' analysis revolves around is "What, in this moment, counts as justification to this student?"

As detailed in Chapter Two, tracking the warrants students use in their mathematics is a valuable window to how they are framing their activity (that is, how they are interpreting their math use and hence focusing on a subset of their total mathematical knowledge). A warrant, again, is the bridge that links data to a claim. Warrants are often unspoken. For example, I might state that Thomas Jefferson is the greatest American founding father (claim) because he wrote the Declaration of Independence (data). The unspoken warrant that allows this data to apply to that claim is that the Declaration of Independence is a cornerstone document in American history, laying out the nascent country's claims for autonomy.

Calculation, Physical Mapping, Invoking Authority, and Math Consistency are the clusterings that arose from this analysis of these students' mathematical warrants. They indicate common framings that emerged from this data set, common ways students interpret their mathematical work and focus on a subset of their mathematical knowledge.

In general, this section will make heavy use of definition by example. Each of these four common framings is defined only briefly before an example is given. This method should ultimately make for a richer introduction to the framing clusters. A more abstracted, stand-alone definition of each of the four framings is reserved for the end of this chapter. Readers first interested in quick examples of these four framings should return to the story about Professors Alpha through Delta in Chapter One, Section 1.1. These four fictional professors were framing their interpretation of $x_f = x_o + v_o \Delta t$ as Calculation, Physical Mapping, Invoking Authority, and Math Consistency, respectively.

4.1 The Calculation Framing

As noted earlier, students' framing of their math use is most evident when there is a shift from one framing to another. This first example makes use of such a shift to illustrate two of the common framings in this dissertation's data: Calculation and Physical Mapping. Lines 1 to 8 are an example of Calculation and will be discussed in this section.

A Calculation framing, like all the other framings that emerged from the data set, is primarily defined by the warrant students use. In a Calculation framing, students rely on computational correctness. Algorithmically following a set of established computational steps should lead to a trustable result. Again, this warrant may often be implicit. If I were deriving $x_f = x_o + v_o t + \frac{1}{2}at^2$ for you from

$\frac{d^2x}{dt^2} = a$, I would probably just carefully explain my steps to you. You would likely accept without further thought. Neither of us explicitly explained, "OK, because carefully following a set of computational steps allows one to trust a result, we should trust this derivation." It would just be an unspoken warrant, one that's shared because both of us frame the discussion as Calculation.

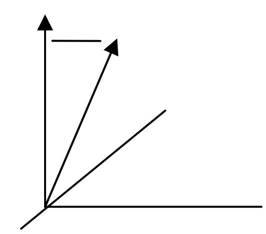
4.1.1 Student 1 Transcript

This first transcript comes from a one-on-one student interview. The student was a junior physics major enrolled in the math methods in physics class. That class had recently devoted several lessons to vector spaces, hence the question posed to the student. He was asked "Let $\vec{r} = 3\hat{x} + 4\hat{y} - 2\hat{z}$. How much $\hat{y}$ was in $\vec{r}$?" The student was standing in front of a whiteboard, writing on it as he answered the question. Focus on the mathematical justification this student offers.

1. **S1**: If you wanted to do it in Dirac, which makes
2. it a little easier, you just define r as a ket vector,
3. and if you wanted to pick out the value of, or rather
4. the scalar component of y, rather the y direction,
5. you just do it with the y, you know, I mean…it's… *writes* $\langle y | r \rangle = 4$
6. spits out 4 because you dot x-hat with y-hat… *writes* $\langle x | y \rangle$
7. or, rather, technically, y-hat with x-hat is zero, *erases* $\langle x | y \rangle$, *writes* $\langle y | x \rangle = 0$
8. y-hat with y-hat is one, y-hat with z-hat is still zero *writes* $\langle y | y \rangle = 1$, $\langle y | z \rangle = 0$

…10 seconds later…

9. **S1**: That's just mathematically, though. If you
10. wanted to do it physically, you have to explain
11. that there's a space, and your vector r, and if this *draws*
12. is x and that's z and that's y, you want to know
13. basically that component of your vector. And
14. you can do that algebraically if necessary.



4.1.2 Calculation: Correctly Followed Algorithms Give a Result Validity

Student 1 uses a calculation scheme in lines 1 to 8 to justify his answer of "four". This framing of the math in play focuses on a common type of mathematical justification. Performing a series of algorithmically correct manipulations should allow one to trust the end result.

S1's speech displays several markers of this calculation framing. He is very concerned with technical correctness, for example. In lines 3 and 4, he corrects his more colloquial choice of words "the value of [y]" with the more technical "rather, the scalar component of y." S1 is very careful with the details of the inner product calculation itself as well. He is using a common physics notation for this operation, Dirac notation, as he mentions in line 1. Briefly, this notational system pictures the y unit vector, $\langle y|$, hitting the vector in question, $|r\rangle = 3|x\rangle + 4|y\rangle - 2|z\rangle$, from the left. The standard visualization is something like seeing $\langle y| \rightarrow |r\rangle$ to distribute among the parts of $|r\rangle$ and give $3\langle y|x\rangle + 4\langle y|y\rangle - 2\langle y|z\rangle$. S1 mentions $\langle x|y\rangle$ in line 6, but corrects himself immediately thereafter in line 7 to $\langle y|x\rangle$ because this ordering of the unit vectors more accurately reflects how $\langle y|$ is brought in from the left. The difference is purely cosmetic in this case, but S1 may not have been consciously aware of this fact. He was, however, consciously concerned with keeping this technical detail straight in his work.

S1 displays another common indicator of his Calculation framing in this bit of math work. His mathematics makes heavy use of symbols but proceeds with very little explicit mention of what these symbols represent physically. The unit vector symbols $|x\rangle$, $|y\rangle$, and $|z\rangle$ figure prominently in lines 1 to 8, but the only explicit reference to what they represent is a brief mention of "direction" in line 4. S1 does his calculation and gets the answer "four". Any physical interpretation that S1 tacks onto that "four" is left implicit.

Saying that S1 is framing his math use in lines 1 to 8 as Calculation is not meant to imply a value judgment, either naivety or sophistication. It merely implies that he is focusing on a manageable subset of his total available math knowledge. Epistemically, this framing choice treats math knowledge as carefully ordered and procedural. Algorithmically following a prescribed set of steps, as in lines 5 to 8, will lead to a trustable result.

This use of mathematics is completely valid and common even in expert physicists' work. Like any of the four framing clusters, however, it can appear naïve if a Calculation framing is either inappropriate for the situation at hand or if it is misapplied, as in the next example.

4.1.3 A Second Calculation Example

This example comes from a student enrolled in an algebra-based Introductory Mechanics course for biological science majors. The student is working with a teaching assistant on a homework problem asking how much time it would take a car traveling 95 feet per second to go 500 feet. This example is taken from a previously published paper (Bing and Redish, 2006) and was collected for an earlier project (Tuminaro, 2004).

1. **S2**: So, I was trying to do a proportion,
2. but that doesn't work. I was like 95 feet
3. per second, oh wait, yeah in 500 feet, like,
4. x would be the time—that doesn't, I get like
5. this huge number and that doesn't make any sense.

Having constructed the equation 95/1 = x/500, the student has arrived at x = 47,500 seconds.

The student's work is organized almost entirely by her attention to a computational scheme. There are words like "time" and "feet" that could sometimes be used to refer to specific physical ideas, but here they function merely as labels on S2's numbers.

Her thinking in lines 1 to 4 is dominated by mathematical formalism. In line 1, S2 states "I was trying to do a proportion." She takes a template, □/□ = □/□, fills in the slots, and solves for the unknown. Her attention is on an implicit Calculation warrant. If she follows this □/□ = □/□ template and cross-multiplies in the usual way, she should get a trustable answer. The result of 47,500 was hence generated almost entirely by calculational means. Yes, she labels her answer of 47,500 as a time and concludes such an answer is unreasonable on physical grounds, but such considerations were post hoc justifications. She justified her in-the-moment production of that answer in lines 1 to 4 almost solely with respect to how her actions aligned with a familiar mathematical algorithm, in this case the algebraic proportion template.

4.2 The Physical Mapping Framing

We now turn to several examples of a Physical Mapping framing. When physics students frame their math use as Physical Mapping, they support their arguments by pointing to the quality of fit between their mathematics and the physical situation at hand. Their warrant concerns how the math we use in physics is only valid insofar as it accurately models the physical world. For example, suppose I was to explain why the expression for the force exerted by a spring includes a negative: $\vec{F} = -k\vec{x}$. I might explain how stretching a spring makes it pull backwards as it tries to contract back to its natural length. If you compress the spring, it'll push back against the compression as it tries to expand. In both cases the spring force is opposite the way the spring is deformed. That is, if $\vec{x}$ is positive (say you extend the spring to the right) then the spring pulls you in the negative (i.e. left) direction. If $\vec{x}$ is negative (say you compress the spring leftwards) the spring exerts a force to the right (positive) direction. Again, I didn't necessarily have to spell out my math-should-model-the-world warrant. It comes along with a Physical Mapping framing.

4.2.1 Physical Mapping: Mathematics in Physics Should Accurately Reflect the Physical World

We now re-quote S1's transcript on the vector projection problem. He was asked to suppose $\vec{r} = 3\hat{x} + 4\hat{y} - 2\hat{z}$ and figure out how much $\hat{y}$ was in $\vec{r}$. He explicitly signals a reframing towards Physical Mapping in lines 9 and 10.

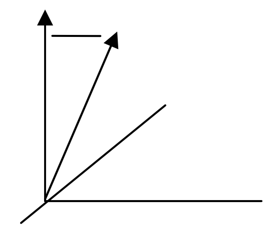
1. **S1**: If you wanted to do it in Dirac, which makes
2. it a little easier, you just define r as a ket vector,
3. and if you wanted to pick out the value of, or rather
4. the scalar component of y, rather the y direction,
5. you just do it with the y, you know, I mean…it's… *writes* $\langle y | r \rangle = 4$
6. spits out 4 because you dot x-hat with y-hat… *writes* $\langle x | y \rangle$

7. or, rather, technically, y-hat with x-hat is zero, *erases* $\langle x | y \rangle$, *writes* $\langle y | x \rangle = 0$

8. y-hat with y-hat is one, y-hat with z-hat is still zero *writes* $\langle y | y \rangle = 1$, $\langle y | z \rangle = 0$

…10 seconds later…

9. **S1**: That's just mathematically, though. If you wanted to do it physically, you have to explain that there's a space, and your vector r, and if this is x and that's z and that's y, you want to know basically that component of your vector. And you can do that algebraically if necessary. *draws*



S1 explicitly indicates that he is going to adopt a different, at least to him at this particular moment, interpretation of the mathematics in play in lines 9 and 10. "That's just mathematically, though. If you wanted to do it physically…" These explicit reframing markers are relatively rare, but this particular one certainly helps illustrate the shift in S1's mathematical thinking. He shifts to framing his math use as "Physical Mapping".

This reframing activates different mathematical resources within S1. He is now concerned much more directly with the physical situation to which this problem refers. S1 draws a diagram, a common indicator of a Physical Mapping framing. A diagram can often act as a mediator between the mathematics at hand and the relevant physical situation. He proceeds to identify the relevant mathematical entities in the problem on his picture, including the x, y, and z directions, and the vector $\vec{r}$ itself, in lines 11 and 12. He even notes what the problem's answer corresponds to in lines 12 and 13 when he draws a little horizontal line from his $\vec{r}$ towards his y-axis and indicates "that component of your vector".

The computational resources that were so prevalent in the Calculation framing of lines 1 to 8 are not nearly as prominent in Physical Mapping of lines 9 to 14. S1 just quickly notes in line 14 that "you can do that algebraically if necessary" in reference to how you could actually find a numeric answer. These calculational details are presently so deemphasized that it's not even clear if he is referring to a trigonometric calculation or an inner product calculation like he did earlier.

One cannot imply, at least based on this lone episode, whether Calculation or Physical Mapping is any better than the other. Each of these framings highlights a certain facet of mathematical knowledge. Both have their strong and weak points depending on the given situation. Calculation can give a neatly packaged result but can lack modeling power. Physical Mapping can trigger physical intuitions but can lack quantitative power.

4.2.2 A Second Physical Mapping Example

This example comes from a group of three physics majors enrolled in PHYS 374, Mathematical Methods. They are working on a homework problem where an object is thrown straight up and falls back down under the forces of gravity and air resistance. In the following excerpts, they are working to understand the expression for the viscous force, $F_V = -bv$, as given in the homework problem. Their conversation includes

1. **S3**: Because the negative means that friction *gestures* ← →
2. operates in the opposite direction of whatever
3. v is

and

4. **S4**: We need to leave this negative in so that it
5. cancels out that one
6. **S3**: Right, because v has that negative built into it,
7. and so we need another negative out here to *points to –bv on board*
8. make sure that the two negatives cancel out and
9. you end up with a positive, which is up, which is
10. the direction of friction because it's going *gestures ↑ and ↓*
11. down—it's falling down, being dragged in the
12. upward direction

and

13. **S4**: Well, but let's do the first one first, if you're
14. going down … what is v gonna be? Is it gonna
15. be negative or positive? *points to –bv on board*
16. **S5**: It's gonna be negative.
17. **S4**: OK, so a negative times a negative
18. **S5**: Is gonna be positive
19. **S4**: Right, and a positive points up. *gestures ↑*

All of these excerpts come from the same two-minute clip. The students are explaining how the mathematical expression for the drag force, $F_V = -bv$, aligns with their physical expectations. Implicit in their conversation is the warrant typical of a Physical Mapping framing: if this math aligns with the physical world, we can trust it.

The students exhibit a bi-directionality in their speech as they examine this math/physics alignment. Sometimes they start with a mathematical statement, "the negative" and translate it to a physical statement, "means that friction operates in the opposite direction…" as in lines 1-3. In lines 13 – 15, a physical statement, "if you're going down" is translated into a mathematical idea, "is [v] gonna be negative or positive?" Examining the role of positive and negative signs in this clip also indicates a Physical Mapping framing. In the second and third excerpts these signs behave according to algebraic rules, but at the same time they encode physical information on the direction of the drag force that is carried along and expressed through these algebraic manipulations.

The students are thus framing their activity as Physical Mapping. Justification for their interpretation of $F_V = -bv$ comes from both a mathematical realm, i.e. the negative signs obey the standard algebraic rule for canceling out, and the physical realm, i.e. if the object is moving down the viscous force must act upwards.

Another piece of evidence for their Physical Mapping framing comes from the abundance of gesture in their work. Gesture is an important modality of communication that can couple closely to how an individual is currently thinking (Goldin-Meadow, 2003). Physics students have been observed to rely heavily on gesture to help articulate their ideas (Scherr, 2008). The students in the example above point to the "*-bv*" inscription on the board and quickly proceed to turn away from the board and gesture up and down with their hands. While the student in the vector projection example relied on a diagram as an intermediary for his Physical Mapping, these students employ gesture to help them establish a connection to the physical world.

The next example of Physical Mapping continues S2's transcript and provides one more example of a Physical Mapping Framing, again in contrast to Calculation.

4.2.3 S2's Shift from Calculation to Physical Mapping

We quickly re-quote S2's previous work as she determines it takes a car 47,500 seconds to go 500 feet if its velocity is 95 feet per second.

1. **S2**: So, I was trying to do a proportion,
2. but that doesn't work. I was like 95 feet
3. per second, oh wait, yeah in 500 feet, like,
4. x would be the time—that doesn't, I get like
5. this huge number and that doesn't make any sense.

Compare the in-the-moment calculation in lines 1 to 4, which was discussed earlier, to what S2 immediately proceeds to do with the teaching assistant's help:

6. **TA**: That doesn't make any sense. So what if I said
7. something like if you're traveling two feet per
8. second and you go four feet. How long would
9. that take you?
10. **S5**: two seconds
11. **TA**: Or, if you tried different numbers, if I was
12. traveling eight feet per second and you traveled
13. sixteen feet, how long would that take you?
14. **S5**: two seconds

Both times S2 answers the TA quickly and correctly without explicit, formal calculation. She was almost certainly not setting up a proportion like 2/1 = 4/x and solving it algorithmically in her head. The simple numbers the TA gives her allows her to answer based on a much more conceptual notion of velocity. If you're going two feet per second, then of course it takes two such seconds to go a total of four feet. Lines 6 to 14 have S2, with the TA's assistance, framing her math use as Physical Mapping. She thinks about the physical situation, being willing to let that justify her mathematics.

S2's case allows a more general point about distinguishing a Calculation framing from a Physical Mapping framing. At some level, all mathematics is ultimately grounded in physical experience. A child learns to associate "1" with a single object, "2" with a collection of two objects, and so on. Higher and higher mathematics are built up by analogy and extension of what are ultimately physically grounded ideas (Lakoff and Nunez, 2000). The distinction between a Calculation framing and a Physical Mapping framing largely concerns a person's in-the-moment awareness of the physical referents of her math. Yes, lines 1 to 4 have S2 working with numbers that technically have relative physical sizes. She even tosses in a few physical labels like "time" and "feet", but her focus is on algorithmically following a computational template: □/□ = □/□. Contrast that Calculation framing with lines 6 to 14, where she is very much more aware of her math's physical referents. Indeed, she is basing her new answers on her reasoning about these physical distances and times.

4.3 The Invoking Authority Framing

This section turns to a different framing observed in upper level physics students' mathematical thought. Some mathematical results are simply used without explicit justification. Quoting a rule or a previously packaged result is often a very appropriate action. Practically speaking, one will not necessarily start a problem from absolute first principals every time.

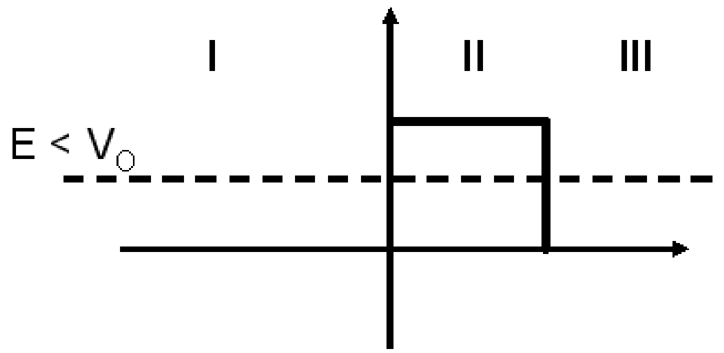
For example, suppose I was trying to convince you what the rotational inertia of a solid sphere was. I might simply pick up an introductory physics book, thumb through the index until I found "rotational inertia", turn to page 253, and point at an entry in a table that says "solid sphere, $I = \frac{2}{5}MR^2$". Perhaps you would accept my argument, also accepting the implicit warrant that underlies my reasoning: information that comes from an authoritative source can be trusted.

An Invoking Authority framing is often closely tied to finding the right level of detail to go into during a problem. It is unreasonable to take every single problem down to absolute first principles every time. Some results will simply be taken for granted. Perhaps you would be more likely to accept my earlier argument for the rotational inertia of a solid sphere if we were engaged with a larger problem like finding the time it would take such a sphere to roll down a given ramp. You might judge the specific value of the sphere's rotational inertia to be sufficiently irrelevant to the problem's main purpose to permit me to quote from the textbook.

4.3.1 Invoking Authority: Remembering a Rule or a Result Can Count As Sufficient Justification

The following example comes from the work of a senior undergraduate physics major in a problem solving interview. He was enrolled in a first-semester graduate quantum mechanics course and agreed to meet with a researcher and work on one of that week's homework problems at a blackboard while the researcher observed and occasionally asked questions.

The student is working on the one-dimensional square-barrier problem, trying to calculate the reflection and transmission coefficients for an incident particle of $E < V_O$. He has sketched the situation.



1. **S6**: So the first thing I wanna do is, um *looks at notebook*
2. write out what the wave equation should
3. look like for each zone here. Um, so for
4. zone I where for x less than 0 we have, I'll just
5. label it psi-I of x and it's gonna be *writes* $\psi_1(x)$*…looks at notebook*
6. an oscillatory wave function, so it's gonna have
7. e to the ikx components and e to the
8. negative ikx components with different
9. coefficients in front of each and this is where *writes* $Ae^{ikx} + Be^{-ikx}$
10. [k] equals square root of *copies* $k = \sqrt{2mE} / \hbar^2$ *from notebook*
11. two m E over h-bar squared.
12. **Interviewer**: Where does all that come from?
13. **S6**: That just came outta my notes. *pauses for 5 seconds, looks at notebook*
14. And the reasoning behind this, *pauses for 3 seconds*
15. I'm not sure if, I'm sure we derived this
16. at some time, at some point. This term, *looks through notebook*
17. but as far as, yesterday in class;
18. we're used to this kind of a form.

S6 is trying to find the relevant wavefunctions for his task. There are several ways to get these wavefunctions, but S6's current framing of his activity causes him to try a search-and-copy approach. His comment about "writing out" the wavefunctions in line 2 attests to how he expects this activity to be about quotation. Compare "writing out" with the implications of a different word choice like "derive" or "produce". He then proceeds to constantly refer to his notebook as he copies out both the wavefunction and the expression for the wavenumber *k*.

When the interviewer asks S6 about how to justify these results, he quickly responds "that just came outta my notes" in line 13. The leaden pause that follows indicates to S6 that the interviewer is likely probing for a different explanation. S6 continues "and the reasoning behind this…" as he vaguely refers to a past derivation of some sort in lines 14 to 17. Line 18 has S6 again falling back on his justification by means of quotation: "We're just used to this kind of a form".

4.3.2 Further Ways to Identify an Invoking Authority Framing

The above episode is an example of the Invoking Authority mathematical framing. This framing has activated a subset of S6's mathematical resources that cause him to focus on treating mathematics as a body of facts to be recalled and used as needed.

As with the other common epistemic framings discussed in this dissertation, Invoking Authority is evidenced through the kind of in-the-moment justification the student calls upon. In this framing, justifications rely on confidently quoting a source of some kind. S6 uses his class notes as the authority figure in this particular episode. If the wavefunction expression appears in his class notes, then he trusts it and uses it in the current homework problem. Other common authority figures observed in students' work include textbooks and the internet. Sometimes, as in the example below, one of the students casts himself as the authority figure as he quotes a result or a rule without any further justification.

Another common trait of the Invoking Authority framing is the absence of extended chains of mathematical reasoning. "Chaining" has been closely tied to students' mechanistic reasoning (Russ, 2006; Russ, Scherr, Hammer, and Mikeska, 2008). When a student links together a series of implications she is chaining. An example would be "adding another resistor in series puts another obstacle in the current's way, so the total resistance goes up, but the battery's push remains the same, so the current flowing decreases". Students engage in more mathematical chaining arguments as well. The Calculation framing often cues reasoning like "A = BC, but we don't know C, but we can use C = EF to get C, then we can use C to get A". The electric current example just above could be a nice example of chaining while in a Physical Mapping framing if the student was simultaneously thinking about the formula $\Delta V = IR$. Chaining is usually absent or severely limited if the student is framing his math use as Invoking Authority. S6 is not considering how to chain together a derivation of the wavefunctions he is quoting. In fact, the Schrödinger Equation from which these wavefunctions are derived is never even mentioned until fifteen minutes later in the episode.

S6 comes across as somewhat unsophisticated in this episode snippet. Of all the general clusters of framings in this dissertation, Invoking Authority can look especially naïve, especially if it occurs in an inappropriate context. It is important to note, however, that Invoking Authority has an important place in expert physicists work. A physicist who turns to Gradshteyn and Ryzhik to simply look up an integral's solution (Gradshteyn, Ryzhik, and Jeffrey, 2000) is likely Invoking Authority, for example.

4.3.3 A Second Invoking Authority Example

Invoking Authority need not always include a student looking up and copying a result. Sometimes it can be as simple as a student stating a rule without offering any further justification or support.

For example, consider the following example from two students in an undergraduate Quantum Mechanics II course. They are working together on one of their homework problems outside of class. This episode is analyzed in detail in Chapter Five, so only a small snippet is given here. The students realize that in the course of their homework problem they will need to perform a partial derivative with respect to a physical constant, $\frac{\partial}{\partial \hbar}$. They move into a discussion of how to interpret a derivative with respect to Planck's constant. In the course of trying to interpret such an operation, one student states:

1. **S7**: You can always take a derivative
2. with respect to anything.

Such a statement appeals to authority, in this case that of the student himself, for its justification. In adopting an Invoking Authority framing, the student is suggesting that the conversation on how to interpret $\frac{\partial}{\partial \hbar}$ be suspended and that they continue along on the authority of the general math rule he has just quoted. Much like the rotational inertia example at the start of this section, we see a close tie of Invoking Authority with the question of what level of detail to open up. These students' homework problem only requires them to perform this partial derivative on the way towards answering another question. S7 is Invoking Authority as he argues to set aside the question of actually interpreting $\frac{\partial}{\partial \hbar}$.

4.4 The Math Consistency Framing

The fourth common clustering that emerged from this study's episodes is the Math Consistency framing. Mathematics has its own internal coherence. The same mathematical structure can underlie two superficially different situations. Citing consistency with a more familiar use of whatever mathematical idea is in play can be valuable source of justification.

Suppose you were trying to explain Coulomb's Law for the electric force, $\vec{F}_E = \frac{1}{4\pi\varepsilon_o}\frac{q_1 q_2}{r^2}\hat{r}$, to a student. You might remind him of the expression for the gravitational force, $\vec{F}_G = -\frac{Gm_1 m_2}{r^2}\hat{r}$, and demonstrate how ideas from this more familiar bit of math map to Coulomb's Law. Both forces depend on the relative strengths (mass or charge) of the two objects in question. Both forces fall of with respect to distance in the same way, and both include a proportionality constant (G or $\frac{1}{4\pi\varepsilon_o}$) that must be experimentally measured. Even disanalogous observations can be illuminating. Gravity is always attractive, hence the negative sign in explicitly included in front of the always positive masses. An electric force can be attractive or repulsive, so the implicit signs on the positive or negative charges, q_1 and q_2, will determine the direction of the Coulomb force.

Implicit in your discussion with the student would be the warrant indicative of a Math Consistency framing. We expect that mathematics has a regularity to it, a consistency of application across different situations. Establishing a common underlying mathematical structure allows one to trust the relevant set of relations and inferences.

Analogies to these more familiar math examples are common indicators of a Math Consistency framing. The first two episodes in this section are examples of such analogies. These analogies can grow into relatively stable categorizations, as this section's third episode illustrates.

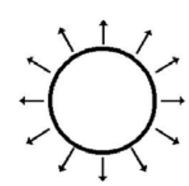
4.4.1 Math Consistency: Use of a Math Idea Should Dovetail with Similar Math Ideas

This example comes from a problem solving interview with a nontraditional student. He had obtained an engineering degree several years prior and had spent significant time in the workforce before returning to the university for another science degree. The student had enrolled in the undergraduate Math Methods course for physics majors and attended the first several lectures before discontinuing. He felt he was already familiar with the material, having completed an engineering degree several years earlier. At the end of the semester, he took the Math Methods course's final exam in an effort to place out of the course requirement. This interview has the student reworking one of these exam problems. The problem at hand reads:

> In class, we derived the integral constraint that expressed the conservation of matter of a fluid: $-\frac{d}{dt}\int_{\tau} \rho d\tau = \int_{\partial\tau} (\rho\vec{v})\cdot d\vec{A}$. Suppose that ρ describes the concentration in a solvent of a chemical compound that could be created or destroyed by chemical reactions. Suppose also that the rate of creation (or destruction) of the compound per unit volume as a function of position at the point $\vec{r}$ at a time *t* is given by $Q(\vec{r},t)$. *Q* is defined to be positive when the compound is being created, negative when it is being destroyed. How would the equation above have to be modified? Explain.

The student is discussing the right hand side of the matter conservation equation when he says:

1. **S8**: So that's equal to the amount of flux *draws*



2. through the area, because when something
3. escapes out of a volume, you can always
4. tell how much has escaped by drawing
5. an area around that volume and *gestures to and around picture*
6. measuring how much is leaving that area. *pauses to look at interviewer*
7. **Interviewer**: OK
8. **S8**: So this is like a flux, and it's a similar *writes "flux" up by equation*
9. principle to Gauss's Law, I think, E&M.

There is a shift in the middle of this snippet, centered around the interviewer's brief interjection in line 7. S8 offers epistemically different types of justification for the matter conservation equation in lines 1 to 6 and in lines 8 to 9. The first part of this snippet is an example of Physical Mapping. S8 has just previously identified $\int_{\partial\tau}$ as an integral over the surface area of a volume. He follows that up with a physical idea. You can treat the surface area of a volume as a boundary and know exactly how much stuff has left the volume by measuring how much crosses that boundary (lines 3-6). Since the math expression at hand matches that physical idea, S8 is arguing for the validity of the matter conservation equation. S8 also draws a picture and gestures around it to help him make his point to the interviewer. As mentioned in an earlier section, diagrams and prolific gestures tend to be markers of a Physical Mapping framing.

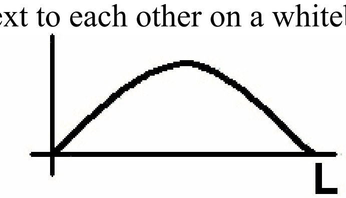
S8 pauses at the end of line 6 and looks away from the whiteboard towards the interviewer. This pause suggests S8 has come to the end, as he sees it, of the explanation at hand. After the interviewer's noncommittal "OK" in line 7, S8 offers another piece of support for the matter conservation equation. "It's a similar principle to Gauss's Law, I think, E&M" (lines 8-9). In contrast to lines 1 to 6, this current justification of the matter conservation equation does not rely directly on a math-to-physics mapping. It is meant as an analogy to a similar occurrence of a surface integral. By calling upon Gauss's Law from electricity and magnetism, S8 is supporting the matter conservation equation with some combination of "this idea has occurred before, so we should trust it" and "import what you recall of Gauss's Law and use that to help you understand this equation". Which flavor of justification S8 is meaning to favor in lines 8 to 9 is both indeterminant and irrelevant. What is relevant is that S8 is making some form of Math Consistency argument as opposed to the Physical Mapping argument in lines 1 to 6.

4.4.2 A Second (Also Analogy-Like) Math Consistency Example

The student in the last example just made a quick analogy to Gauss's Law, a bit of math that was structurally similar to the issue at hand. This next example presents a case of a more extended, detailed analogy. In framing his activity as Math Consistency, this student, like the last one, will support his arguments by citing his work's alignment with a more familiar instance of the math structure in play.

This student is a sophomore physics major enrolled in the Math Methods course. The following transcript comes from an interview that focused on students' understanding of vector spaces, a key component of that semester's Math Methods class. Two problems were written next to each other on a whiteboard.

$$\vec{r} = 3\hat{x} + 4\hat{y} - 2\hat{z}$$



$f(x) = x(L - x)$

How much of $\hat{y}$ is in $\vec{r}$?

How much of $\sqrt{\frac{2}{L}}\sin(\frac{2\pi x}{L})$ is in f(x)?

Figure 1

To a physicist, these questions are structurally identical. Both are examples of vector spaces. The vector $\vec{r}$ is made up of a combination of the basis vectors $\hat{x}$, $\hat{y}$, and $\hat{z}$ in the same way that the function *f(x)=x(x-L)* is made up of a combination of the basis functions $\sqrt{\frac{2}{L}}\sin\left(\frac{n\pi x}{L}\right)$, where *n* is any positive integer. Each question can be answered with an inner product operation.

The student has done such an inner product for the Cartesian problem and is now working on the function problem:

1. **S9**: Since umm, f is, f is kind of like
2. your vector here in this Cartesian coordinate
3. space and in this space of wave on a string, *points to parabola picture*
4. your vector, or your—kind of the way the
5. wave is in this particular moment, is defined *points to f(x) = x(x-L)*
6. as this. Then to pick out how much of this
7. basis vector or normal mode is inside *points to* $\sqrt{\frac{2}{L}}\sin\left(\frac{2\pi x}{L}\right)$
8. this particular wave vector at a time,
9. or function at a time, at this particular moment
10. in time would then be this inner product right here, *points to previously written*

$$\langle e_1 \mid f(x)\rangle = \int_0^L \sqrt{\frac{2}{L}}\sin\left(\frac{2\pi x}{L}\right)x(L-x)dx$$

11. which can be expressed as an integral.

S9 offers justification for the inner product integral, $\int_0^L \sqrt{\frac{2}{L}}\sin\left(\frac{2\pi x}{L}\right)x(L-x)dx$, that he has written. He frames his task as one of Math Consistency, and hence reasons his inner product integral is sound based on its structural similarity to the more familiar Cartesian vector space example. The transcript above is an extended analogy to the Cartesian case. Lines 1 and 2 have S9 pointing out the parallel between $\vec{r}$ and *f(x)*. "*f* is kind of like your vector here." He proceeds to draw a general parallel between the two spaces: "this Cartesian coordinate space and this space of wave on a string" (lines 2-3). S9 lines up the two vector space's bases next. He starts to explain, "Then to pick out how much of this basis vector or normal mode is inside this particular wave vector at a time" in lines 6 to 8. Again, there is evidence that S9 is especially concerned with lining up the Cartesian and function space examples in a consistent way. The "basis vector or normal mode" phrasing in line 7 explicitly juxtaposes his particular words of choice, which he uses throughout the interview, for the basis in the Cartesian and function spaces. Line 8 has an even more striking merging of the two spaces when he mentions a "wave vector".

S9 is thus framing his activity as Math Consistency. He is presently focused on drawing a clear parallel between the familiar Cartesian example and the more exotic function example. The justification of his answer lies within the quality of his analogy.

4.4.3 A Third (Categorization-Like) Math Consistency Example

The two examples above illustrate how analogy can be a powerful indicator of a Math Consistency framing. Often these analogies are quite careful and explicit in physics students' work. If students are thinking about relatively new and unfamiliar ideas, then such careful explicitness is not surprising.

At a certain level, analogies are instances of categorization (Atkins, 2004). Stating that A is like B logically requires that A and B must both be examples of some larger category. Students, however, are sometimes only fleetingly aware or concerned with such larger categories in their real-time thinking with analogies. The student in last example with the Cartesian and function spaces was not overly concerned with talking about the nature of generic vector spaces in general. He was presently concerned with lining up the Cartesian-to-function analogy. Contrast that student with this next one who is also answering the same problem.

1. **S10**: So then you do, you identify the function f
2. with the abstract vector, with the abstract ket f again,
3. and then you do the various different inner products
4. in exactly the same way I did those.

S10's thought resembles a categorization much more closely than the last student's. Whereas S9 explicitly aligned the Cartesian and function examples, S10's speech centers more around a classification idea. Once he asserts that *f(x)* can be considered one of these general vectors, "you identify the function *f* with the abstract vector, with the abstract ket *f* again" (lines 1-2), hence the relevant properties flow from there. "Then you can do the various different inner products in exactly the same way I did those" (lines 3-4).

Because S10 still mentions the relevance of the Cartesian example in line 4, he can still be said to be framing his activity as Math Consistency. One can imagine that as S10 becomes even more familiar with the general idea of a vector space, he might eventually approach such problems by simply asserting "Oh, that's a vector space" and proceed to quote the relevant properties. Such statements would be evidence of an Invoking Authority framing. Strong familiarity with a given concept can lead to explicit analogies being shorthanded via Invoking Authority. That such a bleeding over between Math Consistency and Invoking Authority is possible is not surprising. As discussed in Chapter Three, the labels "Math Consistency" and "Invoking Authority" refer to broad clusterings of observed framings, not mutually exclusive categories.

4.5 A More Complete Definition of the Four Framing Clusters

The previous sections have given several examples of using the mathematical warrants offered by students to identify various framings of their math use. This section now summarizes the four framing clusters.

4.5.1 *Table of Common Framing Indicators*

While the type of justification students offer is taken as the main indicator of their epistemic framing, other pieces of framing evidence exist. The behaviors in the table below were found to occur disproportionately in one of the common framing clusterings.

	Calculation	***Physical Mapping***	***Invoking Authority***	***Math Consistency***
Justification Offered	Correctly following algorithmic steps gives trustable result.	Goodness of fit between math and physical observations or expectations attests to result.	Authoritatively asserting a result or a rule gives it credence.	Similarity or logical connection to another math idea offers validation.
Other Common Indicators	→ focus on technical correctness → math chaining—need this to get that	→ often aided by diagram → more demonstrative gesturing	→ quoting a rule → absence of mechanistic chaining → little acknowledge-ment of substructure	→ analogy to another math idea → categorization

Table 1

Most of the common indicators in this table were illustrated in the examples given earlier in this chapter. Later, more detailed case study episodes will make use of them as well. Again, “Calculation”, “Physical Mapping”, “Invoking Authority”, and “Math Consistency” are meant only as broad groupings of epistemic framings observed in this study’s students. Since they are not rigid classes, one would not expect these “common indicators” to be associated only with one type of framing. Students can certainly gesture, for example, in Math Consistency. S9 points towards pictures and equations he has drawn on the whiteboard. The point of listing these “common indicators” was to merely note that certain behaviors were not observed to be uniformly distributed across the various framing clusters. Hence, these indicators can provide supporting evidence of students’ epistemic framing, in addition to the primary concern of what presently counts as valid proof to the student.

4.5.2 *Varying Timescales for These Framings*

Framing is a dynamic cognitive process. A person’s mind certainly makes an initial judgment regarding the nature of the situation at hand, but that judgment is continually updated and reevaluated. New information comes to the student all the time, whether in the form of a classmate’s comment, an interviewer’s interjection, or simply turning to a different page in a textbook. This new information can lead a student to reframe her activity.

As a result, the epistemic framings observed in these students' work can last for a range of times. S6 stayed in an Invoking Authority framing for about ten minutes as he tried to track down and state the various wavefunctions that were relevant to his tunneling problem. In contrast, S7's Invoking Authority framing was much quicker. He simply stated "You can always take a derivative with respect to anything" in reference to $\frac{\partial}{\partial \hbar}$ before his fellow student made a sufficiently jarring comment that lead to a reframing.

4.5.3 Why an Epistemic Games Analysis Is Not the Best Fit for This Data

This study is not the first to take a manifold view of physics students' thinking and attempt to parse their problem solving. While the relation of many of these studies to this dissertation was discussed in Chapter Two, one significant strand of work was left out. This strand, epistemic games (Collins and Ferguson, 1993), perhaps seems on the surface to be the most natural fit to this dissertation's goals. Indeed, this study began as a study on epistemic games before evolving to its current form. Now that there are several transcript examples in play, we give reasons as to why this dissertation's analysis has not been cast in terms of epistemic games.

Epistemic games are sets of behaviors observed in students' work that are associated with various subsets of their knowledge (Tuminaro, 2004; Tuminaro and Redish, 2007). They are seen to develop over an extended period of a students' education, slowly congealing as a student activates similar conceptual and epistemic resources together again and again in similar situations. Recursive plug-and-chug is a prototypical epistemic game in these studies.

As an epistemic game, recursive plug-and-chug has a certain coherency to it. Any physics teacher reading this paragraph probably immediately knows what it entails. An epistemic game is formally defined by two things: an epistemic form and a knowledge base. The epistemic form is the particular bit of knowledge or information the student is trying to produce, such as an answer of the form $v_f = 5.7$ m/s for the recursive plug-and-chug game. The knowledge base is the group of resources that have become tightly associated with each other and the relevant epistemic form. For the recursive plug-and-chug game, the knowledge base includes such things as algebraic manipulation algorithms, equation indexing strategies, and so on.

An epistemic game encapsulates a goal (the epistemic form), a particular subset of the student's available knowledge (the knowledge base), rules and judgments about how that knowledge base is allowed to be used, and entry and exit rules that tell when the game is to be played and when its goal has been achieved. Repetition and experience make an epistemic game like plug-and-chug into a local attractor for students' thought. Their thought can evolve towards this attractor and even get stuck in it for an extended time, to the exclusion of other problem solving approaches. Phenomenologically, the students in this dissertation sometimes display a similar penchant for getting stuck in a certain mode of thought, missing apparently obvious alternatives. Why, then, has this dissertation not adopted an explicit language of epistemic games for its analysis of students' thinking?

There are two main reasons why this study does not use an epistemic game language. The first centers around the implicit nature of the work of many of this dissertation's upper level students. Tuminaro's epistemic game studies were done exclusively with introductory physics students. These introductory students tend to work more slowly and talk with their classmates more explicitly about the details of their work. Upper level students tend to have more automated subroutines. Mathematical manipulations, graph interpretations, and so on occur both more rapidly and more silently. These upper level students may well be playing similar games to those in Tuminaro's introductory student study, but there is less explicit evidence of them.

Epistemic games are called "games" for a particular reason. Certain "moves" are allowed within each epistemic game. These moves often occur in a repeatable way in the introductory students' work Tuminaro studied. Recursive plug-and-chug, indeed all the epistemic games identified, can be diagramed in a flowchart. Its moves include identifying the target quantity, finding an equation relating that target quantity to other quantities, determining which of the other quantities is known, and (if some of those other quantities are unknown) choosing a sub-target quantity and starting over. The upper level physics students in this study often leave too many of their thoughts and actions implicit to allow a detailed description of their "moves" to be made directly from a transcript. Witness S8's "it's a similar process to Gauss's Law" and S7's "you can always take a derivative with respect to anything". Each statement essentially shorthands a potentially long chain of reasoning. Any move-by-move description of what thoughts went into these statements would involve heavy inferences.

The second, more important, reason for not analyzing these upper level students' thinking in terms of epistemic games centers on epistemic forms. An epistemic form, again, is the specific piece of knowledge, the goal, that a given epistemic game aims to produce. The epistemic form, along with the knowledge base, defines the epistemic game. Focusing on what a student is trying to produce unnecessarily narrows one's description of her work. The epistemic framing analysis in this dissertation focuses on a broader, more vital epistemic issue. What do these students see as the source of the knowledge in play, and how does this stance change throughout a problem solving episode?

This dissertation argues that the type of warrant a student currently sees as sufficient is a valuable parameter to focus upon to describe what they currently see as the source of their knowledge (i.e. how they are epistemically framing their work). The various case studies presented in this dissertation will demonstrate how such a focus provides both a natural way to parse students' thinking and a natural way to study the dynamics of what they consider to be the source of their math and physics knowledge. Focusing on the epistemic forms, or the present goals of these upper level students, presents too mercurial (and often too implicit) a target and also makes many episodes of thought appear more different than they are with regards to students' stances towards the source of their knowledge, as the next paragraph demonstrates.

S6 had a goal of finding the proper wavefunctions to use from a previous in-class example. S7 had a goal of convincing his fellow student that it was permissible to take a derivative with respect to $\hbar$. These goals, in and of themselves, are quite different. Both students, however, are framing their work as Invoking Authority. They are both doing a type of thing familiar to physicists, but that familiarity comes from what the students see as the source of their knowledge. They are searching for an authority to quote and are structuring their arguments accordingly. An epistemic framing analysis captures this "source of knowledge" aspect of students' thinking more explicitly than an epistemic game analysis.

S1, in the second part of his transcript, had a local goal of describing what the y-axis projection of the given vector looked like. S3, S4, and S5 were trying to understand why there was an explicit minus sign in the air drag force, $F_v = -b\vec{v}$. Both arguments, however, were examples of Physical Mapping. Again, their work resonates with physicists not so much because of the (different) epistemic forms they were trying to produce but rather because of the familiar source of knowledge they employ. In these two cases, they are justifying their math by reference to the physical world.

Thus, this study's analysis is not couched within an epistemic game language. The upper level students studied here leave too much implicit and automated in their work, and their local goals can vary too widely. This study does not claim that an epistemic game lens was inappropriate for the introductory students studied by Tuminaro. The epistemic framing analysis presented here that centers on the type of justifications students offer for their mathematics is meant as an extension and generalization of Tuminaro's epistemic games that is especially appropriate for upper level physics students. Tuminaro's epistemic games are seen as particularly familiar and codified examples of the framing clusterings in this study. Recursive plug-and-chug is a familiar, but by no means the only, example of a Calculation framing. Tuminaro's transliteration to math game is a common example of an Invoking Authority framing.

4.6 Inter-Rater Reliability of This Epistemic Framing Analysis Tool

This chapter was meant to describe the details of identifying various epistemic frames in upper level physics students' use of mathematics. Several examples of each common cluster of framings were presented and common indicators, in addition to the central "what counts as justification" criterion, were described.

Science, of course, should be a reproducible enterprise. The value of this dissertation's epistemic framing analysis depends in part on how readily other researchers can apply it consistently. All of this dissertation's episodes themselves are a first indication of the inter-rater reliability of it epistemic framing analysis scheme. Each episode was first selected and analyzed by the author before being shared with other physics education researchers in regular meetings. Only after a consensus on the students' thinking evolved from these meetings was an episode fully incorporated into this study. Thus, any episode's existence in this dissertation indicates a basic level of inter-rater agreement on its content.

A somewhat more powerful statement of inter-rater reliability can be made if this chapter's methodology discussion could be given to another researcher, and that researcher could then parse a new transcript for epistemic framing in a way consistent with the author. Such a test was done.

For this test, a new transcript was selected. A copy is provided in the Appendix. The author and two other researchers independently did a framing analysis. Each tagged evidence for any of the four common framing clusters. After this individual work, the three researchers (A, B, and C) combined their analyses into a single table for comparison. Each colored box in the data tables below represents one piece of evidence, according to that specific researcher, for a given framing. Since the trial transcript had three pages, there are three pre-comparison data tables. The three researchers then met to discuss their analyses, often coming to consensus where they had initially disagreed. There are thus three post-comparison data tables as well.

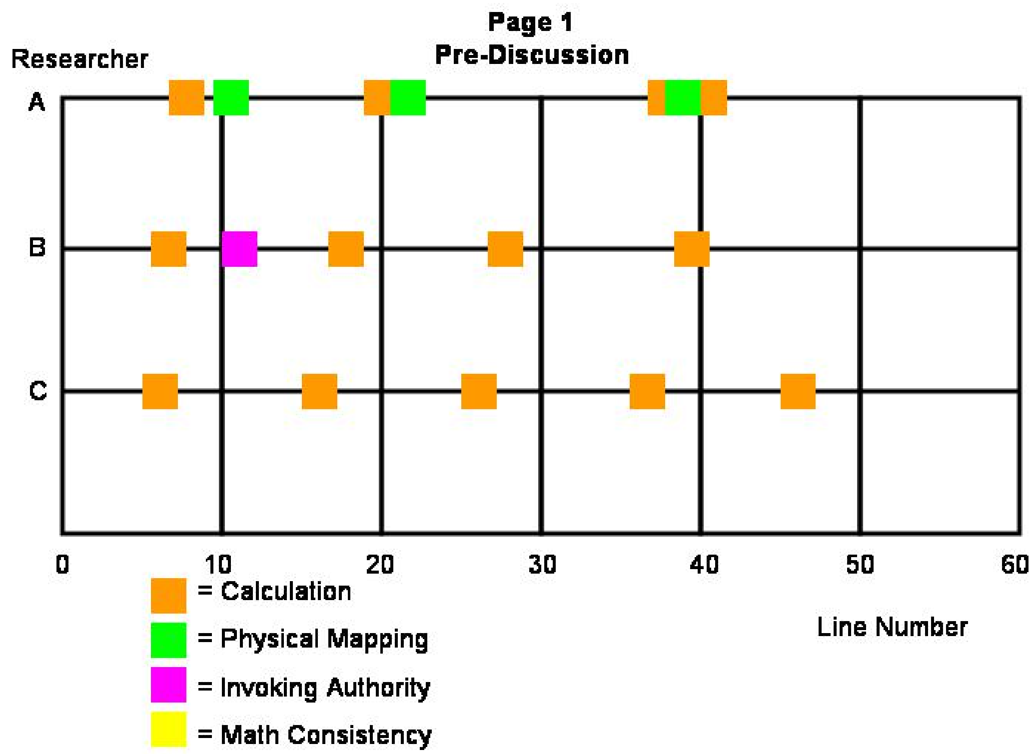
The reader is encouraged to examine the following inter-rater data tables. First, notice that different researchers have different internal cut-offs for what counts as “one piece of evidence” for a framing. Say a student made three consecutive statements, with the first and third being about some calculation algorithm while the second was a shorter, more indeterminant statement. Does that count as two small “Calculation” tags or just one? As shown in the data tables below, different researchers will have different thresholds.

One crude way to address this threshold problem is to simply look at what framing made up the majority of each researcher’s codes on each page. Maybe one researcher flagged on five pieces of evidence whereas another flagged ten on that page, but perhaps each gave most of his codes to a common framing. The first observation, then, is that all three researchers agreed on the dominant framing in each page of transcript, even pre-discussion. Pages 1 and 2 had more Calculation tags than any other type according to all three researchers. Page 3 was seen by all as primarily a Physical Mapping example.

But one can be more precise while still accounting for this evidence threshold issue. Three comparison pairs were made: compare researcher A to B, B to C, and C to A. These comparisons were made for each page, pre- and post-discussion. For each color code that appeared in the first researcher’s work, an “agree” tally was awarded if a similar color code appeared in the other researcher’s work (plus or minus two lines away). In an effort to account for the evidence threshold issue described earlier, only one instance of each color code was counted per ten lines of transcript.

As an example, consider the first page, pre-discussion comparison of researcher A to B. A’s Calculation code at line 8 scores an “agree”. His Physical Mapping code at line 10 scores a “disagree”. His Calculation code at line 20 (it’s greater than ten lines away from the one at line 8) also gets an “agree”, while the Physical Mapping at line 22 gets a “disagree”, as does his Physical mapping code at line 40. His Calculation tag at line 38 gets an “agree”, but his Calculation tag at line 41 is disregarded under the evidence threshold condition (it’s less than ten lines away from the similar code at line 38).

Such an analysis gives a pre-discussion inter-rater reliability of 70% when done for all the data. This figure improves to 80% for the post-discussion codes of the researchers. The pre- and post-discussion data tables for the three pages of inter-rater test transcript are now given.


Page 1
Pre-Discussion
Researcher
A
B
C
0
10
20
30
40
50
60
= Calculation
= Physical Mapping
= Invoking Authority
= Math Consistency
Line Number

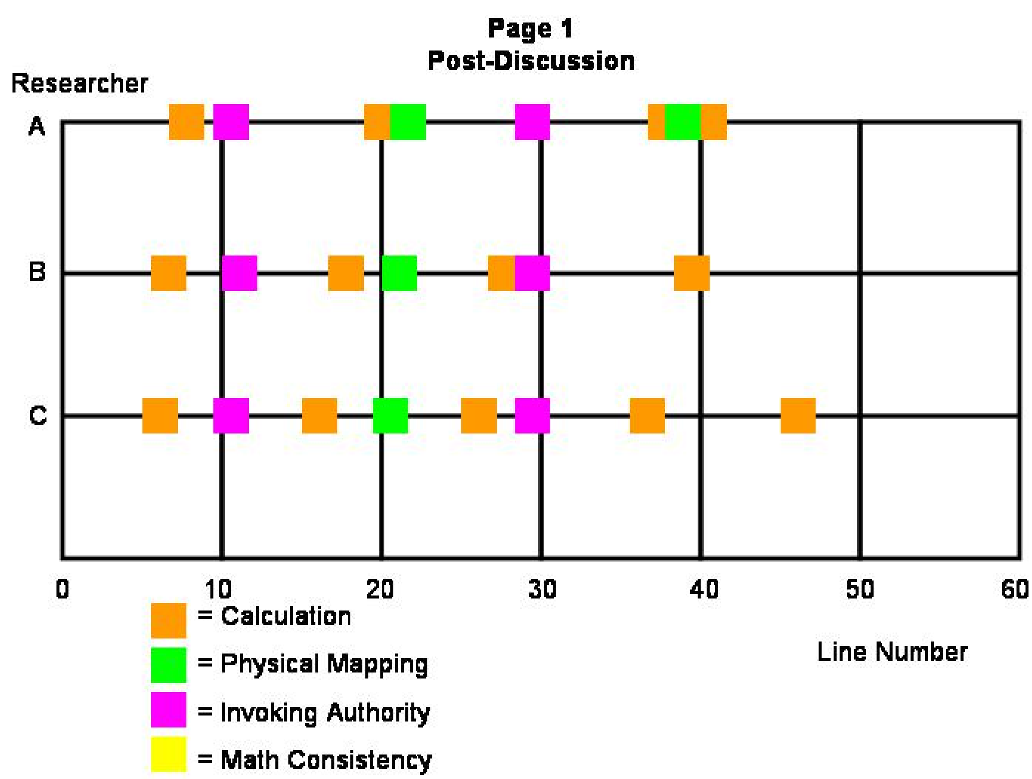

Page 1
Post-Discussion
Researcher
A
B
C
0
10
20
30
40
50
60
= Calculation
= Physical Mapping
= Invoking Authority
= Math Consistency
Line Number

Figure 2

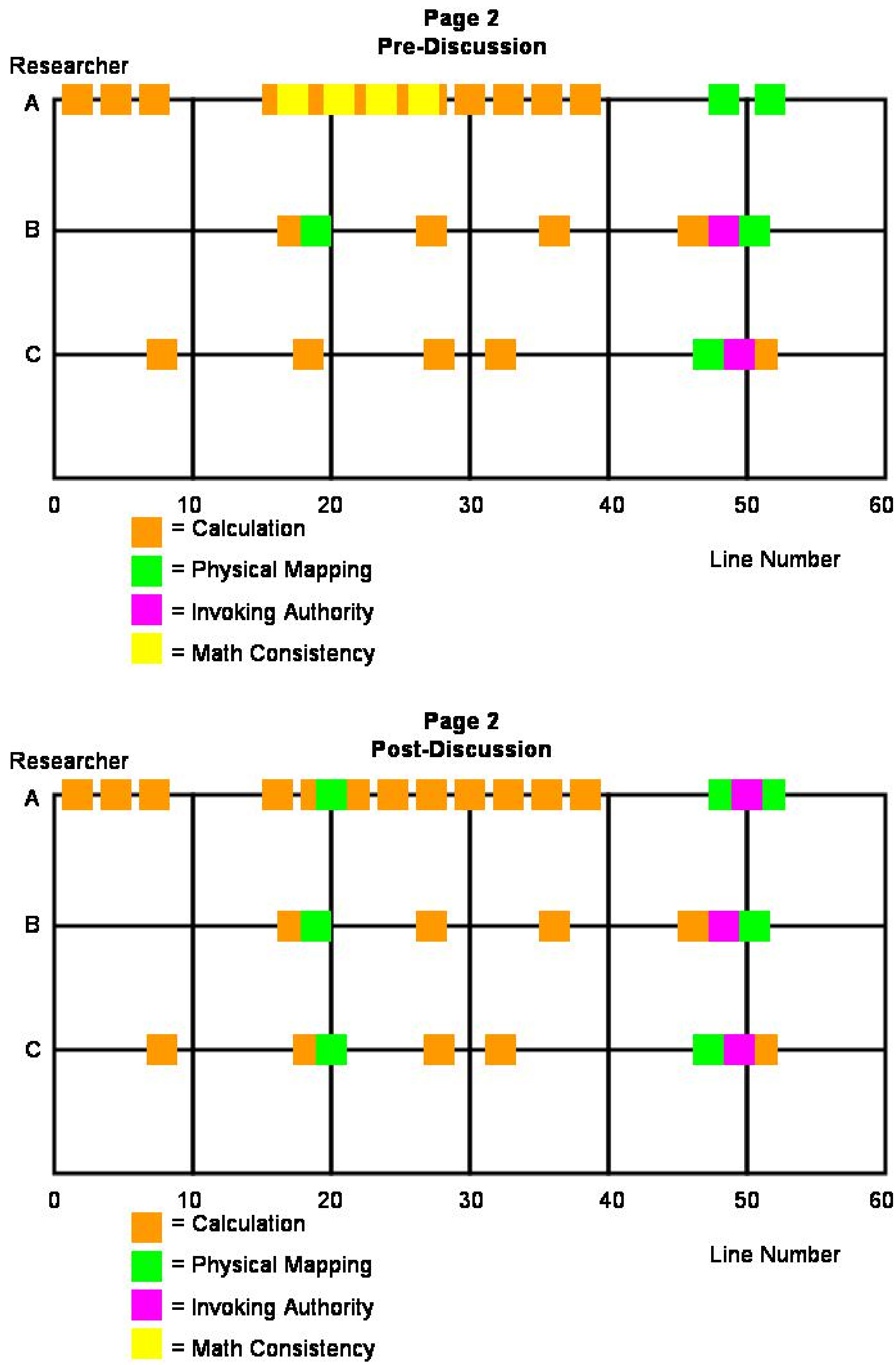
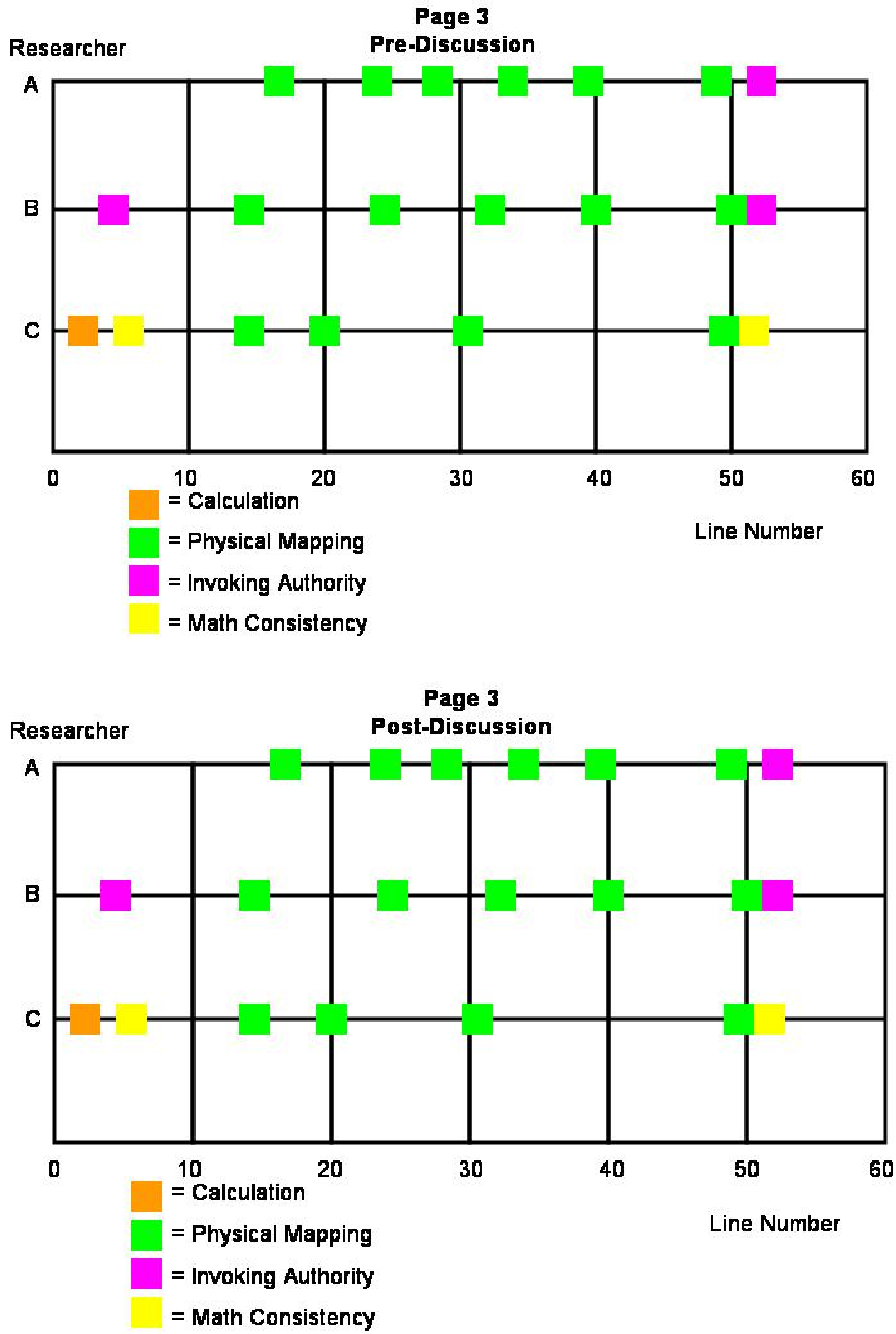

Page 2
Pre-Discussion
Researcher
A
B
C
0
10
20
30
40
50
60
= Calculation
= Physical Mapping
= Invoking Authority
= Math Consistency
Line Number
Page 2
Post-Discussion
Researcher
A
B
C
0
10
20
30
40
50
60
= Calculation
= Physical Mapping
= Invoking Authority
= Math Consistency
Line Number

Figure 3



Page 3
Pre-Discussion
Researcher
A
B
C
0
10
20
30
40
50
60
= Calculation
= Physical Mapping
= Invoking Authority
= Math Consistency
Line Number
Page 3
Post-Discussion
Researcher
A
B
C
0
10
20
30
40
50
60
= Calculation
= Physical Mapping
= Invoking Authority
= Math Consistency
Line Number

Figure 4

There was certainly disagreement in some places, even after discussion, but those disagreements were very often over rather isolated statements that didn't match the dominant framing surrounding them. For example, look at line 52 on the post-discussion chart for page 3. Researchers A and B thought they saw evidence of an Invoking Authority framing while C saw Math Consistency. A large, agreed-upon swath of Physical Mapping stretched from line 15 to 51. Line 52 merely contained a statement that all researchers agreed didn't fit the preceding Physical Mapping framing but had insufficient supporting statements around it to reliably classify. Line 5 on page 3 and line 40 on page 1 are similar cases of post-discussion disagreement on little statements that signal some momentary departure from the surrounding framing. Such departures are completely expected. Human thought is not strictly compartmentalized. The framing process, in general, does not imply a binary switch that makes the students either, for example, do all calculation or none at all.

The only disagreement on a significant swath of students' work was in the pre-discussion analysis of page 2. Researcher A identified a Math Consistency framing from one student's speech in lines 15 to 30. The subsequent discussion with the other researchers brought out the reason for this discrepancy. All three coders had noticed the same thread in the student's thinking, but A had mistakenly thought that thread fell into what had been defined as a Math Consistency framing. It more fit what had been defined as Calculation, as the post-discussion chart indicates.

A final note is in order on the coverage of a given episode that should be expected from Calculation, Physical Mapping, Invoking Authority, and Math Consistency. For clarity's sake, most of the episodes presented in this dissertation fall relatively cleanly into one of these common epistemic framings. A randomly selected transcript of upper level physics students' work would likely be less than half made up of clean examples. Human cognition is a fuzzy process, and these four named epistemic framings were only meant to represent general clusters of similar framings.

Thus, the epistemic coding scheme presented here should not be expected to yield a clean coding of most of a random transcript. This inter-rater reliability test transcript is no different. Students' thinking is simply not found to be that cleanly compartmentalized. Of all the data analyzed for this dissertation, perhaps less than 50% can be cleanly coded under one of these general clusterings. Lines 25 to 45 on page 2 are an example of a fairly "clean" Calculation framing. Lines 10 to 50 on page three are pretty obviously Physical Mapping. Still, those are the only clean examples, accounting for 60 of the 150 lines in these three pages. That ratio is typical.

However, Calculation, Physical Mapping, Invoking Authority, and Math Consistency do a reasonable job of spanning the space of these students' mathematical arguments. 90% of a random episode or more can be seen as made up of behavior indicative of those four landmark framings. Hybrids are very common. Perhaps a student quotes a few computational rules as he performs a long calculation. Maybe a student makes an analogy to both a similar physical situation and a similar math structure. Page 1 of the inter-rater test is a heavily hybridized example of a Calculation framing. So is the last third of page 2.

There are thus two important issues highlighted by this inter-rater reliability test. First is the issue of how well researchers can agree on evidence of the various framings. That was done to 70% agreement pre-discussion in this test and 80% post-discussion. Second is the regular occurrence of hybrid framings. In this test, only 60 of the 150 lines were clear, extended examples of one framing (page 2, lines 25 to 45 and page 3, lines 10 to 50). That ratio is typical of this dissertation's data set, although many of the detailed examples in this document were specially chosen for their clarity.

4.7 Chapter Summary

Chapter Four details the four clusters of framing that emerged from this dissertation's study of physics students' use of mathematics: Calculation, Physical Mapping, Invoking Authority, and Math Consistency. All are identified by looking at the warrants a student is presently treating as appropriate for his mathematical arguments. Calculation focuses on how a result obtained by algorithmically correct computation should be valid. Physical Mapping cites the goodness-of-fit between the mathematics in use and the physical situation at hand. Invoking Authority relies on the confident assertion of rules and previous results. Math Consistency focuses on how a mathematical result should mesh with analogous mathematical ideas.

Other behaviors tend to correlate with these framing clusters. Physical Mapping, for example, is often accompanied by the use of a diagram or an increase in gesturing. These other common indicators of the various framings, in addition to the fundamental mathematical warrants, were summarized.

These framing clusters were never intended to divide 100% of a random transcript of an episode of students' work into clear-cut, exclusive swaths. The physics students in this study simply do not display such absolute compartmentalization in their thinking. All the examples given in this chapter were specifically chosen for their clarity, but less than 50% of a given transcript usually falls cleanly under one or another of these clusters. Hybrids are common.

These four clusters do, however, form a fairly complete set of elements for describing upper level physics students' framing of their math use. While hybrid examples are common, most hybrid examples can be seen as molecular combinations of Calculation, Physical Mapping, Invoking Authority, and Math Consistency. Chapter Five is the first chapter containing extended case study episodes, where we will see several examples of these framing hybrids. Sometimes the students are explicitly looking for a coherent answer from, for example, both a Calculation and a Physical Mapping argument. Other times students are taking more of a brainstorming approach, jumbling several lines of thought quickly together as they try to find an approach to examine more closely.

Chapter 5: Clash of Framings

The previous chapter details this dissertation's method for analyzing physics students' epistemic framing of their math use. We focus on the type of justification a student is currently treating as sufficient for his mathematical claims. Tracking the warrants students use in their mathematical arguments across a wide data set led us to propose to four common epistemic framings for the justification of math claims in physics: Calculation, Physical Mapping, Invoking Authority, and Math Consistency. Chapter Four offered several quick examples of all four of these common framing clusters along with other conversational and behavioral indicators that tend to accompany each of them.

The present chapter now turns the discussion to two extended case studies. That the last chapter could pick out quick isolated examples of various epistemic framings is encouraging. The true test, however, comes from applying this analysis framework to long, uninterrupted episodes of upper level physics students' thinking. This chapter uses the epistemic framing lens to analyze two extended episodes.

Each of these extended episodes is meant to demonstrate the significant role students' framing of their math use can play in a conversation. The principle dynamic in each of these students' conversations concerns how to interpret the math at hand. A significant amount of these students' energy goes into trying to establish the epistemic framing they see as appropriate and communicate this framing to their peers.

Their thinking is dynamic. Different mathematical resources are activated and deactivated as they frame and reframe their activity. Sometimes framing differences have very marked effects. The students in the case studies below sometimes talk past each other, neither one seeming to hear what the other is saying, because they are framing their work differently. Sometimes a student's framing can exhibit considerable resistance to change, as in this chapter's first case study. The second case study shows students being more flexible in their framing.

5.1 Work Is Independent of Path: Framings Can Have Inertia

The first case study comes from a group of three students enrolled in the physics department's Math Methods class (PHYS 374). One is a junior (S2) and the other two are sophomores (S1 and S3). These three students met regularly outside of class to work on their homework together, and this episode was taped during one such homework session.

5.1.1 The Question

Our episode starts in the middle of their work on one of that week's homework problems. The problem reads:

A rocket (mass m) is taken from a point A near an asteroid (mass M) to another point B. We will consider two (unrealistic) paths as shown in the figure. Calculate the work done by the asteroid on the rocket along each path. Use the full form of Newton's Universal Law of Gravitation (not the flat earth approximation "mg"). Calculate the work done by using the fundamental definition of work: $W_{A\to B} = \int_A^B \vec{F}\cdot d\vec{r}$.

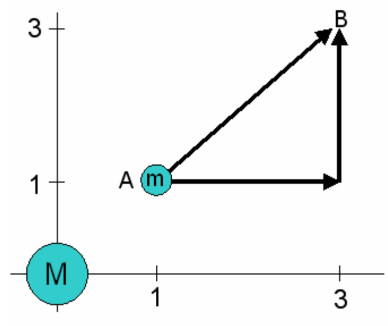


Figure 5

Briefly, a correct solution depends on keeping close track of the angle between the force of gravity, $\vec{F} = \frac{GMm}{r^2}\hat{r}$, and the displacement, $d\vec{r}$, of the small mass m. The force of gravity always points to M at the origin. Finding the work done along the radial path from A to B is relatively easy because there is always a 180º angle between $\vec{F}$ and $d\vec{r}$. The dot product's cosine term (recall $\vec{A}\cdot\vec{B} = |\vec{A}||\vec{B}|\cos\theta$) just gives a negative sign. Thus, $W_{A\to B}^{radial} = \int_A^B -\frac{GMm}{r^2}dr = -\int_{\sqrt{2}}^{3\sqrt{2}} \frac{GMm}{r^2}dr$. The limits of integration are $\sqrt{2}$ and $3\sqrt{2}$ because A is $\sqrt{2}$ from the origin and B is $3\sqrt{2}$ from the origin.

The work done along the two-part path must be found in two separate integrals. Now θ, the angle between $\vec{F}$ and $d\vec{r}$, will be changing as one moves along the path. Hence: $W_{A\to B}^{2-path} = \int_1^3 \frac{GMm}{r^2}\cos\theta dx + \int_1^3 \frac{GMm}{r^2}\cos\theta dy$

$$= \int_1^3 \frac{GMm}{x^2+1}\cos\theta dx + \int_1^3 \frac{GMm}{y^2+9}\cos\theta dy$$

where θ is now a function of your current position in x and y. Expressing $\cos\theta$ explicitly in terms of x and y leads to a final form for the integrals:

$$W_{A\to B}^{2-path} = -GMm\int_1^3 \frac{x}{\left(x^2+1\right)^{3/2}}dx - GMm\int_1^3 \frac{y}{\left(y^2+9\right)^{3/2}}dy$$

Gravity is a conservative force. The work done along each path should be the same, and explicit integration demonstrates this result. Thus (dropping the GMm):

$$-\int_{\sqrt{2}}^{3\sqrt{2}} \frac{1}{r^2}dr = -\int_1^3 \frac{x}{\left(x^2+1\right)^{3/2}} - \int_1^3 \frac{y}{\left(y^2+9\right)^{3/2}}dy$$

$$r^{-1}\Big|_{\sqrt{2}}^{3\sqrt{2}} = \frac{1}{\left(1+x^2\right)^{1/2}}\Bigg|_1^3 + \frac{1}{\left(9+y^2\right)^{1/2}}\Bigg|_1^3$$

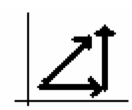
$$-.4714 = -.4714$$

5.1.2 The First Framing Clash

During this episode, the students are trying to decide if the work done should be the same along the two paths from A to B. They had previously suppressed the *G*, *m*, and *M* constants and written the equation $\int_{\sqrt{2}}^{3\sqrt{2}} \frac{1}{r^2}dr = \int_{1}^{3} \frac{1}{y^2+9}dy + \int_{1}^{3} \frac{1}{x^2+1}dx$ on the blackboard to express the work done along the direct and two-part paths, respectively. They have also copied the diagram of the situation, which was given in the previous paragraph, from the problem statement.

The students are standing at the blackboard where all the relevant equations and diagrams appear. Again, focus on the type of justification each student offers for his math arguments:

1. **S1**: what's the problem?
2. You should get a different answer — *Points to each path*
3. from here for this. — *on diagram.*



4. **S2**: No no no
5. **S1**: They should be equal?
6. **S2**: They should be equal
7. **S1**: Why should they be equal?
8. This path is longer if you think about it. — *Points to two-part path*
9. **S2**: Because force, err, because
10. work is path independent.
11. **S1**: This path is longer, so it should have, — *Points to two-part path again*
12. this number should be bigger than
13. **S2**: Work is path independent. If you
14. go from point A to point B,
15. doesn't matter how you get there,
16. it should take the same amount of work.

Lines 1 to 6 introduce the crux of the episode. S1 thinks there should be different amounts of work done on the small mass along the two different paths. S2 believes the work done should be the same.

S1 offers a justification for his claim in lines 7 and 8 when he challenges S2's same-work assertion. "This path is longer if you think about it." The mathematical definition of work, $W_{A\to B} = \int_{A}^{B} \vec{F} \cdot d\vec{r}$, is essentially "force times distance". Since the two-part path from A to B is physically longer than the direct route, it seems to follow that more work is done along the longer path.

In the language of formal argumentation theory (Toulmin, 1958), S1 is "claiming" that more work is done along the two-part path, and he offers the "data" that the two part path is longer. There is an unspoken "warrant" that connects his data to his claim: mathematics should align with the physical systems under study in a physics class. The goodness-of-fit between the math at hand and the physical system attests to the validity of one's conclusions. The work formula seems to say "force times distance" to S1. The two-part path has more "distance", and S1 thus draws justification for his answer.

The warrants used by students offer the primary evidence for their epistemic framing of their math use, as explained in Chapter Two and illustrated in Chapter Four. S1's warrant thus suggests he is framing his activity as Physical Mapping. This characterization is supported by his use of a diagram in lines 1 to 3 and 7 to 8. He gestures to the different paths as he points out that the two-part one is physically longer. Use of a diagram as intermediary between the physical situation and the mathematics is a commonly observed indicator of a Physical Mapping framing, as illustrated in Chapter Four.

S2 not only has a different answer than S1, but he is also framing his use of mathematics in a different way. S2 claims that the work done on the small mass should be the same along the two paths "because work is path independent" (lines 9 and 10). His data is the familiar mantra "work is path independent" (though he omits mentioning how this statement is only valid for conservative forces like gravity). The unspoken warrant that S2 is relying on concerns the common use of rules and definitions in math and physics: sometimes previous results are simply taken as givens for speed and convenience. S2 is framing their math use as Invoking Authority.

After hearing S2's counterargument, S1 repeats himself. In lines 11 and 12, he restates his longer-path justification and again points to the relevant features of the diagram they had previously drawn on the board. S2 responds by restating "work is path independent" in line 13 and again, in a slightly different way, in lines 14 to 16.

The most important observation in this first clip is that S1 and S2 are disagreeing over much more than the answer itself. Explicitly, they are debating whether or not more work is done along the longer path. Implicitly, they are arguing over the most useful way to frame their present use of mathematics. S1 never explicitly says "Please respond to my claim in a way that maps our math to some detail of the physical situation I may have overlooked". His phrasing and gesturing in his initial argument (lines 7 and 8) implies this framing request, though.

When S2 responds with his rule citation, he is not merely arguing for a different answer. He is pushing for a different type of warrant for judging the validity of a given answer. S2's Invoking Authority framing may have even prevented him from really hearing what S1 was saying. S1's framing request may have passed by S2 unnoticed because he was too caught up in the subset of all his math resources that his Invoking Authority framing had activated within his mind. At any rate, S2 responds in lines 9 and 10 with a different type of justification than what S1 was expecting.

When S1 repeats himself in lines 11 and 12, he is implicitly repeating his bid for a Physical Mapping framing. One can imagine a situation when S2's Invoking Authority justification would simply be accepted without incident, but here it did not align with S1's present framing. S2 does not respond to this reframing request and repeats his answer as he remains in Invoking Authority

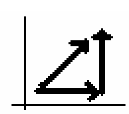
There is thus an intense framing argument going on under the surface of this debate. Sensing that he is not making any headway in the framing battle, S1 now moves to shift both himself and S2 into a third framing.

5.1.3 A Temporary Agreement on a Third Framing

S1 now makes a move toward a third way of addressing the mathematics at hand. S2 accepts for a time.

17. **S1**: OK, that's assuming Pythagorean
18. Theorem and everything else add[s].
19. Well, OK, well is this— what was the
20. answer to this right here? *Points to*

$$\int_{\sqrt{2}}^{3\sqrt{2}} \frac{1}{r^2}dr = \int_1^3 \frac{1}{y^2+9}dy + \int_1^3 \frac{1}{x^2+1}dx$$

21. What was that answer?
22. **S2**: Yeah, solve each integral numerically.
23. **S1**: Yeah, what was that answer?
24. **S3**: Each individual one?
25. **S1**: Yeah, what was
26. **S3**: OK, let me, uhh *S3 starts typing into Mathematica*
27. **S1**: Cause path two is longer than path one, so
28. **S2**: May I, for a minute? *S2 writes on a small corner of the blackboard, but never speaks about what he writes*
29. **S1**: and path one was this.
30. **S2**: Gimme this, I wanna think about something.
31. **S1**: Just add those up, tell me the number for this *Points to integrals again*
32. and I'll compare it to the number of
33. **S3**: OK, the y-one is point one five.
34. **S1**: I, just give me the, just sum those up.
35. I just want the whole total.
36. I just want this total quantity there,
37. just the total answer. *Points to integrals again*
38. **S2**: Oh, it was point four-
39. **S3**: No, that's the other one [direct path].
40. **S1**: you gave it to me before, I just didn't write it down.
41. **S3**: Oh I see, point, what, point six one eight
42. **S1**: See, point six one eight, which is what I said,
43. the work done here should be larger
44. than the work done here 'cause the path *Points to*  *diagram*
45. **S2**: No, no no, no no no
46. **S3**: the path where the x is changing
47. **S2**: Work is path independent.
48. **S1**: How is it path independent?
49. **S2**: by definition
50. **S3**: Somebody apparently proved this before we did.

S1 moves to reframe the discussion in lines 19 to 21. He points to the integrals they've written and asks, "Well, OK…what was the answer to this right here? What was that answer?" He is calling for someone to evaluate each of their expressions for the work so that he can compare the numeric results. This argument relies on another kind of warrant. Mathematics provides one with a standardized, self-consistent set of manipulations. Performing a calculation, or having a computer do it for you, according to these rules will give a valid, trustable result. S1 is moving to reframe their math use as Calculation.

Even though S1 doesn't explicitly detail the new warrant he is proposing, S2 is quick to tune into it. He immediately responds, "yeah, solve each integral numerically" (line 22). Compare this successful epistemic frame negotiation with the struggle of the previous snippet. Lines 1 to 16 had S1 pushing for Physical Mapping while S2 lobbied for Invoking Authority. Both stuck to their positions, resulting in an inefficient conversation. Neither was accepting what the other was trying to say. Lines 19 to 22 have S1 and S2 agreeing, for the moment, on what type of mathematical justification should count.

The Calculation framing negotiated, lines 23 to 41 are mostly about S1 directing S3 to input the proper expressions into Mathematica, a common software calculator package. They finish with Mathematica in line 41. It turns out that the radial path integral, $\int_{\sqrt{2}}^{3\sqrt{2}} \frac{1}{r^2} dr$, is equal to .47 while the two-part path integrals, $\int_1^3 \frac{1}{y^2+9} dy + \int_1^3 \frac{1}{x^2+1} dx$, evaluate to .618. S2 was correct back in lines 1 to 16. The same amount of work should be done along the two paths. While the radial integral is correct as written (within a negative sign), they have neglected the cosine term from the dot product $\vec{F} \cdot d\vec{r}$ in the two-part path integrals.

S1 takes the result of their Calculation argument to support his earlier Physical Mapping framing. "See, point six one eight, which is what I said, the work done here should be larger than the work done here 'cause the path…" (lines 42 to 44). This move is quite impressive. Chapter Seven will make the case that an important part of expertise in physics consists of fluidly combining different epistemic framings, looking for consistency among them. Here, S1 is using his Calculation framing as a subroutine, of sorts. He is nesting his computation within a larger scheme of supporting his Physical Mapping argument of longer-path-means-more-work.

S1 gives another hint that the Physical Mapping framing has not completely decayed while they are calculating. In the midst of the Mathematica work, he tosses in "cause path two is longer than path one" (line 27). This example illustrates the "hybrid" point made in the inter-rater reliability section of Chapter Four. Physics students' thinking is simply not always compartmentalized. Besides, Calculation, Physical Mapping, Invoking Authority, and Math Consistency are only intended to represent general clusters of reasoning. That S1 tosses in a still-active piece from his previous Physical Mapping into the Calculation is neither an anomaly of thought nor a failure of this dissertation's framework. A mark of expertise in physics (as Chapter Seven illustrates) is a fluid movement among framings. Indeed, this problem was set up to encourage students to look for coherency among various framings like S1 is doing here.

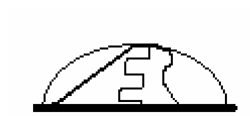
Chapter Four claimed that less than 50% of a random episode of student thinking could be cleanly coded as an elemental form of Calculation, Physical Mapping, Invoking Authority, or Math Consistency. Still, it claimed that about 90% of a transcript could be seen as a molecular combination of overlapping bits of Calculation, Physical Mapping, Invoking Authority, and Math Consistency. Lines 19 to 41 are an example that is mostly Calculation but is fuzzed somewhat by Math Consistency. Chapter Seven details the tendency top combine framings as in this example, casting it as important component of expertise.

S2 responds in a familiar way to S1's recall of Physical Mapping in lines 42 to 44: "See, point six one eight, which is what I said, the work done here should be larger than the work done here 'cause the path". S2 returns to Invoking Authority to justify his equal-work assertion in lines 45 and 47. "No, no no, no no no…work is path independent". When S1 presses him for more detail, "how is it path independent?" (line 48), S2 and S3 respond "by definition" (line 49) and "somebody apparently proved this before we did" (line 50).

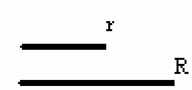
These replies do not contain the type of justification S1 seeks. The next block of transcript begins with S1 making another strong bid for Physical Mapping.

5.1.4 An Even Stronger Bid for Physical Mapping

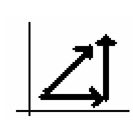
51. **S1**: OK, I don't understand the concept then,
52. because you're saying it's path independent.
53. **S2**: I'm saying, if you're at the bottom of a hill
54. **S1**: all right
55. **S2**: and you want to drive to the top of the hill
56. **S1**: right
57. **S2**: and there's a road that goes like this,
58. a road that goes like this, and a road that's like this, *Draws*



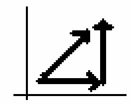
59. it takes the same amount of energy to get
60. from the bottom to the top.
61. It doesn't matter which one you take.
62. **S1**: OK, then you tell me this then;
63. work is force times distance, right?
64. **S2**: It's the integral of f-dr…f-dr, yeah.
65. **S1**: So if you're going this r, and *Draws*



66. you're going to this R, which one has more work?
67. **S2**: If there's constant force?
68. **S1**: Constant force on each one of these.
69. **S2**: This one if it's the same force. *Points to long "R" path*
70. **S1**: OK, now the same force is acting on that
71. **S2**: No. No no. Because this one [radial] has a
72. direct force the whole time. *Gestures at*
73. See, there's lesser force. *two-path diagram*



74. **S1**: OK
75. **S2**: in each one of these [two-part path]

76. **S1**: OK. All right
77. **S2**: your forces are not
78. **S1**: I see what you mean. I see what you mean.
79. Here we're taking
80. **S2**: Here we're supposed
81. to be compensating for that.
82. **S1**: We're just taking the x component *Gestures at* 
83. of the force here, and the y component
84. of the force there. You're probably right.
85. You've probably been right the whole time.
86. Are we thinking about that correctly then?
87. I agree with what you're saying.

S1 begins this last transcript chunk with another bid to frame their math use as Physical Mapping. "I don't understand the concept then, because you're saying it's path independent" (lines 51 and 52).

S2 responds to this newest bid with an interesting hybrid of his own. He is still quoting "work is path independent" but he now couches that rule in terms of a physical situation. He draws a picture of various paths up a hill and asserts "it takes the same amount of energy to get from the bottom to the top. It doesn't matter which one you take" (lines 53 to 61).

S2's latest response still partly reflects an Invoking Authority framing because it offers no physical mechanism for why the work done by gravity should be the same along any of the paths up the hill. Technically, your car would burn more gasoline along the curviest path, but S2 doesn't acknowledge this point and may not have even considered it in light of the inertia Invoking Authority is exhibiting in his thought. Perhaps S2 has a more detailed physical mechanism in his mind, but he doesn't articulate it here.

Nonetheless, S1 recognizes a glimmer of the type of justification he seeks in S2's latest argument. S2 presses further on the longer-path issue. "OK, then you tell me this then; work is force times distance, right? … So if you're going this r, and you're going this R, which one has more work?" (lines 62 to 66) This question is S1's most explicit call yet for a Physical Mapping framing. He closely juxtaposes a mathematical point (work is force times distance) and a diagram-aided observation of a longer path (his r and R picture).

This reframing bid tips S2. His hint of a Physical Mapping framing in lines 57 to 61 asserts itself, putting him in a much better position to understand S2's argument. For the first time in this conversation, S2 explicitly addresses a physical detail relevant to the Physical Mapping S1 is attempting: "if there's constant force?" (line 67) S1 quickly affirms that assumption and S2 correctly concludes that more work would be done on the long "R" path. When S1 quickly moves from this hypothetical r and R case back to the homework problem (line 70), S2 immediately points out the inconsistency. "No. No no. Because this one [radial] has a direct force the whole time. See, there's lesser force … in each one of these [two-part path]…here we're supposed to be compensating for that" (lines 71 to 75 and 80 to 81). S2 gestures to the problem's diagram during this Physical Mapping. The gravitational force vector and the displacement vector are (anti-) parallel for the radial path, hence you need to consider the full magnitude of the gravitational force in calculating the work done along that path. These two vectors do not align perfectly along the two-part path, hence you only consider a component of the force there.

S1 quickly accepts and confirms this argument (lines 78 to 87), which is the first fully articulated Physical Mapping argument S2 had offered during this conversation. His quick comprehension and acceptance occur because S2 has now framed their problem solving in the way S1 has. S1 was mentally ready to accept such an argument.

S2's reluctance to adopt a Physical Mapping framing implies an activation failure, not unsophistication or naivety. His reluctance was certainly not due to simple inability. He was, after all, the one who actually wrote the integrals (which do not contain the necessary cosine factors but, according to S2, were meant to reflect the "lesser force" idea) in the minutes leading up to the presented transcript. S2 quickly generated a Physical Mapping argument once he framed the discussion as Physical Mapping, i.e. once he activated the relevant subset of his mental resources.

5.1.5 Summary of First Case Study

This case study illustrates how epistemic framing negotiation and communication can be a powerful dynamic in physics students' work, even if it is often implicit. S1 and S2 disagreed over much more than whether the gravitational work done was independent of path. Their disagreement over what type of justification was appropriate drove this conversation. Much of this debate was implicit. S1 never came out and said, for example, "please respond to me in a way that points out some detail of the physical situation that I have not mapped correctly to the mathematics we're using." The epistemic framing analysis presented in this dissertation offers a clean way of making this implicit conversation dynamic explicit to teachers and physics education researchers.

Implicit framing miscommunications such as this one between S1 and S2 are also likely to be common in the classroom itself between instructor and students. It is hoped that the framing analysis presented in this dissertation will be of help to physics instructors as they look to understand a student's question or comment. Making these framing issues more explicit should lead to better classroom communication.

This last example had these epistemic framings exhibiting considerable inertia. S2 remained in Invoking Authority despite several prods. S1's commitment to Physical Mapping allowed those prods to keep happening. The next case study shows a student shifting frames much more readily.

5.2 Taking a Derivative with Respect to Planck's Constant: Framings Can Be Flexible

This next case study, like the last one, has two physics students trying to agree on the best way to frame the math use at hand. Like S1 in the last example, S4 will make several framing bids. S5 responds to these bids more readily than S2 did, illustrating how epistemic framing can be a relatively labile process as well.

5.2.1 *The Question*

The two students in this episode are enrolled in a second semester undergraduate quantum mechanics class. Like the students in the previous episode, they are meeting outside of class to work on that week's homework assignment. The case study begins with the students part way through problem 6.32, part b, in Introduction to Quantum Mechanics (Griffiths, 2005), a common undergraduate textbook. That problem deals with the Feynman-Hellmann theorem, $\frac{\partial E_n}{\partial \lambda} = \langle \psi_n | \frac{\partial H}{\partial \lambda} | \psi_n \rangle$, which relates the partial derivative (with respect to any parameter λ) of an energy eigenvalue to the expectation value of the same partial derivative of the Hamiltonian. The problem tells them to consider the one-dimensional harmonic oscillator, for which $E_n = \hbar\omega\left(n + \frac{1}{2}\right)$ and $H = \frac{-\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + \frac{1}{2}m\omega^2 x^2$. They are asked to set λ equal to ω, $\hbar$, and m (the angular frequency of the oscillator, Planck's constant, and the mass of the oscillator, respectively) and hence use the Feynman-Hellmann theorem to get expressions for the oscillator's kinetic and potential energy expectation values.

We begin with S4 noticing an oddity. When she sets $\lambda = \hbar$, the Feynman-Hellmann theorem requires her to consider $\frac{\partial}{\partial \hbar}$. How does one deal with a partial derivative with respect to a constant?

5.2.2 *A Framing Clash and a Quick Shift*

These two students are seated at a table throughout this discussion. They do not gesture towards any diagrams or equations in a shared space.

1. **S4**: If we figure this out, hopefully it'll make
2. the other ones easier. When you say something's
3. a function of a certain parameter, doesn't that mean
4. that as you change that parameter, the function changes?
5. **S5**: mmm-hmm
6. **S4**: OK, so I can change omega, but I can't change h-bar.
7. **S5**: Sure you can.
8. **S4**: I can?
9. **S5**: You can make it whatever you want it to be.
10. **S4**: But
11. **S5**: It's a constant in real life, but it's a funct-, it's,
12. it appears in the function and you're welcome to change its value.
13. **S4**: But then it doesn't mean anything.
14. **S5**: Sure it does. Apparently it means
15. the expectation value of [kinetic energy].
16. **S4**: You don't really know what you're talking about.
17. **S5**: Look, all it is, is you're gonna take the derivative with respect to

18. **S4**: Yeah, I understand what they want me to do here.
19. **S5**: They're just applying the theorem.

S4 begins this passage with a concise check on what a derivative entails. "When you say something's a function of a certain parameter, doesn't that mean that as you change that parameter, the function changes?" (lines 2 to 4). Upon S5's affirmation, S4 points out a mismatch of this mathematical point with a physical reality. H-bar is a constant. Taking a partial derivative with respect to h-bar would imply that Planck's constant can vary. S4 is framing her use of mathematics as Physical Mapping. Her warrant for not accepting the $\frac{\partial}{\partial \hbar}$ operation focuses on how valid uses of math in physics class tend to align with physical reality.

S5 initially responds to S4's concern by asserting a rule. The warrant for his counterargument concerns the practical, common use of statements and previous results without explicit justification. "Sure you can [change h-bar]" he says. "You can make it whatever you want it to be" (lines 7 and 9). In so responding, S5 is lobbying for an Invoking Authority framing. He is suggesting S4 set aside her physically motivated objections and instead judge the validity of $\frac{\partial}{\partial \hbar}$ according to his confidence in his assertions.

Much like the two students in the gravitational work example, S4 and S5 are arguing over something much deeper than whether or not one is allowed to take a partial derivative with respect to ħ. They are disagreeing over what would be appropriate grounds for accepting or rejecting such an operation.

S4 does not accept S5's bid for Invoking Authority. Upon her first protest in line 10, S5 quickly admits "it's a constant in real life" (line 11) but sticks to his Invoking Authority framing. "It appears in the function and you're welcome to change its value" (lines 11 and 12).

S4 protests again; "But then it doesn't mean anything" (line 13). Such a statement's full interpretation relies on acknowledging S4's Physical Mapping framing. In some framings, S4's statement is patently false. The operation $\frac{\partial H}{\partial \hbar}$ can "mean" plenty. For example, it would produce the result $\frac{-\hbar}{m}\frac{\partial^2}{\partial x^2}$. Developing both the calculus machinery and the abstract interpretation of such an operation was the crowning achievement of Newton and Leibniz's mathematical studies. S5 retains his Invoking Authority framing and quickly responds with another "meaning" of $\frac{\partial H}{\partial \hbar}$. Quoting from the textbook's statement of the homework problem, "Sure it [means something]. Apparently it means the expectation value of [kinetic energy]" (lines 14 and 15). Recall the question had told them to set $\lambda = \hbar$ in the Feynman-Hellmann theorem, $\frac{\partial E_n}{\partial \lambda} = \langle \psi_n | \frac{\partial H}{\partial \lambda} | \psi_n \rangle$, and hence obtain an expression for the expectation value of kinetic energy. S5 is thus relying on the authority of the text's question for his interpretation of $\frac{\partial H}{\partial \hbar}$. Only by acknowledging S4's current Physical Mapping framing can we place her claim in the proper context. If one's warrant for judging an operation like $\frac{\partial H}{\partial \hbar}$ concerns the alignment of the mathematics with a physical reality, then yes, that operation can be said not to "mean" much of anything. Planck's constant does not physically vary.

S4 objects to S5's arguments again in line 16. "You don't really know what you're talking about." This perturbation was sufficiently strong to cause S5 to reframe his attempt to justify $\frac{\partial}{\partial \hbar}$. He says "look, all it is, is you're gonna take the derivative with respect to" (line 17) before getting cut off by S4. Coupled with his next statement in line 19, "they're just applying the theorem," these statements can be seen as an attempt to reframe his thinking as Calculation. S5 is suggesting they go ahead and use their calculus machinery to take the partial derivative. As long as they stay true to the rules of calculus, they should be able to trust whatever result appears.

S4 acknowledges this attempt to reframe their work as Calculation. "Yeah, I understand what they want me to do here" (line 18). Lines 17 to 19 nicely illustrate how efficient this implicit epistemic frame negotiation can be. These lines didn't even take five seconds to speak. In those five seconds, S5 made a call for using a different set of warrants. S4 heard that call and her brain quickly activated some of the procedures and techniques that would be associated with such a framing, as evidenced by "yeah, I understand what they want me to do here" (line 18). S5, just as quickly, acknowledges S4's acknowledgment of his reframing suggestion with his "they're just applying the theorem" (line 19).

5.2.3 Another Quick Shift, This Time to a Shared Physical Mapping Framing

S4 still insists on a justification more in line with her Physical Mapping framing. She begins the next chunk of transcript with another reframing objection. S5 responds by nimbly dropping his Calculation framing and adopting Physical Mapping himself.

20. **S4**: But I don't understand how you can take the derivative
21. with respect to a constant.
22. **S5**: Because if you change the constant then the function will change.
23. **S4**: But then it's not, it's not physics.
24. **S5**: So? Actually it is, 'cause, you know,
25. a lot of constants aren't completely determined.
26. **S4**: There's still only one value for it, that's what a constant is.
27. **S5**: The Hubble constant changes. The Hubble constant changes
28. as we improve our understanding of the rate of expansion of the universe,
29. and we use the Hubble constant in equations.
30. **S4**: But there's only one, right, there's only one constant. It does not vary.
31. **S5**: Yeah, but the value's changing as we approach the correct answer.

32. **S4**: It's just gonna get fixed. That's not, that's not helping us with the derivative.
33. **S5**: You can always take a derivative with respect to anything.
34. **S4**: But if you take it with respect to a constant, you'll get zero.
35. **S5**: Not if the constant itself appears in it.
36. The derivative tells you if you change whatever
37. you're taking the derivative with respect to how will the function change?

S4 begins this block of transcript by repeating her discomfort with $\frac{\partial}{\partial \hbar}$ (lines 20 and 21). S5 responds with "because if you change the constant then the function will change" (line 22). This statement does not clearly align with only one of this dissertation's common framings. Its ambiguity comes in large part from its isolation. Perhaps it was a prelude to a Calculation explanation, or perhaps S5 was preparing to use some sort of Math Consistency warrant as he related this $\frac{\partial}{\partial \hbar}$ issue to a more familiar Calculus 101 example. S5's thought could have evolved this way or that, but one cannot assume line 22, by itself, was necessarily the tip of an implicit iceberg of coherence.

S4's next objection, "but then it's not, it's not physics," (line 23) leads S5 to start explicitly searching for an example of a physical constant that varies. In undertaking such a search, S5 has adopted the warrant S4 has been pushing. Valid use of math in physics class should align with physical reality. S5 hopes that by finding an example of a varying physical constant he can convince S4 that it is permissible to take a derivative with respect to Planck's constant. S5 frames his activity as Physical Mapping starting in line 24.

S5 invokes the analogy of the Hubble constant in lines 24 to 31. The Hubble constant is connected to the rate of expansion of the universe. S5 points out that the value of the Hubble constant quoted by scientists has changed over the past half a century as our measurement techniques have improved. He argues that the Hubble constant, variable as it seems, is an important part of many physics equations. By extension, it should be permissible to consider a varying Planck's constant.

S4 offers a much richer response to S5's Hubble constant argument than she has to any of his other attempts in this episode. Up to this point, she had been simply shooting down S5 with comments like "but then it doesn't mean anything" (line 13), "you don't really know what you're talking about" (line 16), and "but then it's not, it's not physics" (line 23). S5's Hubble constant argument marked the first time he adopted S4's warrant concerning the alignment of math and physics, i.e. the first time he and S4 shared a common epistemic framing.

This shared epistemic framing helps S4 engage with S5's chosen example in lines 26 to 32 and points out that he's confusing a measurement variance with an actual physical variance. Sure, she says, our quoted value for the Hubble constant has shifted as our measurements improve, but, presumably, our measurements are tending towards a fixed value. The Hubble constant itself, she says, isn't changing. "That's not helping us with the derivative" (line 32).

This counterargument causes S5 to reframe the situation once again as he turns to a different type of justification. He quotes a rule again in line 33. "You can always take a derivative with respect to anything." S4 misspeaks when she replies. "But if you take it with respect to a constant, you'll get zero" (line 34). This statement seems to confuse her earlier correct interpretation of $\frac{\partial}{\partial \hbar}$ (as in lines 2 to 6) with the Calculus 101 mantra "the derivative of a constant is zero", i.e. $\frac{\partial \hbar}{\partial x} = 0$. S4 responds to this misstatement in lines 35 to 37.

5.2.4 A Final Frame Shift

The final block of transcript from this episode follows S5's quick correction. It begins with S4 objecting yet again and S5 trying out yet another framing.

38. **S4**: So I don't understand how you can change a constant.
39. **S5**: You pretend like it's not a constant.
40. It's just like when you take partial derivatives with respect to,
41. like variables in a function of multivariables.
42. You pretend that the variables are constant.
43. **S4**: Yeah, I don't have a problem with that.
44. **S5**: You're going the other way now.
45. You're pretending a constant is a variable. Who cares?
46. **S4**: It doesn't make sense to me.
47. **S5**: You can easily change a variable—it's not supposed to, I don't think.
48. **S4**: OK, then I believe-
49. **S5**: I don't think, I don't think there's supposed to be
50. any great meaning behind why we get the change h-bar.
51. I think it just-they're like oh look, if you do it
52. and you take its derivative and you use this equation,
53. then all of a sudden you get some expectation of [kinetic energy],
54. and you say whooptie-freekin-do.

S5 responds to S4's latest objection in line 38 via a Math Consistency framing. His newest argument relies on a warrant he hasn't yet tried: mathematics is a self-consistent field of knowledge, so a valid mathematical argument is one that fits in logically with other mathematical ideas.

S5 makes a common move for a Math Consistency framing. He draws an analogy in lines 39 to 45. In order to take a derivative with respect to $\hbar$, one has to "pretend" that the constant is a variable. S5 points out that taking a standard partial derivative with respect to one of the variables of a multivariable function involves "pretending" the other variables are constants. Their $\frac{\partial}{\partial \hbar}$ case, he argues, is "just like" that analogous example, except "you're going the other way now. You're pretending a constant is a variable."

In contrast to her more extended counterargument in the Hubble constant case, S4 rejects this present argument much more coarsely. "It doesn't make sense to me" (line 46). S5 has once again framed their work differently than S4's Physical

Mapping. Each student's mind has activated a sufficiently different subset of their available mathematical resources, which cuts down the depth of their communication and interaction.

When S5 responds "it's not supposed to [make sense], I don't think" in line 47, he is explicitly addressing S4's Physical Mapping framing for the first time. While he had been responsive to her objections throughout this episode, he now argues with her epistemic framing directly. He states that he doesn't think an explanation of the type S4 seeks exists. S4 is possibly about to acknowledge inappropriateness of the Physical Mapping stance when she replies "OK, then I believe-" (line 48), but she gets cut off. S5 then elaborates a hybrid of Calculation and Invoking Authority that he sees as most appropriate in lines 49 to 54. Mechanically take the derivative with respect to $\hbar$, following the familiar calculation algorithms, and then trust the Feynman-Hellmann theorem to relate this derivative to the oscillator's kinetic energy.

5.2.5 Summary of Second Case Study

This case study illustrates how epistemic framing can be a relatively flexible process. The entire episode is essentially many iterations of S4 objecting and S5 saying, "well, all right, how about this other type of explanation?" S4's objections serve as perturbations to S5's mental state. Many of them are of sufficient strength (or occur after he has reached a respectable closure point of his previous argument) to lead him to reframe his thinking. Each reframing results in S5 adopting a different type of warrant for judging the validity of his mathematical claim, that one should accept the operation $\frac{\partial}{\partial \hbar}$ as legitimate within physics, despite the constancy of $\hbar$.

This $\frac{\partial}{\partial \hbar}$ issue is a relatively difficult one. Ordinarily, a Physical Mapping frame is quite valuable in physics. Helping students understand the physical referents or their math is a common, if sometimes difficult, goal of many physics classes. Here, S4 and S5 are being asked to do something even more subtle and difficult: consider an imaginary world, one where $\hbar$ can vary, and see if the mathematics in this imaginary world can inform the real one. That S4 and S5 were willing to engage in an exploration of how to frame this $\frac{\partial}{\partial \hbar}$ issue is commendable, even if the episode ends without an especially satisfying consensus.

5.3 Chapter Summary

The two case studies in this chapter are meant to both align and contrast with each other. Both demonstrate how epistemic framing dynamics can drive a conversation. These framings can have significant inertia (as with S2) or can be relatively flexible (as with S5).

The students in each case study disagree over much more than an answer. They each frame their activity differently and hence try to apply a different type of warrant to judge the validity of their claims. The students exert various pushes and pulls on each other as they try to negotiate a common epistemic framing. Vary rarely are these reframing bids explicit. Nonetheless, these framing debates underlie the speech in both of this chapter's case studies. When a common framing is established, the conversation tends to be richer and more efficient.

There are no rigid time limits for a given framing. Neither a lower nor an upper bound can be set. A framing bid can be made, heard, evaluated, and discarded very quickly. See, for example, S4 and S5's brief Calculation exchange in lines 17 to 19 in the second case study. Those lines took up no more than five seconds of video. Contrast that example with S2's Rule Quoting in the first case study. That framing showed considerable inertia for the entire four-minute episode. Most physics teachers can probably anecdotally remember students getting stuck in some mindset for much longer than four minutes, also. A professional may set up to do a numerical simulation and spend weeks or even months working largely in Calculation. Upper bounds are similarly impossible to set.

Epistemic framing is a vital component of physics students' thinking. One cannot consciously consider all possible mathematical resources at any one time. The mind must pare down all of its mathematical options to some sort of manageable subset. This paring down operation, i.e. epistemic framing, plays a critical, if sometimes implicit, role in how students' thinking evolves during a problem solving episode.

Chapter 6: Application of Framework Towards Describing a Calculator's Effects on Student Thinking

The first part of this dissertation proposes epistemic framing as an analysis tool. In Chapter Four we show examples of how examining the warrants physics students use in their mathematical reasoning helps a researcher describe how the students are framing their activity. In Chapter Five, we show that epistemic frame negotiation can be a powerful, if often implicit, driving force for a conversation as two or more students try to agree on how to interpret the math at hand. Of all their available mathematical resources, which ones should they explicitly consider as they try to make sense of their math?

The previous chapters describe this epistemic framing analysis. They detail how this study's epistemic framing analysis tool was developed and provide several quick and extended examples of different kinds of framing: Calculation, Physical Mapping, Invoking Authority, and Math Consistency. They have also connected this analysis tool to the wider physics education literature. This study works within a manifold model of cognition. It focuses on the real-time evolution of students' thinking rather than attempting to characterize coherent stabilities in thought that students can apply across a wide variety of situations.

The rest of this dissertation applies this epistemic framing tool to address a variety of research questions. This chapter considers what an epistemic framing analysis can show regarding a powerful symbolic calculator's effects on upper level physics students' thinking. Material for this chapter comes from a previously published study, "Symbolic Manipulators Affect Mathematical Mindsets" (Bing and Redish, 2008). Chapter Seven examines how an epistemic framing analysis characterizes expertise in physics.

6.1 Overview of the Symbolic Calculator Study

Symbolic calculators like Mathematica are becoming more commonplace among upper level physics students. The presence of such a powerful calculator can couple strongly to the type of mathematical reasoning students employ. It does not merely offer a convenient way to perform the computations students would have otherwise wanted to do by hand. This chapter presents examples from the work of upper level physics majors where Mathematica plays an active role in focusing and sustaining their thought around calculation. These students still engage in powerful mathematical reasoning while they calculate but struggle because of the narrowed breadth of their thinking. Their reasoning is drawn into local attractors where they look to calculation schemes to resolve questions instead of, for example, mapping the mathematics to the physical system at hand. We model Mathematica's influence as an integral part of the constant feedback that occurs in how students frame, and hence focus, their work.

6.1.1 Introduction to Symbolic Calculators and Number Sense

Recent advances in computers and programming have given today's physics students a new tool. Personal computer programs such as Mathematica and Maple are capable of symbolic calculation, as are many handheld calculators. Whereas calculators were once limited to numeric operations like evaluating the cube root of forty-two, they can now expand $(x+3)^3$ to $x^3+9x^2+27x+27$, evaluate $\int_{-\infty}^{\infty}\frac{\cos(x)}{1+x^2}dx$ as $\frac{\pi}{e}$, and solve $\frac{dy}{dx}=-3y$ as $y(x)=Ce^{-3x}$.

Automated calculation, even when it is strictly numeric instead of symbolic, makes many teachers wary. Almost all physics teachers have anecdotal stories of watching students reach for a calculator to do simple operations like halving a number or multiplying by one hundred. Most have also watched students make obvious errors as they pushed calculator buttons. Teachers worry that these students are neither using nor developing a feel, an instinct, for numbers. Mathematics education researchers have been similarly concerned with this vital sense in students. "Number sense" has at least partially converged to a certain set of meanings in the math education literature, including flexible computing strategies for written and calculator-aided computation, understanding of equivalent representations, and use of equivalent expressions (Reys et al., 1999).

With the expansion of symbolic manipulation capabilities into the teaching of advanced physics, one can raise an analogous question: Does being fluent with a symbolic manipulator damage the advanced physics student's intuition for and ability to make sense of complex mathematics in physics?

In this chapter I begin to address this question by offering two examples from upper level physics majors as they use Mathematica to solve problems in an upper division course in quantum mechanics. Mathematica's presence contributes to the students' difficulties in both cases, but their difficulties do not stem from a stunted or disengaged mathematical intuition. The students show admirable flexibility and creativity as they try different calculation strategies and representational forms. Rather, the difficulties associated with Mathematica use appears to arise from more subtle issues. They arise from a local coherence in their thinking that leads them to focus on computational aspects of the problem while suppressing the connection with the physics and with extended mathematical meanings. I analyze these observations with this dissertation's epistemic framing lens.

6.1.2 Computational Tools and Mathematical Intuition

In considering the student use of symbolic manipulators in advanced physics, it is appropriate to put our considerations in perspective of the use of computational tools at other levels. University instructors often feel that the calculator has done damage to the growth or use of the "number sense" in students. Is this so?

This is a broad question and is complicated by the fact that "number sense" is difficult to define exactly. There is a very large collection of studies with elementary through high school students (see Hembree and Dessart, 1986, Hembree and Dessart, 1992, and Dunham, 2000 for reviews) that mostly suggest numeric calculators help students develop algorithmic computation and problem solving skills. Better rote computation skills, however, do not necessarily imply a better number sense (Reys and Yang, 1998). Still, the relatively few studies that explicitly address the effects of calculators on students' number sense tend to indicate that calculators help (Howden, 1989; Wheatley and Shumway, 1992). A possible explanation for this phenomenon would be to conceptualize number sense as something that evolves out of one's interaction with a conceptual environment (Greeno, 1991). The more a student thinks, works, even plays around with mathematics, the more their intuition, their number sense, evolves. Calculators can, in principle, help streamline this playing around in mathematics, providing quick feedback that can accelerate the development of number sense.

Once students move on to algebra, number sense is extended to include symbol sense. Symbol sense includes healthy intuitions about when introducing symbols can be useful, what passes for proper symbol manipulations, and how symbolic arguments can be general methods of proof (Arcavi, 1994; Fey, 1990). Relatively few studies with symbolic manipulation calculators have explicitly addressed their effects on symbol sense, but some positive correlations exist (Heid and Edwards, 2001). Speaking more broadly, symbolic manipulator use tends to correlate to both better conceptual understanding and better manual calculation skills (Heid, 1988; Palmiter, 1991), just as the numeric calculator studies indicate.

These studies suggest that appropriate instruction using the calculator at a pre-college level may help students develop a sound sense of number and symbol. Whether most of our students have received such appropriate instruction and have such a sense remains to be explored.

In the case of the use of topics such as algebra and calculus in advanced physics, we are concerned with something more than a sense of number and symbol. We want students to develop a "sense of the mathematics," an intuition for the structure of complex mathematical expressions that allows them to interpret and unpack these expressions, providing a capability for transforming equations and quickly recognizing errors. We refer to this extension of the number and symbol senses to more complex mathematics as *math sense*. In the rest of this chapter we explore in depth two examples of students working together to solve authentic physics problems using Mathematica.

6.2 Two Extended Examples

This chapter draws from the same groupwork video data set as the rest of this dissertation. The problems are authentic homework assignments for which the students receive class credit. We have approximately one hundred hours of such videos in upper-division physics-major classes that include intermediate mathematical physics, electromagnetism, and quantum mechanics. Of these, approximately 10% include student use of Mathematica or other symbolic calculator. The two examples below provide the clearest representatives of phenomena that have been observed many times in the full data set.

6.2.1 Example 1: The Feynman-Hellmann Theorem

As one example of Mathematica's influence on student behavior, consider a video of two students doing their homework in a second semester undergraduate quantum mechanics class. They are working on problem 6.32, part b, in Introduction to Quantum Mechanics (Griffiths, 2005). It asks them to use the Feynman-Hellmann theorem, $\frac{\partial E_n}{\partial \lambda} = \langle \psi_n | \frac{\partial H}{\partial \lambda} | \psi_n \rangle$, for the one-dimensional harmonic oscillator with the Hamiltonian $H = \frac{p^2}{2m} + \frac{1}{2} m\omega^2 x^2$. They are asked to set $\lambda = \omega$ to obtain a formula for the expectation value of V, the potential energy.

This question can be quickly answered by noticing that $\frac{\partial H}{\partial \omega}$ and V both go as x^2. Hence $\langle \psi_n | \frac{\partial H}{\partial \omega} | \psi_n \rangle = \langle \psi_n | m\omega x^2 | \psi_n \rangle = \langle \psi_n | \frac{2V}{\omega} | \psi_n \rangle = \frac{2}{\omega} \langle V \rangle$. Since $\frac{\partial E_n}{\partial \omega} = \hbar \left(n + \frac{1}{2} \right)$, the Feynman-Hellmann theorem gives $\langle V \rangle = \frac{\hbar\omega}{2} \left(n + \frac{1}{2} \right)$. The video, however, shows the students engaged in a fifteen-minute effort to program Mathematica to explicitly calculate the expectation value $\langle \psi_n | \frac{\partial H}{\partial \omega} | \psi_n \rangle$.

6.2.1.1 These Students Calculate Sophisticatedly

The students' work is impressive. They identify the Hamiltonian, navigate a complicated general expression for the stationary states of the oscillator, implement Mathematica's predefined Hermite polynomial function, and neatly package all this information into a single line of code that can calculate the expectation value of $\frac{\partial H}{\partial \omega}$. When they hit one of several snags along the way, their error checking and debugging are quick and efficient. To illustrate, consider the following continuous 90-second chunk of transcript from the middle of this episode. S1 and S2 are seated in front of a computer for all of this conversation.

1. **S1**: Umm, Hermite polynomials are all real, right?
2. They're happy? Are there "i"s in the Hermite polynomials?
3. **S2**: Let me check. Remember they had the first set of them way back here.
4. **S1**: Mmm-hmm, they're all real.
5. **S2**: Yeah, they're all real.
6. **S1**: All right, so they're just psi squared.
7. **S2**: Oh, there's, one moment, OK.
8. **S1**: Psi squared, d-omega-H, comma x, comma
9. **S2**: Umm, are those all the different bits? Where's your e to the negative ξ^2?
10. **S1**: It's inside.
11. **S2**: It is? OK. OK good.
12. **S1**: x comma, oh , just x—er well, minus infinity to infinity, right?
13. **S2**: Right.
14. **S1**: Minus escape-n, comma, escape-n.
15. **S2**: That's what you got.
16. **S1**: Yeah, 'cause it didn't do it.
17. **S2**: No it didn't, and it got something. Let's see, negative n, m,
18. didn't even get the same thing I did. It's different.

19. **S1**: Hermite n of x gives a Hermite polynomial h, n of x.
20. **S2**: We got h, n of x?
21. **S1**: What the heck? n is n, oh, 'cause it doesn't know what n is.
22. **S2**: We don't want this for any n. You have to say what n is?
23. **S1**: Well, you can't integrate the Hermite polynomial of x
24. without putting in what the Hermite polynomial is.

It takes a formidable math sense to accomplish these actions, one that goes well beyond the much simpler examples from grade school mathematics on which most of the number sense or symbol sense literature focuses. There are hardly any actual numbers in sight, only variables and constants that stand in for them. Complicated functions, Hermite polynomials and ψ_n, have to be seen as mathematical objects (Sfard, 1991; Sfard and Linchevski, 1994) in and of themselves to be unpacked and operated upon, as in lines 1 to 6. Mathematica has its own protocol for using Hermite polynomials, referred to in lines 19 and 20, that the students have found and interpreted. They have also managed to organize the calculation in an efficient and aesthetic way, hiding some of the details behind user-defined symbols in the Mathematica code. S2 asks about one such move in line 9. After this snippet ends, the students even set up an array in Mathematica to perform ten of these expectation value calculations, one for each of the first ten stationary states, at once.

Their debugging is efficient as well. Lines 17 to 19 show the students reacting to Mathematica's evaluation of their first coding. The program has balked at the Hermite polynomial function call. S1 quickly interprets the error in lines 21 to 24. They have not indicated which specific Hermite polynomial Mathematica should use. Such a quick debugging demonstrates S1's engaged grasp of the Hermite polynomials. They are an articulated set of specific mathematical objects to him, not merely some nebulous symbol upon which to operate.

This transcript, and the larger fifteen-minute episode that surrounds it, is quite a display of flexible computation and representation, two hallmarks of math sense. The students are framing their task as Calculation. They are focusing on *drilling down into a calculation*. In this framing, students pay attention to the computational details, look for ways to achieve a result, unpack mathematical structures, and manipulate expressions within the problem that they have identified. They are assuming that producing and justifying a final answer will require a lengthy, technically correct calculation. Their minds are focused on their corresponding computational resources for the entire fifteen-minute episode from which this except is taken.

6.2.1.2 But They Don't Consider Alternate Framings

Absent from all this work, however, is any discussion of how they plan to connect their calculation's result to $\langle V \rangle$ as the question requires. Their thinking was drawn into this Mathematica calculation, which sustained itself for fifteen straight minutes even when difficulty arose. Making the calculation work became a goal in and of itself, irrespective of the original homework question. Their excellent calculation eventually yields the truism $\hbar\left(n+\frac{1}{2}\right)=\hbar\left(n+\frac{1}{2}\right)$ from the Feynman-Hellmann theorem, but then they are stuck. The never step back to notice that $\frac{\partial H}{\partial \omega}$ is proportional to x^2, as is V. Simply shuffling a few constants around in the Feynman-Hellmann theorem can yield an expression for $\langle V \rangle$. No explicit calculation of expectation values is required.

That is not to say that thinking about how both $\left\langle \frac{\partial H}{\partial \omega} \right\rangle$ and $\langle V \rangle$ are proportional to $\langle x^2 \rangle$ is not also an application of math sense. It is, however, an application driven by a search for a different kind of mathematical justification (i.e. a different epistemic framing). When these students were programming Mathematica to compute the expectation value of $\frac{\partial H}{\partial \omega}$, they were focusing on how convincing mathematical arguments are procedurally correct. A technically correct calculation should lead to a trustable result. Their math sense is projected along this computational axis and manifests itself as the flexible calculation and representation strategies seen in their work.

Noticing that $\left\langle \frac{\partial H}{\partial \omega} \right\rangle$ and $\langle V \rangle$ are both proportional to $\langle x^2 \rangle$ focuses on a different aspect of mathematical justification. Instead of being concerned with drilling down into a detailed calculation, it entails packaging parts of an expression together and seeing how the various packages relate to each other. It searches for an analogy, of sorts, between the expressions for $\left\langle \frac{\partial H}{\partial \omega} \right\rangle$ and $\langle V \rangle$.

If the previous Calculation framing was "drilling down" then this second framing is more of a "moving across". This framing is an example of Math Consistency. Math Consistency also entails an important type of mathematical justification. Mathematical systems involve many parts, and understanding how each part interacts and relates to the other parts is essential for comprehending the system as a whole. Whereas the earlier Calculation framing brought out certain facets of a student's math sense, the Math Consistency framing would highlight other aspects of math sense, such as proportionality and functional dependence.

In this example, Mathematica seems to have facilitated the students entering and sustaining a Calculation framing, ignoring broader and more direct mathematical approaches to the problem. This is not to say that the students' thought would not have evolved towards a Calculation framing without Mathematica. Indeed, we have seen many students overly rely on computation at many levels. But having Mathematica seems to remove a barrier to entering Calculation that is explicitly illustrated in the next example.

6.2.2 *Example 2: An Expectation Value*

The second example comes from a video recording of six junior and senior physics majors meeting to work on their homework for a second semester undergraduate quantum mechanics class. They are working on Problem 5.6 in the same text (Griffiths, 2005). The problem asks them to calculate $\left\langle (x_1 - x_2)^2 \right\rangle$ for two particles in arbitrary stationary states of a one-dimensional infinite well, where x_1 is the coordinate of the first particle and x_2 is the coordinate of the second. Three successive parts of the problem ask them to assume the particles are distinguishable, identical bosons, and identical fermions, respectively. In the course of this calculation, the students realize they need to evaluate $\int x_1^{\,2} |\psi_n(x_1)|^2 dx_1$. This notational shorthand, which doesn't specify the limits of integration, is taken from the hints the text gives in the pages preceding this problem. The transcript begins with a student in the group explicitly mistaking the limits of integration to be from negative infinity to positive infinity instead of just over the width, L, of the well. They are thus led to try to evaluate $\frac{2}{L}\int_{-\infty}^{\infty} x^2 \sin^2\left(\frac{n\pi x}{L}\right) dx$.

6.2.2.1 Calculation Identifies a Problem, More Calculation Ensues

All these students are sitting in desks arranged in a circle for this entire episode.

1. **S3**: The integral is from negative infinity
2. to infinity, right?
3. **S4**: Yeah.
4. **S3**: So we have x squared — *Types into Mathematica*

…one minute later…

5. **S3**: It's telling me it doesn't converge. — *Sets Mathematica aside,*
6. What if I tried — *begins trying to integrate by parts with pencil and paper*
7. **S5**: So what's the integral equal to?
8. **S3**: It wasn't happy, so let me just try something else.
9. **S5**: Oh, we got undefined?
10. **S3**: It said it didn't converge.

S3 is the main focus. She is one of the top students in her class and graduated with honors and significant research experience. Our analysis of her thinking centers around five times when she explicitly hits a roadblock in her work during the seven-minute stretch from which these transcript chunks are drawn. *Hits a roadblock* means that her current line of thinking has either come to a result that does not satisfy her or that she judges has become too complicated to justify continuing. The most important aspect shared by the roadblocks S3 encounters is that they all necessitate her picking a new approach and are hence provide likely opportunities for reframing her work.

S3 encounters five roadblocks and makes five choices about what is appropriate to try next. All of her choices result in strategies aimed at producing a technically correct calculation except for one ambiguous case at the end. Mathematica is an integral part of her thinking during each of the events we observe, playing an active role in sustaining her Calculation framing.

S3 encounters the first of these roadblocks above in lines 5 and 6. She mistakenly sets the limits of integration in line 1, and Mathematica correctly informs her that $\frac{2}{L}\int_{-\infty}^{\infty} x^2 \sin^2\left(\frac{n\pi x}{L}\right)dx$ diverges. Faced with this unexpected result, S3 now faces a choice of how to proceed. She chooses to try evaluating the integral by hand. This choice may or may not have been a result of conscious reflection. Note that whether S3 consciously thought of an alternative way to continue and then suppressed it in favor of integrating by parts manually is not directly relevant. What is relevant is the fact that her antidote to the failed Mathematica calculation is another form of calculation. No large-scale reframing has occurred.

S3 started by trying to answer the question "What is the value of $\frac{2}{L}\int_{-\infty}^{\infty} x^2 \sin^2\left(\frac{n\pi x}{L}\right)dx$?" The initial Mathematica computation was aiming to produce and justify a result by means of a technically correct calculation. She keeps her search for justification in the calculation realm even though the roadblock has now transformed the original question with the refinement "Does $\frac{2}{L}\int_{-\infty}^{\infty} x^2 \sin^2\left(\frac{n\pi x}{L}\right)dx$ really diverge?" Calculation is by no means the only useful framing for answering this question. Math Consistency would work well. Is it possible to categorize the various subparts of that integral? A squared sine function is neither negative nor does it tend asymptotically to zero, and x^2 certainly tends to infinity as x approaches positive and negative infinity. Both subparts of the integral belong to the class of functions whose value does not tend to zero as x increases. That integral must therefore blow up. S3, however, keeps her search for proof in the calculational realm.

<u>6.2.2.2 More Students Drawn to Calculation</u>

This strip of transcript picks up about ten seconds after the end of the previous strip.

11. **S3**: I mean, this is an integral
12. that's quite do-able by — *Brings back computer with Mathematica*
13. **S5**: trig substitution
14. **S3**: by parts
15. **S5**: oh, by parts
16. **S4**: Yeah.
17. **S3**: So — *Starts typing again*
18. **S6**: Can you break it up into

19. different parts and then do it on a TI-89?
20. That's what I usually do, a combination
21. by hand, by calculator.
22. **S5**: Well, integrate it indefinitely and plug in.
23. **S7**: Are you not substituting
24. a value in for n and L, or are you?
25. **S3**: Umm, no, but I just tried doing
26. x-squared, sine of x squared, and it's not happy.

S3 implicitly encounters her second roadblock in lines 11 and 12. She had been trying to manually integrate by parts with pencil and paper but decided such a computation would be too involved to reasonably continue. Again a choice of new direction confronts S3, and she again opts for a computational approach. She continues to frame her work as Calculation, looking to build her answer via computation schemes. S3 reaches for Mathematica again and tries evaluating a simpler form of the integral, $\int_{-\infty}^{\infty} x^2 \sin^2 x dx$, as she reports in lines 25 and 26.

This incident is an example of how having Mathematica as a tool can "open channels" to calculational approaches that might not have been chosen had it not been available. It costs S3 a very small investment of effort to try evaluating this slightly different integral. Mathematica lowers the potential barrier to the evaluation of $\int_{-\infty}^{\infty} x^2 \sin^2 x dx$, allowing S3 to explore the problem space more freely. The downside, as this example will illustrate, is this calculation enabling can make it that much easier to get stuck in a Calculation framing.

Also noteworthy is how the local tendency to solve this dilemma solely by further calculation spreads through the group. The rest of the group sees S3 reach for Mathematica a second time in line 12 and infers she needs help with the divergent result. Three other students offer potential solutions, all of which are calculation strategies. S6 suggests a hybrid approach in lines 18 to 21. Do the potentially complicated work of rearranging $\int udv$ into $uv - \int vdu$ by hand, and only then call on the computer to work on the simpler integrals. Line 22 has S5 suggesting Mathematica might be having trouble evaluating the antiderivative at the positive and negative infinity limits. Try just letting Mathematica find the indefinite integral of $x^2 \sin^2\left(\frac{n\pi x}{L}\right)$ and then plug in the limits by hand. S7 offers, in lines 23 and 24, that maybe Mathematica is being confused by an undefined parameter.

All of these suggestions, in addition to the one S3 has tried in lines 25 and 26, reflect a developed, engaged math sense. They treat the calculation at hand as a malleable thing, as something that can be rearranged, simplified, and executed in different ways. The explicit representation of the integral is changed as the students work.

The suggestions of all these students reflect a sophisticated perception that Mathematica is a fallible tool whose precise usage can be deconstructed and tailored to suit the situation at hand. Framing the activity as Calculation does not imply naivety or unsophisticated reasoning. Their difficulty, like the students in the Feynman-Hellmann example, does not stem from a math sense muted by Mathematica. It comes from the relative narrowness of their search. They are trying to resolve a calculational difficulty with more computation instead of catagorizing the subparts of the integral itself and their relations (evidence of a Math Consistency framing) or asking how the integral they are trying to calculate aligns with the physical situation at hand (evidence of a Physical Mapping framing). These alternate framings would bring out different facets of their math sense.

6.2.2.3 Another Reframing Opportunity Passes

With the failure of her simpler Mathematica calculation in lines 25 and 26, S3 encounters her third roadblock. She again elects to try more calculation to resolve it and proceeds to follow some combination of S5 and S6's suggestions. She types some more into Mathematica, produces the antiderivative of the integrand, and then spends nearly a minute copying the antiderivative from her computer screen onto her paper. S3 is looking at this antiderivative when she next begins speaking.

27. **S3**: I can see why it says that doesn't converge.
28. **S4**: Yeah, but I know it…we've done it.
29. **S8**: We're like, but I know it does.
30. **S4**: We've done that integral so many times.
31. **S3**: Find me one, cause see, this [indefinite integral]
32. **S4**: Yeah.
33. **S3**: Is equal to that [antiderivative],
34. and so you know there's a whole number of places
35. where it'll shoot to infinity.
36. **S4**: Like, how else do we find
37. the expectation value of x-squared?
38. **S3**: Yeah.
39. **S4**: Like, I know we've done it
40. for the infinite square well. *S3 starts paging back through textbook*

S3 succeeded in drilling down into the calculation Mathematica does when it tries to evaluate $\frac{2}{L}\int_{-\infty}^{\infty} x^2 \sin^2\left(\frac{n\pi x}{L}\right)dx$. In line 27, she is looking at the various places in the antiderivative where plugging in the infinity limits leads to an infinite result. When both S4 and S8 respond to her work by asserting the result must be finite in lines 28 to 30, S3 faces her fourth roadblock. She elects to trust her result. No reframing occurs as she counts her technically correct calculation as sufficient justification. In line 31, she challenges S4 to find an example of the "so many times" they've allegedly done this integral and proceeds to summarize her calculation for him.

S3 faces a fifth roadblock when S4 refines his finite-value assertion in lines 36 to 40. This specific integral has occurred much earlier in their quantum mechanics coursework when they were simply calculating $\langle x^2 \rangle$ for a single particle in an infinite well. That result was not infinite. Her response to this final roadblock is ambiguous. She does not say anything more but begins paging back through her textbook. By cognitive inertia, one might expect she is looking back to the book's original infinite well treatment, searching for an explicit calculation of $\langle x^2 \rangle$. Maybe an Invoking Authority framing is coming into her reasoning as she searches the text. There is no evidence to confirm or deny these assumptions, however, because thirty seconds later S5 speaks.

6.2.2.4 A Reframing Occurs

41. **S5**: Hey, it's not negative infinity to infinity.
42. **S3**: What is it?
43. **S5**: Is it? Well, we just have to integrate it
44. over the square well, 'cause it's the infinite square well.
45. **S4**: Oh yeah, so it's zero to [L].
46. **S3**: You're right. *Chuckles*
47. **S5**: Yeah, that's why it's not working.
48. **S3**: Well, is it zero to [L] or negative [L] to [L]?
49. **S4**: Uhh, it's defined in [chapter] 2.2 as zero to [L].
50. **S5**: So yeah, that would be why we're [dumb]. *Laughs*
51. **S7**: Oh. We're awesome.
52. **S5**: Yeah, none of us know how to do a square well anymore. *Laughs*
53. **S8**: We really know what we're doing.
54. **S6**: What are you guys talking about?

In line 41, S5 tracks down the cause of the group's difficulties. He framed the task differently, looking for a different type of justification for his mathematics. Instead of looking towards more and more detailed calculation as S3 and the group have been doing, S5 has now looked at the fit of the mathematics they are using with the physical system under consideration. The negative to positive infinity limits of integration do not match the finite span of the infinite well. S5 had shown inklings of this shift towards a Physical Mapping framing about ninety seconds earlier when, in an unquoted part of the transcript, he had asked which of the group's derivations were meant to correspond to distinguishable particles, fermions, and bosons. When the camera panned over to S5 directly after his pivotal comment in line 41, his calculator was not obviously positioned around him.

How does one justify calling S5's new mode of thought a significant reframing? Yes, he was using a different, non-calculational type of justification, but is there further evidence this shift was significant? Most importantly, the students give clues in their speech that indicate they feel S5 has done a different type of thinking than they have been doing. S3 chuckles in line 46 as she acknowledges S5's answer. This laughter could indicate several things about S3's thought. Perhaps, like many other instances of laughter, it indicates surprise or a violation of an expected action. S3 was expecting more and different types of calculation, and S5's new contribution fell outside of that expectation. Perhaps it is an embarrassed laugh. S3 is maybe a little ashamed of how she was temporarily blinded to this relatively straightforward solution. In either case, her laughter indicates that she feels she has been doing a different type of thinking than was needed. S5 also laughs as he pokes fun at himself and the group in lines 50 and 52.

Two other students react with sarcasm, a close cousin of the S3 and S5's laughter. S7 and S8 sarcastically compliment themselves and the group in lines 51 and 53. This sarcasm, regardless of whether it is more indicative of embarrassment or exasperation, indicates that S7 and S8 are also aware of the temporary blindness that has affected the group.

S6 provides a final piece of evidence that the group is itself aware of a shift in their thinking. He has been reading the text silently for most of this last snippet, but he asks the rest of the group what has just happened in line 54. His question suggests he has noticed the sudden change in the conversation's composition, the laughter and sarcasm described earlier. While he has missed the content of the shift, the change in tone that accompanies the other students' reframing still communicates "something different is going on here" to S6. The new tone communicates so strongly that S6 is compelled to explicitly ask what just happened.

Again, the focus at the end of this transcript illustrates a different framing from Calculation or Math Consistency. This third framing is Physical Mapping. It entails examining the interplay between the physical system at hand and the mathematics used to model it. This framing highlights how mathematics in physics class is only valid insofar as it reflects the physical system under study. Physical Mapping highlights still different components of a student's math sense, those focusing especially on the physical meaning behind numbers and their operations.

6.2.3 Summary of the Two Examples

In these two examples, we have observed effects of framing, a ubiquitous process in students' (or anyone's) thinking. Framing indicates that the student, or group of students, is temporarily focusing on a limited subset of her available tools and skills. Of all the common types of framings in this dissertation, Mathematica couples most strongly to Calculation.

In the first example, the students spent fifteen minutes calculating $\left\langle \frac{\partial H}{\partial \omega} \right\rangle$, persisting even through difficulties. Their work was neither naïve nor silly; their math sense was engaged. However, this Calculation framing, influenced by Mathematica's presence, highlighted certain aspects of their math sense at the expense of others.

The second example illustrates how Mathematica plays a role in providing feedback that encourages the students to remain in a Calculation framing. Again, their trouble doesn't come from lacking math sense but rather stems from applying that math sense narrowly towards computational issues. Mathematica continually reinforces this preference for calculation over other possibilities like mapping the mathematics to the physical system at hand or packaging and evaluating parts of expressions in a search for mathematical consistency.

Did S3 realize she had to calculate something and then reach for Mathematica? Or did the chain start the other way, with Mathematica being within her reach, causing her to look preferentially towards calculation? Given the place we chose to start providing transcript, the former perhaps seems the most likely. However, S3 had promptly announced she had brought Mathematica with her back when she entered the room ten minutes earlier. Then again, maybe she had been vaguely aware of the tendency of quantum problems to involve calculation when she was packing her bag back at home that morning. It's a chicken and egg dilemma that we are not interested in teasing apart.

The important theme of these case studies is that Mathematica is an active participant in how these students continually interpret and reinterpret their physics work, not merely a passive tool that offers them a convenient way to do whatever calculations they would have encountered on their own. This stickiness Mathematica gives the Calculation framing is a significant source of difficulty in and of itself, even when a robust math sense is present in student thought.

6.3 What Insights Does a Framing Analysis Give?

What is gained by using an epistemic framing language to describe how a symbolic calculator affects upper level physics students' thinking? This dissertation spent five chapters developing the epistemic framing analysis tool used in this calculator study. There are two main insights this tool offers towards the analysis of the students' thinking in this chapter. First, it highlights the evolving nature of students' thinking as their minds repeatedly take in new information, assess the situation, and decide (sometimes subconsciously) on the most suitable type of future action. It is easy to picture a tool like Mathematica playing an active role in such a dynamic cognitive system. Second, an epistemic framing analysis focuses on the activation of various bits of students' knowledge, not on whether they do or do not possess various bits of knowledge. Such a focus allows a researcher to make a more refined assessment of a calculator's impact beyond a blunt force it-stunts-math-sense argument.

6.3.1 Framing Highlights the Actively Updating Nature of Thinking

Framing is a cognitive process that the mind repeatedly updates. A person does not simply frame a new situation once at the beginning of the activity. That initial framing is repeatedly reevaluated and altered as the situation evolves. The person's thinking similarly evolves as time goes on.

There were examples of this continual updating of the framing process in the roadblock analysis of S3's work. In lines 5, 11-12, 25-26, and 28-30, S1 is made explicitly aware that the calculation she had tried was not satisfactory. Each of these roadblocks presents a possible updating point for S3's thought. Her solution in each case was to try a different type of calculation. While her overall framing didn't change, she did update her specific approach within Calculation several times. The behavior of others also influenced S3's thinking during the episode. When three other students offered more calculation suggestions in lines 18-24, those responses fed back into S3's thought and maintained her commitment to calculation.

Given such a dynamic view of students' thinking, it is relatively easy to conceptualize a symbolic calculator as an active input. Many other inputs exist, but Mathematica's input plays an especially strong role in determining the trajectory of a student's thinking.

How can one justify this "active" interpretation? Basically all physics teachers have anecdotal stories of their students tending to prefer calculation over other modes of thought like mapping mathematics to the physical systems at hand. Perhaps these students were simply following this general trend, and it just happened that Mathematica was there. The strongest evidence against this passive interpretation comes from lines 18 to 24 in the second episode. These lines have three other members of the group, in addition to S3, chiming in with suggestions on how to resolve the infinite result problem. All of these suggestions refer to using Mathematica in different ways. At least at this particular time, Mathematica has become utterly ingrained in the students' thinking. Mathematica made them aware of the diverging integral problem in the first place, and all four of their resolution strategies involved using Mathematica again.

This cycle where a calculator both alerts students to a problem and subsequently becomes part of the attempted solution as well can also be seen in this chapter's first example. S1 and S2 spent fifteen productive minutes getting Mathematica to efficiently calculate $\langle\psi_n|\frac{\partial H}{\partial\omega}|\psi_n\rangle$. Many calculation hurdles arose throughout those fifteen minutes, but the students found ways around them. All their hurdling methods, however, merely involved other calculation techniques. We conjecture that their thinking got stuck in a Calculation framing in part because of Mathematica's active role in the framing process. Again, a Calculation framing, in and of itself, is not a sign of unsophistication. The students in these case studies employ some advanced computational schemes and employ them quickly and efficiently. The unsophistication these students do show is mostly due to their hesitancy to look to alternate framings, to look for other arguments to support (or reject) their computational arguments. Chapter Seven will treat this point about looking for coherency among different framings in much more detail.

6.3.2 Framing Sees Activation, Not Acquisition, As the Critical Issue

When a student frames her math use in a particular way, her mind primes a subset of their vast store of mathematical resources. Given a certain framing, certain mathematical resources are much more likely to bubble up to conscious consideration. The reverse is also possible. An overall framing (like Calculation) can arise from the obvious cuing of certain mathematical resources (like how Mathematica explicitly focuses on integration schemes).

Much like the students in Chapter Five's case studies, the critical dynamic here is a framing-level issue. The two students in Chapter Five's gravitational work example were predominantly arguing over how to best approach their problem, not over some conceptual issue one understood while the other didn't. The two students in the $\frac{\partial}{\partial\hbar}$ example were similarly dancing among different framings as they argued about the best way to interpret their derivative.

While there are no such obvious arguments in this chapter's examples, framing considerations are very relevant. S3 isn't struggling because she is poor at calculation. She shows evidence of a strong math sense at several different times as she rearranges and simplifies her diverging integral. She and her partners come up with several different ways to incorporate Mathematica's power into their work. S3 struggles because of a framing issue. She is not activating a set of her mathematical resources that are more likely to help her make progress on this particular problem. Her formidable math sense is not being projected along a particularly useful axis.

S1 and S2's difficulties stem from a similar framing source. Their calculations are well constructed and efficient. An expert would be hard pressed to do better. Mathematica, again, has not wiped out their math sense. It has, however, disproportionately highlighted the calculational part of their math sense. Their success depends on their reframing their work, bringing other manifestations of their math sense to bear.

Thus, an epistemic framing lens offers a researcher several insights into the effects of symbolic calculators on physics students' thinking. This point does not imply that other theoretical lenses are not applicable. Metacognition, for example, could almost certainly play a role in helping us understand these students' thinking. These students do not step back and ask, at least explicitly, questions like "What exactly are you doing? Why are you doing it? How does it help you?" (Schoenfeld, 1985b). There is likely a connection between frequency of metacognitive events and flexibility of one's framing, but investigating such a claim is beyond the scope of this chapter.

Activity theory (Engestrom, 1987; Engestrom and Miettinen, 1999) advocates a much more socio-cultural analysis perspective. It holds that the use of tools, like Mathematica, is fundamentally a culturally determined process. Physics students tend towards calculation mindsets when they use Mathematica because that is how the tool is primarily treated among their social group. Again, we do not see this alternate perspective as orthogonal to the one we considered in detail. For conciseness, we have focused our analysis more at the individual cognitive level rather than the broader social level. A full treatment of the intersection of these two perspectives is also beyond the scope of this dissertation.

6.4 Further Results of Applying an Epistemic Framing Analysis Towards Examining a Symbolic Calculator's Effects on Thinking

I have spent the effort making a theoretical connection to framing for two reasons. First, it suggests a process by which Mathematica can help lead to the stickiness of the calculation tendencies that we have observed. This process, this framing the mind conducts, is much more general than some mental operation specific to a physics classroom alone. In a sense, it helps make the actions and shortcomings of the students in the earlier examples seem natural and reasonable. If our brains are indeed always involved in framing, always assessing situations, relating those assessments to groups of expectations, and allowing those expectations to limit our possible responses, then the temporary attentional blindness these students exhibit becomes a plausible error.

Second, using framing to help model student thought, like any scientific theory, affects subsequent hypotheses one makes about students' thinking. This influence is especially important in our real time interactions with students. As physics teachers, we will continue to encounter students using calculators and relying, at least in isolated episodes, too heavily on computation, as do the students in this chapter. How should we address this issue in our classrooms? Being aware of these framing effects at least highlights an alternate cause of students' trouble, beyond simple inability. They may possess the relevant knowledge to solve their problem but are being actively bracketed away from this knowledge by their focus on their calculator. An appropriate response by the instructor might be to search for a trigger to activate this latent knowledge. This does not imply that all difficulties our students encounter can be adequately addressed by helping them reframe the issue. Sometimes there are gaps in their understanding and more direct instruction methods are appropriate. This chapter is, however, arguing that framing issues are disproportionately often present when powerful calculators are involved because of the active role they assume in the dynamics of student thought.

In the interpretation of the events discussed here, Mathematica is an active participant in the students' framing of their approach. It provides feedback that encourages them to stay in a calculation mindset.

While this chapter argues that Mathematica is an active influence on students' thinking, it does not advocate its removal from the undergraduate physics curriculum. Such powerful calculators certainly speed up computation, and their graphing abilities can provide quick and detailed visualizations. The students in these case studies demonstrated impressive sophistication with respect to some aspects of its use, like efficient programming with quick debugging, treating it as a fallible tool whose precise computational scheme can be tailored, etc. The purpose in presenting the analysis in this chapter is twofold. To researchers, this chapter argues that Mathematica and similar calculators have the ability to drive students' thinking, often towards framing their activity as Calculation. To teachers, I hope that detailing this phenomenon will make us more sensitive to its occurrence in our classrooms. If Mathematica is indeed an epistemologically potent tool, there is no reason not to explicitly address its power in class. Rather than simply suggesting students use Mathematica and leaving it to exert whatever influence it defaults to with each student, we could explicitly model specific uses of the program, using Mathematica to explore a function's behavior, to quickly test physically meaningful cases, to merely confirm a mathematical conclusion instead of generating it, and so forth. All the while, if we talk with students about how we are using the program in a way other than straightforward calculation, we may help them begin to see how to integrate their use of Mathematica with other framings that can broaden their math sense and make it more effective. Making students more explicitly aware of Mathematica's potential roles in their thinking is a first step to their learning to harness its full power for themselves.

Chapter 7: Application of Framework Towards Characterizing Expertise

Chapter Six applies this dissertation's epistemic framing analysis towards describing the effects of a calculator on physics students' thinking. This analysis lens allowed a more nuanced description beyond the blanket claim that a powerful calculator simply shuts down physics students' math sense. Tools like Mathematica couple to the Calculation framing. Students' effort is projected along this calculation axis. They often demonstrate a powerful math sense in this framing, but sometimes struggle if the physics problem at hand is not especially suited for a calculation-based argument, one where solution strategies and their justifications depend on an algorithmically correct computation.

Chapter Seven now turns this framing analysis tool towards another research question. What insights regarding expertise in physics problem solving does this analysis lens provide? Certainly, physics experts value correctness in problem solving, but they also value the consideration of multiple lines of reasoning. Finding more than one way to solve a problem, finding several lines of argument that all point to the same answer, is very powerful indeed. The common framing clusters in this dissertation (Calculation, Physical Mapping, Invoking Authority, and Math Consistency) are refinements of what is meant by "lines of argument".

Briefly, a framing analysis will highlight how experts are especially adept at looking for coherency across several different arguments (i.e. epistemic framings) when solving a physics problem. Calculation, Physical Mapping, Invoking Authority, and Math Consistency are treated as subframings nested within a broader framing that values inter-argument coherency.

7.1 Brief Review of Two Foci on What Makes Experts Good Problem Solvers

Chapter Two detailed the two main strands of study in the physics and math problem solving literature. Each strand focuses on a different component of experts' problem solving ability. First, experts tend to have a larger and better organized knowledge bank. Second, experts are usually more adaptive and better in-the-moment navigators during the problem solving process. These two stands of research are briefly reviewed here, along with details on how they align with this dissertation's framing analysis.

7.1.1 Knowledge Breadth and Organization: Relatively Static Characteristics of Expertise

Experts are good problem solvers, in part, because they simply have more knowledge that is indexed efficiently (Maloney, 1994; Hsu, Brewe, Foster, and Harper, 2004). They are thus both more likely to be familiar with the relevant bit of knowledge required and more likely to be able to quickly retrieve it from long term memory.

What is commonly called an expert's "physical intuition" likely arises, in large part, from the breadth and efficient indexing of the expert's knowledge base

(Larkin, McDermott, Simon, and Simon, 1980). Upon reading a familiar type of problem, an expert can quickly, even subconsciously, recall a wide variety of similar problems and their results and respond in a way consistent with his stored examples. An expert's large knowledge base also allows him to categorize problems more efficiently according to general physics principles (Chi, Feltovich, and Glaser, 1981), again leading to faster, more complete solutions.

Even the most straightforward upper-level physics problems relate to a large amount of mathematical and physical information (Manogue, Browne, Dray, and Edwards, 2006). The standard mechanics problem of two coupled harmonic oscillators, for example, draws upon Newton's Second Law, a careful analysis of the spring forces to get the positive and negative signs correct, a solution method for coupled second-order differential equations, a vector space language for describing the completeness of the two normal modes, and so forth. Experts can quickly solve the two coupled oscillator problem due, in part, to their impressive knowledge bank. They have lots of experience with vector spaces, so the mathematical structure of the normal modes and the space of all possible motions that they span is easily recalled and applied. The mathematical mechanics of actually solving the coupled linear differential equations are so familiar to the expert physicist that the process is nearly automatic. Checking the signs of the spring force terms against the physical system is also a familiar operation for the expert, one that can likely be done very fluidly as well.

The epistemic framing analysis in this dissertation allows a parsing of the expert's access to this knowledge bank. Experts' experience and practice allows them to operate efficiently and fluently within a given framing like Calculation, Physical Mapping, Invoking Authority, or Math Consistency. Experts can calculate faster than novices. They can model a physical system mathematically with greater depth and ease. Experts can draw from a wider store of rules to invoke at opportune times. They have a wider base of mathematical knowledge and can hence draw comparisons to analogous bits of math more easily than novices.

7.1.2 Knowledge Use: Relatively Dynamic Characteristics of Expertise

Besides their wider scope of knowledge, expert physicists tend to be better in-the-moment navigators as they solve problems. Experts are more likely to realize they are on a dead-end path with respect to a given problem and are hence more likely to navigate away from that particular approach. Novices have a greater tendency to drift along in whatever problem solving current they happened to enter (Schoenfeld, 1992; Redish, 1999; Sabella and Redish, 2007).

The particulars of what exactly experts are doing when they navigate from moment to moment are difficult to define precisely. Some studies have attempted to map out these details in large problem solving prescriptions, including calls to "focus on the problem", "describe the physics", and "plan the solution". These prescriptions, however, tend to be overly linear when compared to expert thought (Heller and Reif, 1984) and not especially helpful in making novices better problem solvers (Huffman, 1997).

Nonetheless, expert physicists tend to have good instincts for recognizing one of their own (or a student who is at least on a path to become one of their own). An expert can watch a student working on a problem and almost can't help but form an opinion on how much or how little that student's behavior models an expert's. Some of this judgment can simply be attributed to whether the student is familiar with Topic X, as described in the previous section. Other parts of the expert's assessment of the student's performance, however, are more difficult to pin down.

This chapter will use an epistemic framing analysis on several episodes of upper level physics students' problem solving. These episodes all pass the expert's gut feeling test; they show students working in ways that most experts would approve. The framing analysis, however, will highlight a characteristic of these students' work that goes beyond the static knowledge-bank issues described earlier. In fact, this chapter's students will all be wrong, technically. Their knowledge banks seem to be failing them, at least in the strict sense of correct vs. incorrect.

These episodes will highlight fluency *among* different framings, as opposed to fluency *within* the different framings. The common thread running through all of this chapter's expert-like examples of students' thinking is that all these students are framing their math use in several different ways. That is, they are using several different types of mathematical warrants in building their arguments. Most importantly, these students are then looking for coherency among the results of their various framings. They do not simply disregard the result of one framing and move on to another framing.

I suggest that this tendency to both look for and eventually insist on coherency among different framings of the same problem comprises an important part of experts' in-the-moment navigation during problem solving. Experts have a super-framing of sorts, one that prioritizes this coherency across different arguments. The epistemic framing analysis is a tool that can concretize this idea of "different arguments" and make it more explicit.

7.2 Expert-Like Examples: Students Looking for Coherency Among Different Framings

We now turn to two examples of the work of upper level physics students that demonstrate a search for coherency among different epistemic framings. These students look to apply several different types of warrants in their mathematical arguments. The students are not always immediately successful, but this search for coherency among different strands of argument comprises an important component of expertise in physics problem solving.

7.2.1 Quickly Trying Three Different Framings Upon Encountering a Confusion

Our first example comes from a strong nontraditional student who had enrolled in PHYS 374, Math Methods, at the beginning of the semester. This student already held an undergraduate science degree and had spent several years in the workplace before returning to the university to study for another degree. Upon attending the first several classes, he discovered that he was already familiar with most of PHYS 374's content. He decided to look for an option to place out of the class, which was technically required for his major. As part of the agreement reached, he took that semester's final exam some months after the course ended.

When the student sat for this problem solving interview, he had already taken the exam but hadn't yet seen how it was scored.

In the interview, the student was given a blank copy of one of the exam problems he had worked on a few days earlier. This problem dealt with three-dimensional vector calculus and was designed with an eye towards the analogous Continuity Equation the students would soon encounter in their Electricity and Magnetism class. It read:

> In class, we derived the integral constraint that expressed the constraint that expressed the conservation of matter of a fluid: $-\frac{d}{dt}\int_\tau \rho d\tau = \int_{\partial\tau}(\rho\vec{v})\cdot dA$.
>
> Suppose that ρ describes the concentration in a solvent of a chemical compound that could be created or destroyed by chemical reactions. Suppose also that the rate of creation (or destruction) of the compound per unit volume as a function of position at the point $\vec{r}$ at a time t is given by $Q(\vec{r},t)$. Q is defined to be positive when the compound is being created, negative when it is being destroyed. How would the equation above have to be modified? Explain.

One good way to begin this problem would be to do a dimensional analysis. Both the $-\frac{d}{dt}\int_\tau \rho d\tau$ and the $\int_{\partial\tau}(\rho\vec{v})\cdot dA$ terms have dimensions of amount of compound divided by time. Q is already a rate, so there shouldn't be an additional time derivative involved. Integrating Q (which is a concentration as well) over the volume would give a dimensionally consistent third term for the equation: $\int_\tau Q d\tau$.

What relative sign should be given to this third term? One way to find out would be to consider the case where there is a source of the chemical inside the volume (so $Q >$ 0 by the problem's definition) but the total amount of solvent in the volume (i.e. $\int_\tau \rho d\tau$) is not changing in time (so $\frac{d}{dt}\int_\tau \rho d\tau = 0$). Chemical must then be flowing out of the volume, so $\int_{\partial\tau}(\rho\vec{v})\cdot dA$ is positive. Thus, the Q term goes on the left side with a positive sign:

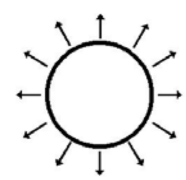
$$\int_\tau Q d\tau - \frac{d}{dt}\int_\tau \rho d\tau = \int_{\partial\tau}(\rho\vec{v})\cdot dA.$$

At the point where this clip picks up, Student 1 (S1) has already read through the problem and copied the main equation, $-\frac{d}{dt}\int_\tau \rho d\tau = \int_{\partial\tau}(\rho\vec{v})\cdot dA$, onto the blackboard and added a Q term to the equation giving,

$$Q(\vec{r},t) - \frac{d}{dt}\int_\tau \rho d\tau = \int_{\partial\tau}(\rho\vec{v})\cdot dA,$$

although he is not yet sure of that term's proper sign. He is not yet aware of the dimensional inconsistency of the way he included this Q term. He has also already drawn a sketch showing an outflow of chemical from a region of space, to which he will refer in the upcoming transcript:

1. **S1**: yeah the one thing I was confused about
2. on the exam and I continue to be confused
3. about it now, is the sign of this here, *writes "+/-" in front of Q*
4. like whether this is going to be a plus or a minus
5. because, rate of creation, so if it's getting created,
6. and then it's-Yeah, I'm not sure about this one,
7. about this sign.
8. **Interviewer**: OK, so if, let's say you pick the positive sign
9. **S1**: Right.
10. **I**: OK?
11. **S1**: Yeah.
12. **I**: What does that then entail, that you could go check,
13. try to check if it's right or wrong?
14. **S1**: Uhhh, yeah, if it's a, if it's a positive sign
15. then the right hand side has to increase *points to* $\int_{\partial\tau}(\rho\vec{v})\cdot dA$
16. because something is getting sourced
17. inside this volume. So for this to increase- *points to picture:*



18. Yeah, so it cannot be a positive, it has to be a negative,
19. because then that's going to increase-
20. for these signs to match, for the magnitude to increase, *points to signs in front of +/-Q and* $-\frac{d}{dt}\int_{\tau}\rho d\tau$
21. like these signs have to match, *Erases "+/-" and writes "-Q"*
22. so it's probably negative.
23. Although on the other hand, when I think of a source
24. I think of a positive sign and sink is a negative sign.
25. Yeah so that's where my confusion lies.

This clip begins with S1 acknowledging his confusion over the sign of the Q term. Lines 1 to 7 have him putting a "+/-" notation next the Q in his equation and noting how he wasn't particularly sure how to handle this issue several days before on the exam itself. The interviewer wanted to see how S1 would address this confusion, so he prompted S1 with a guess-and-check strategy. "Let's say you pick the positive sign…What does that entail, that you go check?" It so happens that the interviewer suggests the correct answer. Q should be positive, given the side of the equation on which S1 wrote it.

The noteworthy part of this clip concerns how S1 responds to the interviewer's suggestion. S1 tries to frame the question in several different ways, trying different mathematical warrants with each framing. He doesn't disregard the previous framings' results but instead looks for consistency among the answers he gets with these different framings. His confusion on the sign of *Q* persists, but it persists because he can't align the results his various framings. S1 is confused, but in a sophisticated way.

S1 exhibits an overarching framing, one that values coherency among multiple lines of reasoning. Physical Mapping, Calculation, and Invoking Authority can be seen as subframes nested within this larger coherency-valuing framing.

S1 begins with a Physical Mapping framing in lines 14 to 17. He argues that if there is a source of the chemical inside the volume (i.e. if *Q* is positive) "then the right hand side has to increase because something is getting sources inside this volume." He had previously spent (before the quoted transcript) nearly a minute describing how $\int_{\partial\tau}(\rho\vec{v})\cdot dA$ represented a flux, an outflow of chemical from the volume. S1 is arguing that if there is a source of the chemical inside the volume, then you'd physically expect more to flow out of the volume. He juxtaposes a mathematical expression (when he points to $\int_{\partial\tau}(\rho\vec{v})\cdot dA$) with a diagram-aided physical observation of more material flowing out of the volume.

S1 makes an expert-like move when he then turns to another type of argument to hopefully support the positive-*Q* conclusion of his Physical Mapping. His reframing is not complete. S1 is not about to simply disregard his previous reasoning in a Physical Mapping framing. He keeps his answer from the Physical Mapping framing (*Q* should be positive) on hold to compare with what his upcoming Calculation argument will give. Lines 18 to 22 have him quickly reframing the problem as Calculation. He shifts his focus to the arithmetic signs in front of the various terms in his equation: $\pm Q(\vec{r},t)-\frac{d}{dt}\int_{\tau}\rho d\tau=\int_{\partial\tau}(\rho\vec{v})\cdot dA$. He notes that computationally, a positive sign in front of the *Q* and a negative sign in front of the $\frac{d}{dt}\int_{\tau}\rho d\tau$ won't have the same effect with regards to increasing the $\int_{\partial\tau}(\rho\vec{v})\cdot dA$ on the right side. "For these signs to match, for the magnitude to increase, like these signs have to match, so [*Q*] is probably negative." A negative and a negative will "match" and can work together to change the value of the right hand side.

S1's expertise does not lie in the argument he constructs in his Calculation framing. Technically, his argument is flawed. A positive *Q* will increase the total value of the equation's left side regardless of the negative sign in front of the $\frac{d}{dt}\int_{\tau}\rho d\tau$ term. S1's expertise lies in the fact that he looked to Calculation in addition to Physical Mapping. He is framing the question in different ways, nesting these framings within a larger search for coherency. Unfortunately, his two framings have produced opposite answers, so he tries a third approach.

Lines 23 to 25 have S1 reframing his work again, this time as Invoking Authority. He quickly recalls a common convention in physics (and one quoted in the problem itself). "Although on the other hand, when I think of a source I think of a positive sign and sink is a negative sign". This line of reasoning would put a positive sign in front of Q, contradicting the result from his Calculation framing. Still unable to find a satisfactory coherence among his arguments, S1 finishes with "yeah, so that's where my confusion lies."

This example demonstrates an important component of expertise that an epistemic framing analysis can especially bring to the fore. On the one hand, S1 isn't showing much sophistication. He hasn't answered the question of the sign of Q, after all. On the other hand, S1 is demonstrating a very impressive component of expertise among physicists. He is approaching the problem from several different angles, trying out several different types of arguments. He is confused because he is searching for coherence among these different arguments, and he isn't finding it. Nonetheless, he is implicitly valuing this coherency. An epistemic framing analysis helps bring out this important component of expertise.

7.2.2 Using One Framing to Confirm the Results of Another

Our next example also comes from an interview of a student enrolled in PHYS 374, Math Methods. This student was a junior and not a physics major. His enrollment in PHYS 374 was not out of the ordinary, as that class fulfills a requirement for several applied mathematics and computer science degrees as well. Much like S1, this student exhibits an overarching value on coherency. He frames the question at hand in different ways, looking for coherence among his different arguments.

Since PHYS 374 devotes a large amount of attention to vector spaces, the interviewer asked this student a Cartesian vector problem.

> If $\vec{r} = 3\hat{x} + 4\hat{y} - 2\hat{z}$, how much of $\vec{r}$ is in $\hat{y}$?

That question was written on the blackboard alongside an analogous vibrating string question.

> If a string of length L is tied down on both ends and bent into a parabola shape given by $f(x) = -x(x-L)$, how much of the second normal mode, $\sqrt{\frac{2}{L}}\sin\left(\frac{2\pi x}{L}\right)$, is in *f*?

The student does not address this string problem in the transcript quoted below. A good way to answer either of these problems involves an inner product. Since $\hat{y}\cdot\vec{r} = 4$, one can say that four units of $\vec{r}$ are in the $\hat{y}$ direction. The question is somewhat ambiguous, and seeing how the student resolved the ambiguity was a part of what was being explored. This student works towards more of a proportion or fractional answer.

1. **S2**: All right, so I want to find for this one,
2. how much of r is in the y direction.
3. That goes back to, that'd be the projection of r onto y,

4. I guess I would say … first we'll set a vector for r.
5. That's 3,4, -2 equals r, and y is just 0,1,0. *Writes* $\vec{r} = \langle 3,4,-2 \rangle$ *and* $\hat{y} = \langle 0,1,0 \rangle$
6. So projection, we're doing—think it's
7. just r dot y over y dot itself.
8. I'll see if it makes any sense afterwards.
9. **Interviewer**: OK
10. **S2**: 3, 4, -2; 0,1,0 … So up here we just get *Writes* $\frac{\langle 3,4,-2 \rangle \langle 0,1,0 \rangle}{1}$
11. four over one, and that's just four. *Appends "* $= \frac{4}{1}$ *" onto above line*
12. Let's see, does that make sense to me?
13. See how much of r is in the y direction,
14. so, if we have … Oh, we have to do it over,
15. let's see, I think we have to do it over r dot r *Erases "1" in denominator and* $= \frac{4}{1}$
16. on the bottom here. So that would just
17. be 3,4,-2; 3,4,-2. That's four over 9 + 16 + 4 *Writes* $\frac{\langle 3,4,-2 \rangle \langle 0,1,0 \rangle}{\langle 3,4,-2 \rangle \langle 3,4,-2 \rangle} = \frac{4}{29}$
18. is 29. Four, twenty-nine, that doesn't really
19. make sense to me either. 4 over 29,
20. how much of r is in y direction… I'm thinking…
21. four, twenty-nine is about five, two,
22. think we have to square root this. *Inserts radical sign:* $\sqrt{\frac{4}{29}}$

S2 begins by framing his work on this problem as Calculation. He identifies this "how much $\vec{r}$ is in $\hat{y}$" question as referring to a projection operation in line 3. "That'd be the projection of r onto y." After quickly writing out expressions for the vectors $\vec{r}$ and $\hat{y}$ in line 5, S2 starts trying to produce an answer via computation. He struggles with finding the proper projection formula from which to start his calculation. The correct expression for the projection of $\vec{a}$ onto $\vec{b}$ is $\frac{\vec{a} \cdot \vec{b}}{\sqrt{\vec{b} \cdot \vec{b}}}$. S2 gets it nearly correct in line 7, where he forgets the square root in the denominator. He says the projection of $\vec{r}$ onto $\hat{y}$ is "just r dot y over y dot itself". Lines 10 and 11 have S2 realizing that his $\frac{\vec{r} \cdot \hat{y}}{\hat{y} \cdot \hat{y}}$ expression gives an answer of "four".

As an aside, note that the projection of $\vec{r}$ onto $\hat{y}$ is indeed four. Dropping a perpendicular down from $\vec{r}$ onto the y-axis will mark a segment along the y-axis that is four units long. Since $\hat{y} \cdot \hat{y}$ is one, S2's projection formula still works. S2 does not feel that "four" correctly answers "how much of $\vec{r}$ is in $\hat{y}$". As will become clear later in lines 24 to 40, S2 is looking for more of a fractional answer. He interprets the question as asking what fraction of $\vec{r}$ lies in the $\hat{y}$ direction.

S2's expectation of a fractional answer to "how much of $\vec{r}$ lies in $\hat{y}$?" leads him to try two further projection calculations. He tries calculating $\frac{\vec{r}\cdot\hat{y}}{\vec{r}\cdot\vec{r}}$ in lines 14 to 18. This projection calculation yields $\frac{4}{29}$. Still not satisfied, S2 decides to include a square root in his projection calculation in line 22, yielding $\sqrt{\frac{4}{29}}$.

The most striking part of these first twenty-two lines, however, is that S2 repeatedly indicates that he is thinking along a second thread in addition to the pure computation described earlier. This second thread relates to his expectation of a fractional answer, but that interpretation won't be clear until the next chunk of transcript. He first acknowledges this parallel track in line 8. After quickly outlining the calculation he plans to do, S2 notes "I'll see if it makes any sense afterwards." S2 goes on to repeat this interjection two more times. Upon finishing his "$\frac{4}{1}$" calculation in line 11, S2 says "let's see, does that make sense to me?" It is not clear what "make sense" entails in this moment to S2, however. He repeats the problem in line 13 while he thinks, but doesn't vocalize anything else before he apparently decides that $\frac{4}{1}$ doesn't "make sense". Line 14 then has S2 doing an amended computation that gives an answer of $\frac{4}{29}$. S2 then acknowledges his parallel line of thought a third time in lines 18 and 19 when he notes that $\frac{4}{29}$ "doesn't really make sense to me either." He then decides that a square root should be involved, again apparently by this still tacit "sense-making". He is nesting both his Calculation thread and his "sense-making" thread within a larger framing that values coherency among different arguments.

Wanting to hear more details of this parallel line of thought, the interviewer asks S2 to explain what made him insert a square root into his calculations.

23. **Interviewer**: What made you say that?
24. **S2**: I would just say that because I'm just looking
25. at the magnitude of r is the square root of r dot r.
26. So I would say that's why on the bottom we have
27. this magnitude of r. Up on top, what made me kind
28. of think the square root would be there just be
29. looking at it is saying—well, we're comparing.
30. We have 4 units in the y direction, and we basically
31. have 9 units total. This is just completely non-mathematical
32. whatsoever. But just saying, looking at it, I would say that,
33. the, like 4/29 is a lot smaller, than, yeah is a lot smaller

34. than just would seemingly make sense for something
35. that takes up, that's basically almost, the biggest component
36. of a function should not be less than a third of what
37. the amount of it is in that direction. So when I do this
38. it's approximately two, two over about 5.5, somewhere
39. between 5 and 6 for the square root on the bottom.
40. So I'll go with that for what's in the y direction.

S2 explains why he changed his projection formula from $\frac{\vec{r}\cdot\hat{y}}{\vec{r}\cdot\vec{r}}$ to $\sqrt{\frac{\vec{r}\cdot\hat{y}}{\vec{r}\cdot\vec{r}}}$, i.e. from $\frac{4}{29}$ to $\sqrt{\frac{4}{29}}$. The idea to use a square root seems to come from a sudden recollection that the magnitude of a vector is given by the square root of its dot product with itself. Why one would use the magnitude of $\vec{r}$ instead of $\vec{r}\cdot\vec{r}$ in the projection formula is not clearly articulated by S2. S2 gives no indication he thought it through completely. He likely decided he wasn't happy with $\frac{4}{29}$ (by the reasoning detailed in the next paragraph) and hence started looking for a way to change his answer. A quick recollection that square roots come up when one is talking about the magnitude of vectors probably led him to simply try square rooting. His verbalized explanation in lines 24 to 27 isn't really an explanation and hence supports this interpretation: "I'm just looking at the magnitude of r is the square root of r dot r. So I would say that's why on the bottom we have this magnitude of r."

S2 then goes on to explain why he thinks the numerator should be square rooted as well. In so doing, he provides evidence of what his earlier tacit "does that make sense to me?" line of thought included. He again signals that he sees, at least in this moment, this line of thought as being separate from his earlier calculation thread. "This is just completely non-mathematical whatsoever," he notes in lines 31 and 32. Since $\vec{r} = 3\hat{x} + 4\hat{y} - 2\hat{z}$, S2 notes in lines 30 and 31 that the y-direction has four out of the nine (3 + 4 + 2 = 9) total units in the vector. He then explains why $\frac{4}{29}$ "[didn't] really make sense to me either". In lines 33 to 37, S2 notes that the y coordinate is the largest of the three components of $\vec{r}$ and that "the biggest component…should not be less than a third". More than four twenty-ninths of $\vec{r}$ should lay in the y-direction. S2 goes on to explain that square rooting $\frac{4}{29}$ gives two over about 5.5. Since that fraction is greater than one third, S2 decides to "go with that for what's in the y direction" (line 40). This explanation of his tacit thinking aligns with his speech back in line 21 where he said "four, twenty-nine is about five, two".

We can thus reasonably say S2 was thinking about these proportions tacitly back in the first chunk of transcript when he was asking "Does that make sense to me?" He likely disregarded the $\frac{4}{1}$ answer because he was expecting his projection calculation to yield the fraction of $\vec{r}$ that is in the y-direction. Four is not a fraction. S2 similarly rejects his $\frac{4}{29}$ on proportional grounds. Four twenty-ninths is less than a third, and $\vec{r}$ is at least one third in the y-direction.

While most of S2's explicit work reflected a Calculation framing, this parallel line of proportional reasoning is more indicative of Physical Mapping. When a student frames his activity as Physical Mapping, his arguments rely on aligning his mathematics with the physical situation at hand. S2 is not especially concerned with the physical vector $\vec{r}$ itself. He has not drawn a picture to illustrate $\vec{r} = 3\hat{x} + 4\hat{y} - 2\hat{z}$, for example. He is, however, concerned with what the various fractions in play represent physically. How big is $\frac{4}{29}$? Is $\sqrt{\frac{4}{29}}$ a larger piece of the whole than $\frac{1}{3}$?

This episode thus has S2 framing his work as both Calculation and Physical Mapping. He looks for coherency between them, amending his computations until he gets an answer that represents a physically reasonable proportion of $\vec{r}$ that lies along the y-axis. This insistency on coherency among different framings is a mark of sophistication in S2's reasoning, much like S1's earlier work with the matter conservation equation.

Why is S2's proportional reasoning, which he finally explains in lines 30 to 40, taken as evidence of a different framing from his calculating? An expert could legitimately claim that one can't help but think of the relative sizes when fractions like $\frac{4}{29}$ and $\frac{1}{3}$ come up in thought. Why label S2's explanation of these relative sizes "Physical Mapping" and then praise his expert-like search for coherency among framings when all he's doing is comparing $\frac{4}{29}$ and $\sqrt{\frac{4}{29}}$ to $\frac{1}{3}$? Isn't that really closely tied to the argument he was making in his Calculation framing?

The critical issue is that S2, himself, gives repeated clues that he sees his computations and his fractional reasoning as two distinct lines of thought in this particular moment. He begins by outlining his computational scheme in lines 4 to 7 before noting "I'll see if it makes any sense afterwards" in line 8. After doing his first projection calculation and getting $\frac{4}{1}$ he echoes his earlier statement. "Let's see, does that make sense to me?" (line 12). He gives no detail on his sense-making evaluation before altering his calculation to produce $\frac{4}{29}$. S2 then again signals his second strand of thought with "that doesn't really make sense to me either" in lines 18 and 19, and amends his calculation again to get $\sqrt{\frac{4}{29}}$. When asked to explain his "sense-making" line of thought, S2 gives an even stronger signal that he sees his proportional reasoning as separate: "this is just completely non-mathematical whatsoever" (lines

31 and 32). Granted, an expert might often treat calculation with fractions and thinking about their relative physical sizes as inseparable, but S2 is clearly distinguishing between them in this episode.

This episode thus illustrates an important feature that an epistemic framing analysis can contribute to characterizing expertise. It does not rely on external, normative judgments of what type of things are appropriate to think about in a given problem. Nor does this framing analysis focus on the correctness of the student's math use. S2 misremembers the projection formula. He also consciously separates a fraction calculation from thinking about those fraction's relative sizes, where an expert might be much more likely to do both automatically. The critical idea of this framing analysis is that it highlights the value S2 is placing on coherency among different lines of thought. This insistence on coherency, this super-framing that nests (in this case) Calculation and Physical Mapping within it, is an important component of expertise that can be discussed in addition to considerations of technical correctness or aptitude. The earlier clip of S1 had a similar distinction between technical correctness and coherency searching.

7.3 Novice-Like Examples: Getting Stuck in a Certain Framing

We now turn to several examples illustrating the other end of the spectrum. The previous two examples had students explicitly expecting to find coherence among several different strands of mathematical argument. They fluidly switched among different framings as they attempted to reinforce one type of argument with another. This commitment to coherency is an important component of expertise in physics problem solving, a component that can be discussed in addition to the breadth and veracity of a students' knowledge. Novice problem solving more frequently has students getting stuck in some framing, persisting with a given approach to a problem for too much time.

"Too much time" needs an important clarification. If you tell a first-year physics student and a professor to solve $\frac{dx}{dt} = v_o + at$, of course the professor will arrive at $x(t) = x_o + v_o t + \frac{1}{2}at^2$ sooner than the student. Has the first-year student spent "too much time" in Calculation? No, he hasn't. Experts have much more problem solving experience than novices, and this extra experience leads not only to a more comprehensive store of knowledge but also to a more efficient organization of that knowledge. Even apparently simple physics problems can involve complicated ideas that the expert has long ago packaged into easily recallable chunks, whereas the novice must, through no fault of his own, spend much more time thinking carefully through these ideas (Redish, Scherr, and Tuminaro, 2006).

"Too much time" must then entail something besides a simple clock reading. The particulars of the given situation must be taken into account. Specifically, there has to be some sort of prod away from the students' current framing. Perhaps a student is trying and trying to quote a rule, while a fellow student's bids to reframe their work as Physical Mapping go apparently unnoticed. The stronger and more repeated these reframing bids are, the more the first student can be said to be "stuck" in Invoking Authority.

7.3.1 *An Example from Introductory Physics*

This first example of students getting stuck in a certain framing comes from an introductory algebra-based physics class. The group of students is meeting outside of class to work on that week's homework. They are meeting in a room that is staffed by their course's teaching assistants during normal working hours. The idea is for the students to naturally form groups and help each other while the TA offers occasional guidance.

An earlier portion of this episode was included in an older study on introductory physics students' uses of mathematics (Tuminaro, 2004). While these students' work is naïve compared to most of this dissertation's upper level physics majors, this episode is included in an attempt to illustrate the extreme of getting stuck in a certain framing. An example of more advanced students having trouble reframing their work will follow.

The introductory students in this episode are working on a question about air pressure.

> Use the physics we have learned to estimate the difference in air pressure between the ceiling and floor of your dorm room. Be sure to clearly state your assumptions and how you came to the numbers you estimated. (You may take the density of air to be 1 kg/m^3.)

Each one of their weekly homework assignments included an estimation problem of some sort. The purpose of these estimation problems is to help students learn to make quick order of magnitude calculations while providing reasonable justifications for their estimates. These Fermi-like problems were also meant to send an epistemic message: Physics is not just about formula chasing. Many times physicists have to make reasonable estimates of a real world situation that is simply too complicated to admit a straightforward plug-and-chug answer.

A reasonable answer to this problem could begin by noting the extra air pressure at the room's floor is due to the weight of the air between the floor and ceiling. A typical dorm ceiling is about three meters high since a tall person is about two meters and you can imagine stacking one and a half of them to reach the ceiling. We can imagine laying down these same tall people around the room's perimeter to get a length and width of about five meters each. That gives a volume of about 100 m^3. Since the density of air is about 1 kg/m^3, the air has a mass of 100 kg and hence weighs about 1000 N. Spreading this weight of air out over the floor's area gives about a 40 Pa pressure difference between floor and ceiling. (Or, they could use the formula for the increase of pressure with depth, $P = P_0 + \rho g d$.)

These students, however, have begun their work by selecting a formula with "pressure" in it that turns out to be irrelevant: $PV = nRT$. In the minutes leading up to this clip, they have been trying to decide on values to use for the various quantities in this equation so they can calculate a number for P, the pressure. A teaching assistant has come over to the group and follows them along their chosen path for a while. He either does not notice their $PV = nRT$ approach cannot answer the question or merely decides to let them come to that realization on their own. The question that concerns this clip is found in line 3 below. How can we find the volume of the dorm room? He follows their Calculation framing attempt to find the volume before attempting stronger and stronger reframing bids. These bids to reframe their math use as Physical Mapping go unnoticed by the students as they continue along in their Calculation framing.

1. **TA**: Normally, what is a room? Room temperature.
2. Ok, so you know the temperature you know
3. this constant R. Do you know the volume?
4. **S3**: No.
5. **S4**: No, we can find it though.
6. **TA**: How?
7. **S4**: We're doing mass over density,
8. but we need to know the mass.
9. **S3**: Oh, duh!
10. **TA**: So, you would need to know the mass. OK.
11. **S3**: One kilogram per...
12. **TA**: It's says um consider a dormitory room and
13. they tell you the density, so you're saying if all—
14. if I knew the mass I could find the volume.
15. **S4**: Right.
16. **S3**: Well, it's one kilogram per meters cubed,
17. so it's kind of easy.
18. **TA**: That's the density?
19. **S3**: Yeah, but that's kind of giving it to us easy,
20. right, 'cause it's
21. **S4**: It's saying that mass is one, one kilogram.
22. Is that what you're saying?
23. **S3**: One kilogram per one meter cubed.
24. **TA**: Right, so if you lived in a room that was this big, *Gestures to a $1m^3$ size*
25. one meter cubed, there would be one kilogram of air there.
26. **S3**: Yeah.
27. **TA**: I don't think you live in a room that big.
28. **S3**: Yeah, I feel silly. OK. So, it's one kilogram.
29. **TA**: So, what um
30. **S4**: So, the mass is one kilogram, is what you're saying?
31. **TA**: Would you agree with me this is an estimation problem?
32. **S3**: Um.
33. **S4**: Yes.
34. **TA**: OK.
35. **S3**: To a certain extent, yeah.
36. **TA**: What this problem is about is a dorm room.
37. How big is a dorm room?
38. **S3**: Oh!
39. **S4**: Not big at all.
40. **S3**: He gave it in another problem, like another homework.
41. **TA**: So, let me ask you another question. You're trying

42. to figure out what is the mass so you can find the volume.
43. Is there another way that you could just tell me
44. what the volume is?
45. **S3**: One meter cubed.
46. **S4**: Yeah.
47. **TA**: That's this big. We're talking about a dorm room. *Gestures to a 1m³ size*
48. What's the volume of a dorm room?
49. **S3**: What would make it—what would make everything one?
50. Oh! Would it be a hundred? Or, a thousand?
51. **S4**: A thousand that's really big.
52. **S3**: A hundred maybe?
53. **TA**: Alright, ok.
54. **S3**: 'Cause I'm looking what's going to make this equa—
55. what's going to make this num—what's going to make it one.
56. You know.
57. **TA**: Let me jump in again. You're trying to make
58. numbers work out, instead of thinking. Just stop and
59. think for a moment.
60. **S3**: OK.
61. **TA**: What's the volume of a room?

The TA's words in lines 1 to 3 give a good sense of what has occupied the group for the several minutes before the quoted transcript. They have identified the Ideal Gas Law, $PV = nRT$, as an equation containing pressure and have been listing its terms, checking off the ones for which they have numeric values. R is a known constant. T stands for the temperature, which is presumably known to be room temperature. The TA points out that they haven't yet found a value for the volume, V, so they can't yet use $PV = nRT$ to directly find a value for pressure (never mind that such a value wouldn't actually answer the question of the pressure difference between floor and ceiling).

The students have a ready response for finding the volume of the dorm room. S4 notes that they can find the volume via the density formula, $\rho = \frac{m}{V}$. The volume is "mass over density, but we need to know the mass" (lines 7 and 8). This pattern of equation chaining is a common indicator of a Calculation framing. We need X to find Y, but we can find X via its relation to W and Z, etc. Correctly manipulating these chains of calculations should yield an answer that can be trusted. The TA acknowledges this Calculation framing in lines 13 and 14 when he repeats the students' proposed calculation chain.

S3 and S4 are thus looking for the mass of the air in the room so that they can use the known density of air to get the volume of the dorm room. S3 proposes a method in lines 16 to 17 that is difficult to understand. She says, "Well, it's one kilogram per meters cubed, so it's kind of easy." The TA is confused at this statement, and he responds in line 18 with a question, "That's the density?" S3 is most likely trying to find simple numbers that will make her estimation calculations easier. Since the number "one" is very easy to deal with computationally, her attention was drawn to the quantity of one kilogram per cubic meter.

Several pieces of evidence support the inference that S3 is focusing on "one kilogram per meters cubed" because of the computationally easy "one". First, she will echo this search for computationally easy numbers later in lines 49 and 50. "What would make [the volume]—what would make everything one? Oh! Would it be a hundred? Or, a thousand?" These lines make it especially obvious that her thinking is dominated by a search for computationally easy numbers, at least during lines 49 and 50. It is not unreasonable to assume a similar type of argument is at work back in lines 16 and 17. A second piece of evidence for this easy-number interpretation of lines 16 and 17 comes from its alignment with the teaching in her class. Since S3's class always has an estimation problem on every week's homework, the instructor devotes considerable effort to demonstrating estimation techniques. One common one he illustrates is to choose numbers that are computationally easy. S3 is likely trying to mimic a technique she has observed her professor do many times.

The third and strongest piece of evidence that S3 is framing her work as Calculation and hence searching for numbers that will be computationally easy to deal with comes from her response to the TA's next comment. S3 and S4 talk back and forth in lines 19 to 23 about using the "one" before the TA makes a bid to consider what "one kilogram per meters cubed" means physically. "Right, so if you lived in a room that was this big, one meter cubed, there would be one kilogram of air in there" (lines 24 to 25). He gestures by marking out a one cubic meter volume with his hands. After S3's noncommittal "yeah", the TA continues with "I don't think you live in a room that big" (line 27). S3's immediate response is to say "Yeah, I feel silly. OK. So, it's one kilogram." Clearly, a miscommunication has taken place. The TA's comment about what a density of one kilogram per cubic meter would correspond to physically, coupled with his gesture about the physical size of a cubic meter and his comment on the size of a dorm room, constitutes a bid for Physical Mapping. S3 hears the TA's dismissal of her "one kilogram per meters cubed" answer and "feels silly," but she is not jarred sufficiently out of her Calculation framing. She does not adopt the TA's Physical Mapping bid and instead responds with a different label on the "one" that she wants to calculate with, stating that "OK. So, it's one kilogram" (line 28). Had S3 been framing her math use as Physical Mapping back when she originally proposed the confusing "one kilogram per meters cubed" answer for the mass of the air in line 16, she would have likely responded more coherently to the TA's Physical Mapping comment in lines 24 to 27. Their miscommunication around lines 24 to 27 is good evidence of a framing mismatch, as has been seen many other times throughout this dissertation.

After a few further moments of miscommunication in lines 29 and 30, the TA makes another bid to pull S3 and S4 away from their Calculation framing, or at least to get them to consider an alternate line of reasoning in addition to their computational scheme. "Would you agree with me this is an estimation problem?" (line 31) He follows with "what this problem is about is a dorm room. How big is a dorm room?" (lines 36 and 37) These lines constitute another, stronger push from the TA to get S3 and S4 to reframe their math use as Physical Mapping. Such a framing would have the students justify their estimates by reference to the physical size of their dorm rooms. S3 responds to the TA's newest bid by getting a new idea, as evidence by her quick "Oh!" exclamation in line 38. She remembers that the professor "gave [the size of a dorm room] in another problem, like another homework" (line 40). This one line is more indicative of an Invoking Authority framing instead of her earlier Calculation framing. S3 suggests taking the size of her dorm room from an authoritative source, from a previous homework question that included the size of a dorm room as data. While it is encouraging that S3 has been momentarily shifted away from her Calculation framing, this Invoking Authority approach neither aligns with the TA's Physical Mapping bid nor provides an obvious way to tie in her previous line of thought in a search for coherency among framings.

The TA quickly responds with an even stronger bid for Physical Mapping in lines 41 to 44. "So, let me ask you another question. You're trying to figure out what is the mass so you can find the volume. Is there another way that you could just tell me what the volume is?" He restates the Calculation framing's approach of chaining equations they've been taking to find the room's volume before asking the students, again, to think of another approach. When they again fall back to an easy-number answer of "one meter cubed" in lines 45 and 46, the TA again tries to draw their attention to a Physical Mapping framing. "That's this big (gestures to one cubic meter). We're talking about a dorm room. What's the volume of a dorm room?" (lines 47 and 48)

S3 still does not shift to thinking about her physical experience with the size of dorm rooms. Her robust Calculation framing again distorts her interpretation of the TA's question. This Calculation framing causes her to hear the TA's latest Physical Mapping question in lines 47 and 48 as asking her to choose better, more convenient numbers for computation. She responds "What would make it—what would make everything one? Oh! Would it be a hundred? Or, a thousand? … 'Cause I'm looking what's going to make this equa—what's going to make this num—what's going to make it one?" (lines 49 to 55)

The TA's next comment provides one more piece of evidence that he is not satisfied with the students' line of thinking, i.e. he is trying to get them to reframe their work. He gives his strongest reframing bid yet for breaking them out of Calculation in lines 57 to 61. "Let me jump in again. You're trying to make numbers work out, instead of thinking. Just stop and think for a moment … what's the volume of a room?" Soon after this last prompt, the TA simply tells the group that their $PV = nRT$ equation won't help them answer the original question about the pressure difference between floor and ceiling, so the conversation's topic takes a major turn. We end this episode here.

These introductory students thus illustrate a striking case of getting stuck in one framing of their math use. "Getting stuck" is defined with respect to missed bids for reframing, not with respect to a simple clock reading. That S3 and S4 spent essentially all their time in this clip in Calculation is not relevant. What is relevant is that the TA made approximately half a dozen bids towards Physical Mapping, trying to get S3 and S4 to think about the numbers they were using with respect to a physical dorm room. While several of the TA's comments communicated "No, your answer is incorrect" to the students, the Physical Mapping reframing bid itself went unnoticed. The students sometimes changed their answers but never adopted a Physical Mapping framing. Contrast this framing stubbornness with the fluid, coherency-seeking frame shifting of S1 and S2 earlier in this chapter. All of this chapter's students are giving wrong answers, but this fluid search for coherency among different framings highlights an important component of expertise.

7.3.2 An Example from Upper Level Physics: A Student Getting Stuck in Calculation

The last example from introductory students' work was meant to illustrate an extreme case. S3 and S4 persisted in their Calculation framing despite repeated, ever-louder bids for Physical Mapping from the TA. We now turn to a case from upper level physics students. The students in this example, most notably S5, also get stuck framing their work as Calculation.

Again, "get stuck" must mean something besides taking a clock reading of their calculating time and judging it to be too long. There must be some sort of easily identifiable perturbation, some reframing bid or opportunity, that passes by. In the $PV = nRT$ example, the TA was providing very noticeable calls for Physical Mapping. The perturbations to these upper level students' Calculation framing will be somewhat more subdued but still offer reasonable markers for our analysis.

This episode was analyzed in detail in Chapter Six. That chapter gave a detailed description of how the presence of a powerful symbolic calculator like Mathematica tended to couple to a Calculation framing. We will not repeat that detailed symbolic calculator analysis here. An important part of the earlier analysis that is relevant here is the idea of a "roadblock". When a student hits a roadblock, her current line of thinking has either come to a result the does not satisfy her or has become too complicated to justify continuing. S5 hits four roadblocks in the transcript below. Each roadblock offers a reframing opportunity, but S5 persists through all of them in Calculation.

This transcript comes from a video recording of six junior and senior physics majors meeting to work on their homework for a second semester undergraduate quantum mechanics class. They are working on Problem 5.6 in <u>Introduction to Quantum Mechanics</u> (Griffiths, 2005). The problem asks them to calculate $\left\langle (x_1 - x_2)^2 \right\rangle$ for two particles in arbitrary stationary states of a one-dimensional infinite well, where x_1 is the coordinate of the first particle and x_2 is the coordinate of the second. Three successive parts of the problem ask them to assume the particles are distinguishable, identical bosons, and identical fermions. In the course of this calculation, the students realize they need to evaluate $\int x_1^{\,2} \left|\psi_n(x_1)\right|^2 dx_1$. This notational shorthand, which doesn't specify the limits of integration, is taken from the hints the text gives in the pages preceding this problem. The transcript begins with a student in the group explicitly mistaking the limits of integration to be from negative infinity to positive infinity instead of just over the width, L, of the well. They are thus led to try to evaluate $\frac{2}{L}\int_{-\infty}^{\infty} x^2 \sin^2\left(\frac{n\pi x}{L}\right) dx$.

1. **S5**: The integral is from negative infinity
2. to infinity, right?
3. **S6**: Yeah.
4. **S5**: So we have x squared *Types in Mathematica*

…one minute later…

5. **S5**: It's telling me it doesn't converge. *Sets Mathematica aside,*
6. What if I tried *begins trying to integrate by parts with pencil and paper*
7. **S7**: So what's the integral equal to?
8. **S5**: It wasn't happy, so let me just try something else.
9. **S7**: Oh, we got undefined?
10. **S5**: It said it didn't converge.

…ten seconds later …

11. **S5**: I mean, this is an integral
12. that's quite do-able by *Brings back computer with Mathematica*
13. **S7**: trig substitution
14. **S5**: by parts
15. **S7**: oh, by parts
16. **S6**: Yeah.
17. **S5**: So *Starts typing again*
18. **S8**: Can you break it up into
19. different parts and then do it on a TI-89?
20. That's what I usually do, a combination
21. by hand, by calculator.
22. **S7**: Well, integrate it indefinitely and plug in.
23. **S9**: Are you not substituting
24. a value in for n and L, or are you?
25. **S5**: Umm, no, but I just tried doing
26. x-squared, sine of x squared, and it's not happy.

… S5 spends 2 minutes using Mathematica to produce the antiderivative of $x^2 \sin^2 x$ *and copies that antiderivative onto her paper …*

27. **S5**: I can see why it says that doesn't converge.
28. **S6**: Yeah, but I know it…we've done it.
29. **S10**: We're like, but I know it does.
30. **S6**: We've done that integral so many times.
31. **S5**: Find me one, cause see, this [indefinite integral]
32. **S6**: Yeah.
33. **S5**: Is equal to that [antiderivative], *points to paper with antiderivative*
34. and so you know there's a whole number of places
35. where it'll shoot to infinity.

36. **S6**: Like, how else do we find
37. the expectation value of x-squared?
38. **S5**: Yeah.
39. **S6**: Like, I know we've done it
40. for the infinite square well. *S3 starts paging back through textbook*

S5 begins her attempt to evaluate $\frac{2}{L}\int_{-\infty}^{\infty} x^2 \sin^2\left(\frac{n\pi x}{L}\right)dx$ by framing her activity as Calculation. Correctly following an algorithmic set of computationally steps (or having Mathematica do them for her) should lead to a trustable result. This framing is an almost obviously appropriate one to try first. The goal, after all, is to compute the value of an integral. S5 thus begins typing the integral into Mathematica in line 4.

The first noticeable roadblock occurs a minute later at line 5. Mathematica has returned the (correct) evaluation of $\frac{2}{L}\int_{-\infty}^{\infty} x^2 \sin^2\left(\frac{n\pi x}{L}\right)dx$ as infinity. This roadblock offers a reframing opportunity to S5. She realizes her Mathematica computation does not match what she expected, and she now has the opportunity to try a different way of producing and justifying an answer. No reframing occurs, however, as she sets Mathematica aside and begins trying to perform a manual integration by parts in line 6. She continues framing her work as Calculation, still looking to produce an answer by means of a technically correct computation. The only difference is that she is now doing the computational steps herself.

S5 quickly decides that a manual integration by parts would be too complicated around line 12. This decision marks a second roadblock. Two attempts in a Calculation framing have now failed, at least in S5's eyes. She again chooses another approach, but again no reframing occurs. Line 12 has her bringing Mathematica back onto her desk and trying to have it evaluate a simpler form of $\frac{2}{L}\int_{-\infty}^{\infty} x^2 \sin^2\left(\frac{n\pi x}{L}\right)dx$, specifically $\int_{-\infty}^{\infty} x^2 \sin^2(x)dx$. This newest approach is still an attempt to find and justify an answer via a technically correct computation. S5 is still framing her work as Calculation.

Calculation is certainly not the only possibility for this problem. A Physical Mapping framing would have the student examining how the integral in play, $\frac{2}{L}\int_{-\infty}^{\infty} x^2 \sin^2\left(\frac{n\pi x}{L}\right)dx$, aligns with the physical situation of the problem, that of a particle in an infinite well of length L. The limits of integration do not match the spatial boundaries that confine the particle, which is causing the infinite result. The proper integral to consider is $\frac{2}{L}\int_{0}^{L} x^2 \sin^2\left(\frac{n\pi x}{L}\right)dx$.

S5 faces a third roadblock in line 26 when Mathematica correctly informs her that $\int_{-\infty}^{\infty} x^2 \sin^2(x)dx$ diverges as well. She responds to this third failed attempt in a

Calculation framing with yet another computational approach. This time she follows the suggestion of two of her classmates, S7 and S8. She has Mathematica produce the antiderivative of $x^2 \sin^2 x$ which she then copies onto her paper before trying to evaluate it at the positive and negative infinity limits.

Two minutes later, S5 announces that she "can see why [Mathematica] says that doesn't converge" (line 27). There are several places in the antiderivative of $x^2 \sin^2 x$ that will blow up if she tries to substitute $x = \pm\infty$. She faces a fourth reframing opportunity when S6 and S10 point out that their integral must converge because "we've done that integral so many times." S5 responds to this fourth roadblock by keeping her faith in her computation which, after all, is indeed correct. She challenges S6 to find an example of the "so many times" they've supposedly done this integral and proceeds to summarize her latest computation for him in lines 31 to 35. S6 responds by noting the problem of finding $\langle x^2 \rangle$ for a single particle in an infinite well. That old problem led to an integral that was identical to the current one and which did not have an infinite result.

S5 can thus be said to be "stuck" in a Calculation framing in a similar, if less striking, way to the introductory students S3 and S4. The prompts for S5 to reframe her work are not as obvious as a TA verbally lobbying for Physical Mapping, but she does face four places where her computation did not give the result she was expecting. No reframing occurs around any of these four roadblocks as S5 merely tries different types of computations.

7.4 Chapter Summary

All of the students in this chapter on expertise are technically wrong. S1 does not find the correct sign to put in front of his Q term. S2 misremembers the projection formula. S3 and S4 are working with an equation that couldn't possibly lead to a reasonable answer to their pressure difference homework question. S5 slaves over an integral that is irrelevant to her homework question.

As summarized at the beginning of this chapter, and detailed in Chapter Two, there are two main threads of research on expertise in physics and math problem solving. First, experts have larger and better-organized banks of knowledge. Second, experts are better in-the-moment navigators during the problem solving process. Since all the students in this chapter are wrong, it could be said that their knowledge banks are failing them.

S1 and S2, however, still display a hallmark of expertise that S3, S4, and S5 do not (at least in these given episodes). S1 and S2 consciously frame their math use in different ways, looking for consistency across the different arguments their reframings produce. They display a larger, overarching framing that values this coherency and can hence nest Calculation, Physical Mapping, Invoking Authority, and Math Consistency within it. S3, S4, and S5 display a stubborn commitment to a single frame, even in the face of reframing bids or opportunities. This dissertation's epistemic framing analysis tool, then, provides a window for investigating the in-the-moment navigation component of expertise in physics problem solving. An overarching framing that values coherency among different lines of reasoning is an important component of expertise, one that can be discussed independent of the strict correctness or incorrectness of students' reasoning—an important message for both physics instructors and physics education researchers.

Chapter 8: Dissertation Summary and Future Directions

This dissertation has developed an analytic framework for analyzing how upper level physics students frame their math use. Focusing on the mathematical justifications these students offer allows for a moment-to-moment description of how they are interpreting the mathematics at hand. Do they presently see their math as being about calculation? Are they examining the physical referents of their equations? Are they seeing math as a vehicle to invoke a rule or previous result? Are the students thinking about how their math fits in with other analogous mathematical ideas?

The main benefit of a framing analysis is that it makes explicit a vital conversational dynamic that, despite its importance, is very often implicit in normal conversation. Many examples in this dissertation have illustrated how students' work, especially their arguments and debates, tend to be a series of framing pushes and pulls as they try to establish a common framing, a common way of approaching the problem at hand. Miscommunications can often be traced to two students framing their task differently, each focusing on a sufficiently different subset of their mathematical resources as they try to make sense of the math in play.

Before turning to a more extended discussion of this dissertation's implications to both physics education researchers and physics teachers, let us return to our four hypothetical professors from Chapter One.

8.1 Professors Alpha Through Delta and Their Different Framings

Recall the four professors who were each going to teach a section of their university's introductory, calculus-based physics course. They had decided to put special emphasis on the role of mathematics in physics. When each was preparing for the first lecture of the semester, the equation $x_f = x_o + v_o \Delta t$ quickly appeared in the lecture plan. For the convenience of the reader, each professor's musings as they planned how they would interpret this equation for their students are repeated below.

Professor Alpha looks at $x_f = x_o + v_o \Delta t$ and thinks to himself, "All right, that equation encodes a calculation scheme. If v_o is 4, Δt is 2, and x_o is 3, then that equation tells us how to calculate x_f. It's just 4*2 + 3." He plans on working a few sample calculations for his class and refreshing them on some simple algebra techniques. If you wanted to solve $x_f = x_o + v_o \Delta t$ for Δt, for example, there is a certain algebraic order of operations that must be observed. First subtract the x_o from each side and only then divide by v_o.

Professor Beta has a different reaction when $x_f = x_o + v_o \Delta t$ appears in her lecture plan. She sees that equation and plans to show her students how appropriate uses of math in physics correctly model whatever physical system is at hand. Dr. Beta plans on talking with her class how $x_f = x_o + v_o \Delta t$ encodes a physical idea. Velocity is how far an object moves for every, say, one second. The quantity $v_o \Delta t$ is how many seconds' worth of motion you're dealing with. Tack that distance traveled onto x_o, which is where you started from, and you'll have where the object must end up, x_f.

Professor Gamma thinks something still different when he's sitting at his desk, planning his explanation. "Oh, the point of $x_f = x_o + v_o\Delta t$ is that it's a convenient rule for kinematics," he muses. "There are several other rules too, like $x_f = x_o + v_o t + \frac{1}{2}at^2$ and $v_f^2 = v_o^2 + 2a\Delta x$. I'll present these various rules and talk with my students about how important it is to make sure you're quoting a rule that is applicable to your current problem. $x_f = x_o + v_o\Delta t$, for example, is only true if your acceleration is zero." Professor Gamma also plans on talking about how math, in general, provides a convenient and time-saving system for physicists. No one, practically speaking, starts every physics problem from absolute first principles every single time. Physicists sometimes take shortcuts, quoting previously packaged mathematical results. Mathematics is powerful, in part, because it allows such packaging.

Professor Delta's mind goes in yet another direction when she realizes that $x_f = x_o + v_o\Delta t$ is going to come up in her lecture. "The great thing about using math in physics," she thinks to herself, "is that you get this whole big web of interconnected math ideas. Math gives a formal, logical structure that connects superficially different applications. I'm going to emphasize to my students how $x_f = x_o + v_o\Delta t$ fits in with a web of other math ideas." Dr. Delta plans on talking about how $x_f = x_o + v_o\Delta t$ can be derived from the definition of average velocity: $\langle v \rangle = \frac{\Delta x}{\Delta t}$. She also wants to note how $x_f = x_o + v_o\Delta t$ has a base-plus-change structure to it just like, for example, $v_f = v_o + a\Delta t$. Stepping way back, $x_f = x_o + v_o\Delta t$ is a solution to a general class of differential equations: $\frac{d^2x}{dt^2} = k$.

Professors Alpha through Delta reflect the four common framing clusters that emerge from this dissertation's analysis of upper level physics students and their use of mathematics in physics problem solving. Each professor focuses on a subset of his or her total store of mathematical resources. Dr. Alpha frames his interpretation of $x_f = x_o + v_o\Delta t$ as Calculation as he considers the computational algorithms implicit in the equation. Dr. Beta thinks about how $x_f = x_o + v_o\Delta t$ encodes information about a physical situation; she frames her activity as Physical Mapping. Professor Gamma frames his interpretation of $x_f = x_o + v_o\Delta t$ as Invoking Authority as he considers how the quotable equation neatly packages together a variety of other information. Dr. Delta frames her work as Math Consistency, leading her to consider analogous mathematical instances.

This dissertation has described how these four common framing clusters emerge from an analysis of the work of upper level physics students. Chapter Two sets the theoretical background for this work. It describes a manifold model of mind wherein many small-grained knowledge elements activate and chain together in complicated, context dependent ways. Framing is a (often subconscious) process that helps an individual navigate all these manifold possibilities. It primes a certain subset of a person's total knowledge for conscious consideration. Chapters Three and Four lay out a method for identifying how these upper level physics students are framing their math use: look at the justification they are currently offering for their mathematics. The type of mathematical justification a student offers correlates to what aspect of mathematics they are currently focusing on, that is, how they are framing their math use. Calculation, Physical Mapping, Invoking Authority, and Math Consistency are the four common framing clusters that emerged from the data set.

Chapter Five provides the first detailed applications of this framing analysis tool. Two case studies illustrate how frame negotiation can constitute an important, if usually tacit, conversational dynamic. Much of these students' conversation can be seen as a series of pushes and pulls as each attempts to establish a common framing of the mathematics at hand. Miscommunications result when this frame negotiation fails. The conversations often became richer and more involved when several students shared a frame, interpreting their work in that moment with similar subsets of mathematical resources. Chapter Six illustrates how an external tool, Mathematica in this case, can preferentially couple to a certain framing. Mathematica became an especially powerful part of the context of the students' work, cuing and sustaining a Calculation framing at the expense of alternate framings. Chapter Seven describes how a framing analysis highlights an important component of expertise in physics problem solving. Experts are especially adept at framing a problem in several different ways, insisting on coherency among the resulting arguments.

8.2 The Value of This Dissertation to Physics Education Researchers: It's a Detailed Description of the Analysis and Importance of Framing in Regard to Physics Students' Conceptual Reasoning with Math

This dissertation contains many detailed episodes of upper level physics students' thinking. Analysis always starts from actual transcript of the students' conversations as they work on their physics. These authentic episodes, as a whole, illustrate the dynamic, moment-to-moment flow of these students' thinking.

These students have been shown to shift their focus during a problem solving episode, sometimes quickly and repeatedly. Perhaps they try a computational approach for a little while, focusing on performing an algorithmically correct calculation. A quick shift often follows, with the students becoming aware of an alternate approach. Their conscious attention is then occupied differently. Perhaps they examine the physical referents of their math or search for an authoritative source of information to quote. Many episodes have highlighted a temporary blindness in students. They appear oblivious to an alternate course of action until some cue helps it spring to attention. Once cued, this new approach is often carried out quite efficiently. See especially both case study episodes in Chapter Five.

While many of the students in this dissertation struggle at times, their struggles are largely due to these issues of selective attention. The relevant issue here is the activation of knowledge, not the lack of knowledge.

This dissertation thus argues for, and supports, a manifold model of students' thinking. The work and speech we observe in physics students are best modeled as the product of a complex activation, deactivation, and coordination of a great many smaller knowledge elements. A unitary model of mind, in contrast, would speak in terms of students either having or not having a large, coherent conceptual idea. As a quick specific example, recall S2 from Chapter Five's gravitational work problem. For the first part of the episode, S2 was seemingly oblivious to S1's longer-path-means-more-work-done argument. He simply kept asserting the works should be the same along the two paths. A unitary model would explain that S2 has a poor conception of a dot product. S2, however, gives an excellent answer to S1's question later in the episode once the relevant resources have been activated in his mind. He then seems to have an excellent conception of the dot product. If this unitary dot product conception is in fact so great, why didn't it show up right from the beginning? A unitary model of mind does not capture the context dependence observed in this dissertation's examples. A manifold view's emphasis on activation and deactivation is a much better fit.

This dissertation has proposed a method for analyzing this complicated activation and deactivation of knowledge elements in a manifold model of mind. It uses the idea of framing to describe how the mind, often subconsciously, primes a certain subset of a person's available knowledge for explicit consideration. The idea of framing is not new. Many different academic fields have spoken of similar processes concerning selective attention and the influence of expectations, including linguistics, psychology, art, and sociology. This dissertation's main contribution is its system for analyzing how upper level physics students frame their math use. Look to the type of justification the student offers for the math at hand, and this classification of warrants will offer a particularly good window onto his framing. Such an analysis yielded four general framing clusters common to upper level physics students: Calculation, Physical Mapping, Invoking Authority, and Math Consistency.

8.3 The Value of This Dissertation to Physics Teachers: It Makes Explicit an Important Component of Our Students' Thought Process

This dissertation's analysis explicitly focuses on how physics students frame their math use. Bringing out this framing dynamic allowed for a natural parsing of many episodes of students' work. Conversations among students, or between a student and an interviewer, often reduced to a series of framing pushes and pulls: framing bids that were either accepted or passed over.

These framing bids and cues were rarely explicit. A framing bid that is too implicit or too quick often passes by unnoticed. If a given student is especially committed to a certain framing at a certain time, he can be that much harder to nudge out of it. This dissertation has examples of students talking past each other, neither seeming to acknowledge what the other is saying. Framing differences were often at the root of these miscommunications.

These framing differences, and the miscommunications that accompany them, could certainly occur in a physics classroom as well. Both the teacher and the students will naturally frame what occurs during a lesson, but there is no guarantee they will frame each part of the lesson in the same way. A teacher may calculate for a while and then want to make a point about how an equation matches a physical expectation. The teacher may even offer a signal that he's switching approaches, but perhaps that signal isn't sufficient to tip the students. They may merely try to interpret his Physical Mapping comments through a Calculation lens. Perhaps a professor gives an extended Math Consistency discussion, carefully explaining how the math at hand is analogous to a more familiar math idea. Maybe his students are framing his discussion as Invoking Authority and instead hear a series of math facts to be accepted on faith.

There are two readily apparent ways to combat such teacher/student framing misunderstandings. The first is for a teacher to simply exaggerate her framing cues. If the situation calls for conveniently quoting a rule, spend a little extra time explaining your reasons for doing so. If it's obvious to you, as a teacher, that a Physical Mapping discussion is in order, make that (and your reasons for believing so) more explicit to your class. More explicit framing cues might lessen the probability of miscommunication due to a framing mismatch.

The second antidote to teacher/student framing mismatches is for the teacher to gather more evidence, in real time, of her students' framing. In a traditional lecture, information tends to only flow from the professor to the students. Such a lecturer will have scant evidence available for how her students are framing her lesson. Asking questions that have simple phrase-like answers may give a teacher evidence of the simple correctness or incorrectness of the class's answers, but is only of marginal help for deducing the students' framing. Engaging one's students in extended discussions during class is the best way to get valuable framing evidence. Asking open-ended questions that give students a wide range of possible responses will require them to explain their reasoning to a much greater depth. As they explain their justifications for their claims, their framing will become much more apparent to the teacher. Framing mismatches will become much easier to diagnose in real time.

Professors Alpha through Delta were all preparing lectures with material that could have been appropriate depending on the particulars of their students' thinking. What's most importantly missing from each of their lesson plans is any way of gathering evidence, in real time, of how their students are framing the math use in the lecture.

Finally, Chapter Seven's observations bear repeating here. Students' reasoning should be judged by richer and more sophisticated criteria than a simple labeling of their answers as correct or incorrect. Framing considerations can add considerable depth to a teacher's evaluation of her students' thinking. Are the students only framing their activity in one way, or are they making an effort to approach the problem with several different framings? Are they valuing coherency among the different arguments they produce for the same problem? Even some incorrect student answers are very sophisticated from this multiple framing viewpoint, like several of the examples in Chapter Seven. As teachers, we should make a special point of modeling this search for coherency among framings for our students.

8.4 Future Directions

This dissertation has detailed a system for analyzing upper level physics students' framing of mathematics. It has put forward both a lens to look through (examine the justification students offer for their math) and a clustering scheme for common types of framings (Calculation, Physical Mapping, Invoking Authority, and Math Consistency).

Since we now have a system for identifying these framings, a logical extension is examining how to cue them most reliably. What is the best form for one of these framing cues? Do textual cues in the problems themselves have more or less an effect than the setting (interview, homework group, exam, etc)? Does a student's framing of mathematics respond equally well to verbal and visual cues?

Any investigation of cuing is very likely to quickly encounter issues with individual student differences. A certain textual prompt might be especially likely to cause a specific student to start calculating, whereas the same textual prompt may tend to pass by two other students unnoticed. A large sample size will likely be necessary before any authoritative claims of a specific framing cue's efficacy can be made.

Analyzing episodes from a physics class itself is another interesting extension of this dissertation's work. All of the episodes presented in this dissertation come from homework groups or interviews. The social setting of a formal physics classroom combined with the professor's presence likely exerts a strong influence on how students frame their activity. Carefully analyzing this influence will lead to a still better understanding of how to cue various framings most reliably. How one might collect information on this issue in a traditional lecture is problematical. The need for such feedback argues strongly for beginning to reform upper division physics classes to include more active student engagement in the way that some introductory classes have been reformed.

Chapter Seven also raises an important issue for future investigation. That chapter focuses on valuing students' attempts to find coherency across several different lines of argument, but there are still grain-size issues to explore with this integration of different framings. For example, consider what an expert might think of when you tell him "add three resistors in series". Chances are, our expert will quickly think of something typical to both an Invoking Authority framing (quoting the rule $R_{eq} = R_1 + R_2 + R_3$) and a Physical Mapping framing (if you put more resistors in series, the current must go through all of them in turn, so their effects should add).

But how did this integration of justifications come to be in our expert? If we want to teach our students about resistors in series, is it easiest to learn if we cast Physical Mapping as subservient to Invoking Authority (i.e. state up front "here's the rule" and apply it several times before tossing in a physical motivation to help students better remember the rule)? Or is the reverse hierarchy easier for students to grasp (i.e. starting with a physically motivated reason resistances in series should add and later on giving the equivalent resistance formula, casting it as nothing more than a shorthanded way to write this physical intuition)?

Answering such a question will certainly depend on the particular students involved. If one presentation method more closely aligns with what they are used to in their typical physics class, then that method will likely have an advantage. The particular physics subject matter at hand probably also plays a role. Some uses of math in physics simply have clearer, more obvious physical interpretations than others. It would be very surprising if a blanket statement like "Always start with a physical interpretation and only give a rule quickly later on" always held the best approach. Still, specific situations can be examined.

All of these possibilities for future work would help further advance the two main goals of this dissertation: providing researchers with an effective way to model students' thinking and providing teachers with a practical tool for enhancing their interactions with students.

Appendix: Transcript for Inter-Rater Reliability Test

These students were juniors and seniors enrolled in the Advanced Electricity and Magnetism course (PHYS 411). The students are given a square loop of wire, length "a" to a side, that sits in the y-z plane and is centered on the origin. It has a current *I* running counterclockwise in it. They are trying to find the total magnetic force on the loop if it is placed in a magnetic field of the form $\vec{B}(\vec{r}) = kz\hat{x}$.

The transcript that was actually used for the inter-rater reliability test was of a different font and type size with different margins than this document. The line numbering below only numbers the start of each students' speech and corresponds to that used in the actual test (and on the inter-rater charts in Chapter Four).

Page 1

1. **S1:** I know what I did before. OK, what's the question? Where did I get -I/16 a?
5. **S4:** I have no idea. You must have—
7. **S1:** OK, the total, the total J for the whole thing is I/4a. There's some I for the four of these. Then the total...for each side it's one fourth of that.
11. **S4:** J is total current divided by 4a? OK, what's the current? What is J—that's the cross sectional right?
14. **S1:** Yes
15. **S4:** So you're saying the total current divided by 4 times the a, which is the length.
19. **S1:** And then on each side of these there's one fourth of the total J so you another 4 again. now this here it goes in the plus y-hat direction.
23. **S4:** What equation are you using for that integral? Which problem is this...5.4? So you used just—cause we're trying to find all the ones for F.
27. **S1:** It's F = the integral of J cross B d tau. [Equation] 5.27
29. **S4:** 5.27

Off task

32. **S4:** See the reason I didn't integrate is because the B is constant along here. That's the reason I didn't integrate.
36. **S1:** Well, it is. The integral goes away.
38. **S4:** What are you integrating?
40. **S1:** The integral dy. Here it's dy. But when you do the cross product there is no more y. The integral does go away.
43. **S4:** So you just end up with constant number times Ia/2?
46. **S1:** Yeah.

Off task

48. **S1:** [S3] we did something wrong on 5.4.
50. **S3:** What's that?

Page 2

1. **S1:** [S4] and I worked it out. I agree with him.
3. **S3:** So what did we—
5. **S4:** Of course I still did it wrong but apparently the way I did it would have been right.
8. **S1:** If you look at the two y integrals, when we plugged, when we
11. **S3:** The y integrals meaning in the y-hat direction or dy?
13. **S1:** The ones that had dy.
15. **S3:** OK
17. **S1:** When you do the, you notice that the bounds of integration are different because we are integrating around the loop. When we did both of the integrals we just had an a, one of them should be a minus a.
22. **S3:** Why should there be a minus a?
24. **S1:** Because for the side we called three—
26. **S3:** Oh, no, there is a minus a, it's right there. I have a minus a in mine.
29. **S1:** Where?
30. **S3:** It was minus a/2. The first one. So the side which was the bottom. The bottom had the minus a/2.
34. **S4:** (to s3) You got everything to go to zero right?
36. **S3:** And minus z-hat. No.
38. **S1:** (to s4) Yes.
39. **S4:** [S1] said that they summed all to zero. That's what [S1]—
42. **S3:** Oh yeah, the total sum is zero, yeah.
44. **S2:** The force should be something.
46. **S4:** Yeah, it should be going upwards, because there's an exact example of it somewhere.
49. **S1:** OK, no you're—I'm integrating the bounds in the wrong order. You're right.
52. **S3:** OK, there's a B going down and a B going up.

Page 3

1. **S1:** Our minus sign is in the first side.
3. **S4:** This one. If you look at this example they get the sides drop out but—
7. **S3:** So where is there something wrong with this?
9. **S4:** I have no idea I did it a completely different way and got a completely different answer and it may be completely wrong.
12. **S1:** [S3] you were right. I was integrating the bounds—
14. **S3:** But why are we claiming there should be, so kz x is the magnetic field, right?
17. **S1:** Yeah.
18. **S2:** OK look.
19. **S3:** And we have a loop centered on the xy (sic) plane basically because we don't care about the ones running up and down.
23. **S2:** Can we do a little—
25. **S3:** So we're doing left and right. So there's a minus z and a plus z. Right?

28. **S4:** Right.
29. **S3:** So in one the B field is pointing in one direction and in the other it's pointing in the other direction and the current is, yeah I agree it should move up.
34. **S2:** Now, cause, look—
36. **S4:** Does the B field not point in a different direction.
39. **S3:** OK guys here's my thought guys just so you know, oh wait B depends on z.
42. **S3:** Yeah.
43. **S4:** Right.
44. **S2:** Well, then of course it should move somewhere. There should be a force.
47. **S3:** It should.
48. **S2:** The reason why I was going to say there shouldn't be a force is because it's a dipole in a uniform field but it's not a uniform field.

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